Chapter 7

Euclidean Algorithm and Linear Diophantine Equations

7.1 \( \gcd(a, b) \) as an integer combination of \( a \) and \( b \)

It is well known that the gcd of two (positive) integers \( a \) and \( b \) can be calculated efficiently by repeated divisions. Assume \( a > b \). We form two sequences \( r_k \) and \( q_k \) as follows. Beginning with \( r_{-1} = a \) and \( r_0 = b \), for \( k \geq 0 \), let

\[
q_k = \left\lfloor \frac{r_{k-1}}{r_k} \right\rfloor, \quad r_{k+1} = \text{mod}(r_{k-1}, r_k) := r_{k-1} - q_k r_k.
\]

These divisions eventually terminate when some \( r_n \) divides \( r_{n-1} \). In that case, \( \gcd(a, b) = r_n \).

If, along with these divisions, we introduce two more sequences \( (x_k) \) and \( (y_k) \) with the same rule but specific initial values, namely,

\[
x_{k+1} = x_{k-1} - q_k x_k, \quad x_{-1} = 1, \quad x_0 = 0;
\]
\[
y_{k+1} = y_{k-1} - q_k y_k, \quad y_{-1} = 0, \quad y_0 = 1.
\]

then we obtain \( \gcd(a, b) \) as an integer combination of \( a \) and \( b \): \(^1\)

\[
\gcd(a, b) = r_n = a x_n + b y_n.
\]

\(^1\)In each of these steps, \( r_k = a x_k + b y_k \).
It can be proved that $|x_n| < b$ and $|y_n| < a$.

**Theorem 7.1.** Given relatively prime integers $a > b$, there are unique integers $h, k < a$ such that $ak - bh = 1$.

**Proof.** Clearly, $x_n$ and $y_n$ are opposite in sign. Take $(k, h) = (x_n, -y_n)$ or $(b + x_n, a - y_n)$ according as $x_n > 0$ or $< 0$. \[\square\]

**Corollary 7.2.** Let $p$ be a prime number. For every integer $a$ not divisible by $p$, there exists a positive integer $b < p$ such that $ab - 1$ is divisible by $p$. 

<table>
<thead>
<tr>
<th>$k$</th>
<th>$r_k$</th>
<th>$q_k$</th>
<th>$x_k$</th>
<th>$y_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-1$</td>
<td>$a$</td>
<td>$*$</td>
<td>$1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$0$</td>
<td>$b$</td>
<td>$\lfloor \frac{a}{b} \rfloor$</td>
<td>$0$</td>
<td>$1$</td>
</tr>
<tr>
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<td>$\lfloor \frac{a}{r_1} \rfloor$</td>
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<td>$y_1$</td>
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<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
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<td>$q_{n-1}$</td>
<td>$x_{n-1}$</td>
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<tr>
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<td>$q_n$</td>
<td>$x_n$</td>
<td>$y_n$</td>
</tr>
<tr>
<td>$n+1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
</tbody>
</table>
7.2 Nonnegative integer combinations of $a$ and $b$

Example 7.1. Find the largest positive integer which cannot as $7x + 11y$ for integers $x, y \geq 0$.

Let $S := \{7x + 11y : x, y \text{ nonnegative integers}\}$. Arrange the positive integers in the form

1 8 15 22+ 29 36 43 50 57 64 71 ...
2 9 16 23 30 37 44+ 51 58 65 72 ...
3 10 17 24 31 38 45 52 59 66+ 73 ...
4 11+ 18 25 32 39 46 53 60 67 74 ...
5 12 19 26 33+ 40 47 54 61 68 75 ...
6 13 20 27 34 41 48 55+ 62 69 76 ...
7 14 15 28 35 42 49 56 63 70 77 ...

Observations: (i) Every number in the bottom row, being a positive multiple of 7, is in $S$.
(ii) Among the first 11 columns, along each of the first 6 rows, there is a unique entry (with asterisk) which is a multiple of 11. This entry with asterisk, and those on its right along the row, are in $S$.
(iii) None of the entries on the left of an entry with asterisk is in $S$.
(iv) The entries with asterisks are on different columns.
(v) The rightmost entry with an asterisk is 66. From this, the largest integer not in $S$ is $66 - 7 = 59$.

Theorem 7.3. Let $a$ and $b$ be given relatively prime positive integers. Every integer greater than $ab - a - b$ can be written as $ax + by$ for nonnegative integers $x$ and $y$.

Proof. Let $S := \{ax + by : x, y \text{ nonnegative integers}\}$.

Suppose, for a contradiction, $ab - a - b = ax + by$, $x, y \geq 0$. Then $ab = a(x + 1) + b(y + 1)$. Note that $a|b(y + 1)$. Since $\gcd(a, b) = 1$, we must have $a|y + 1$. But $y + 1$ is a positive integer smaller than $a$. This is clearly a contradiction. From this $ab - a - b \notin S$.

Every integer $t$ in the range $0 < t < a$ can be written as $t = au - bv$ for $0 < u < b$ and $0 \leq v < a$. (Choose $u \in \{1, 2, \ldots, b - 1\}$ such that $au \equiv t \pmod{b}$. Then $0 < au - t < ab$. It follows that $au - t = bv$ for some $1 \leq v < a$. Thus, every positive integer $< a + b$ is of the form $au - bv$, $0 < u < b$, $0 \leq v < a$. Suppose $(a - 1)(b - 1) \leq n < ab$. Then $ab - n < a + b$. Write $ab - n = au - bv$ for $0 < u < b$ and $0 \leq v < a$. From this, $n = a(b - u) + bv$. This shows that $n \in S$. \qed
Exercise

1. Suppose \( n \geq 3 \). Show that an even number of the fractions

\[
\frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}
\]

are in lowest terms.

2. Show that \((n! + 1, (n + 1)! + 1) = 1\).

3. Instead of successive divisions, the gcd of two positive numbers can be found by repeated subtractions. Make use of this to find \( \gcd(2^a - 1, 2^b - 1) \) for positive integers \( a \) and \( b \).

4. Find a parametrization of the integer points on the line \( 5x + 12y = 3 \).

5. In how many ways can a number of 44-cent and 90-cent stamps were purchased with exactly 50 dollars.

6. Somebody received a check, calling for a certain amount of money in dollars and cents. When he went to cash the check, the teller made a mistake and paid him the amount which was written as cents, in dollars, and vice versa. Later, after spending $3.50, he suddenly realized that he had twice the amount of the money the check called for. What was the amount on the check?
Chapter 8

Pythagorean triangles

8.1 Construction of Pythagorean triangles

By a Pythagorean triangle we mean a right triangle whose side lengths are integers. Any common divisor of two of the side lengths is necessarily a divisor of the third. We shall call a Pythagorean triangle primitive if no two of its sides have a common divisor. Let \((a, b, c)\) be one such triangle. From the relation \(a^2 + b^2 = c^2\), we make the following observations.

1. Exactly two of \(a, b, c\) are odd, and the third is even.

2. In fact, the even number must be one of \(a\) and \(b\). For if \(c\) is even, then \(a\) and \(b\) are both odd, and \(c^2 = a^2 + b^2 \equiv 1 + 1 = 2 \pmod{4}\), an impossibility.

3. We shall assume \(a\) odd and \(b\) even, and rewrite the Pythagorean relation in the form

   \[
   \frac{c + a}{2} \cdot \frac{c - a}{2} = \left(\frac{b}{2}\right)^2.
   \]

   Note that the integers \(\frac{c + a}{2}\) and \(\frac{c - a}{2}\) are relatively prime, for any common divisor of these two numbers would be a common divisor \(c\) and \(a\). Consequently, each of \(\frac{c + a}{2}\) and \(\frac{c - a}{2}\) is a square.

4. Writing \(\frac{c + a}{2} = u^2\) and \(\frac{c - a}{2} = v^2\), we have \(c = u^2 + v^2\) and \(a = u^2 - v^2\). From these, \(b = 2uv\).

5. Since \(c\) and \(a\) are both odd, \(u\) and \(v\) are of different parity.

We summarize this in the following theorem.
**Theorem 8.1.** The side lengths of a primitive Pythagorean triangle are of the form $u^2 - v^2$, $2uv$, and $u^2 + v^2$ for relatively prime integers $u$ and $v$ of different parity.

Figure 8.1: Primitive Pythagorean triangle

Here are some simple properties of a primitive Pythagorean triangle.

1. Exactly one leg is even.
2. Exactly one leg is divisible by 3.
3. Exactly one side is divisible by 5.
4. The area is divisible by 6.
Example 8.1. How many matches of equal lengths are required to make up the following configuration?

Suppose the shape of the right triangle is given by a **primitive** Pythagorean triple \((a, b, c)\). The length of a side of the square must be a common multiple of \(a\) and \(b\). The least possible value is the product \(ab\). There is one such configuration consisting of

(i) two Pythagorean triangles obtained by magnifying \((a, b, c)\) \(a\) and \(b\) times,

(ii) a square of side \(ab\).

The total number of matches is

\[(a + b)(a + b + c) + 2ab = (a + b + c)c + 4ab.\]

The smallest one is realized by taking \((a, b, c) = (3, 4, 5)\). It requires 108 matches.

Example 8.2. For which positive integers \(n\) does the equation \(x(x + n) = y^2\) have solutions in positive integers \(x\) and \(y\)?

**Solution.** We make use of the fact that every positive integer \(\geq 3\) is the side of a Pythagorean triangle.

(i) If \(n = 2k\), \(x(x + 2k) = y^2 \Rightarrow (x + k)^2 = k^2 + y^2\). This clearly has solutions in positive integers \(x\) and \(y\) except when \(\frac{k}{2} = k = 1, 2\).

(ii) If \(n\) is odd, \(x(x + n) = y^2 \Rightarrow 4x(x + n) = (2y)^2; (2x + n)^2 = (2y)^2 + n^2\). This has solutions in positive integers \(x\) and \(y\) for every \(n > 1\).

Therefore, the equation \(x(x + n) = y^2\) has solutions except when \(n = 1, 2, 4\).
Exercise

1. Show that for every integer \( a \geq 3 \), there are positive integers \( b \) and \( c \) such that \( a^2 + b^2 = c^2 \).

2. How many matches are required in the next smallest configuration?

3. Find the least number of toothpicks (of equal size) needed to form

4. A man has a square field, 60 feet by 60 feet, with other property adjoining the highway. He put up a straight fence in the line of 3 trees, at \( A, P, Q \). If the distance between \( P \) and \( Q \) is 91 feet, and that from \( P \) to \( C \) is an exact number of feet, what is this distance?

5. Here is the smallest square which can be dissected into three Pythagorean triangles and one with integer sides and integer area.

   (a) What is the length of a side of the square?
   (b) What are the lengths of the sides of the Pythagorean triangles?
6. In rectangle $ABDF$, $AC = 125$, $CD = 112$, $DE = 52$, as shown in the figure, and $AB$, $AD$, $AF$ are also integral. Evaluate $EF$.

7. Three relatively prime numbers $a, b, c$ are such that $a^2$, $b^2$, $c^2$ are in arithmetic progression. Show that they can be written in the form

$$a = -p^2 + 2pq + q^2, \quad b = p^2 + q^2, \quad c = p^2 + 2pq - q^2$$

for relatively prime integers $p, q$ of different parity.
Chapter 9

The area of a triangle

9.1 Heron’s formula for the area of a triangle

Theorem 9.1. The area of a triangle of sidelengths $a$, $b$, $c$ is given by

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$

where $s = \frac{1}{2}(a + b + c)$.

Proof. Consider the incircle and the excircle on the opposite side of $A$. From the similarity of triangles $AIZ$ and $AI'Z'$,

$$\frac{r}{r_a} = \frac{s - a}{s}.$$ 

From the similarity of triangles $CIY$ and $I'CY'$,

$$r \cdot r_a = (s - b)(s - c).$$
From these, 
\[ r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \]
and the area of the triangle is 
\[ \triangle = rs = \sqrt{s(s-a)(s-b)(s-c)}. \]

9.2 Heron triangles

A Heron triangle is one whose sidelengths and area are both integers. It can be constructed by joining two integer right triangles along a common leg. For example, by joining the two Pythagorean triangles \((9, 12, 15)\) and \((5, 12, 13)\), we obtain the Heron triangle \((13, 14, 15)\) with area 84.

Some properties of Heron triangles

1. The semiperimeter is an integer.
2. The area is always a multiple of 6.

Example 9.1. Heron triangles with sides < 50:

<table>
<thead>
<tr>
<th>((a, b, c, \triangle))</th>
<th>((a, b, c, \triangle))</th>
<th>((a, b, c, \triangle))</th>
<th>((a, b, c, \triangle))</th>
<th>((a, b, c, \triangle))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 4, 5, 6)</td>
<td>(5, 6, 12)</td>
<td>(5, 8, 12)</td>
<td>(5, 12, 13, 30)</td>
<td>(10, 13, 13, 60)</td>
</tr>
<tr>
<td>(4, 13, 15, 24)</td>
<td>(13, 14, 15, 84)</td>
<td>(9, 10, 17, 36)</td>
<td>(8, 15, 17, 60)</td>
<td>(16, 17, 17, 120)</td>
</tr>
<tr>
<td>(11, 13, 20, 66)</td>
<td>(7, 15, 20, 42)</td>
<td>(10, 17, 21, 84)</td>
<td>(13, 20, 21, 126)</td>
<td>(13, 13, 24, 60)</td>
</tr>
<tr>
<td>(12, 17, 25, 90)</td>
<td>(7, 24, 25, 84)</td>
<td>(14, 25, 25, 168)</td>
<td>(3, 25, 26, 36)</td>
<td>(17, 25, 26, 204)</td>
</tr>
<tr>
<td>(17, 25, 28, 210)</td>
<td>(20, 21, 29, 210)</td>
<td>(6, 25, 29, 60)</td>
<td>(17, 17, 30, 120)</td>
<td>(11, 25, 30, 132)</td>
</tr>
<tr>
<td>(5, 29, 30, 72)</td>
<td>(8, 29, 35, 84)</td>
<td>(15, 34, 35, 252)</td>
<td>(25, 29, 36, 360)</td>
<td>(19, 20, 37, 114)</td>
</tr>
<tr>
<td>(15, 26, 37, 156)</td>
<td>(13, 30, 37, 180)</td>
<td>(12, 35, 37, 210)</td>
<td>(24, 37, 37, 420)</td>
<td>(16, 25, 39, 120)</td>
</tr>
<tr>
<td>(17, 28, 39, 210)</td>
<td>(25, 34, 39, 420)</td>
<td>(10, 35, 39, 168)</td>
<td>(29, 29, 40, 420)</td>
<td>(13, 37, 40, 240)</td>
</tr>
<tr>
<td>(25, 39, 40, 408)</td>
<td>(15, 28, 41, 126)</td>
<td>(9, 40, 41, 180)</td>
<td>(17, 40, 41, 336)</td>
<td>(18, 41, 41, 360)</td>
</tr>
<tr>
<td>(29, 29, 42, 420)</td>
<td>(15, 37, 44, 264)</td>
<td>(17, 39, 44, 330)</td>
<td>(13, 40, 45, 252)</td>
<td>(25, 25, 48, 168)</td>
</tr>
<tr>
<td>(29, 35, 48, 504)</td>
<td>(21, 41, 50, 420)</td>
<td>(39, 41, 50, 780)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
9.3 Heron triangles with consecutive sides

If \((b-1, b, b+1, \triangle)\) is a Heron triangle, then \(b\) must be an even integer. We write \(b = 2m\). Then \(s = 3m\), and \(\triangle^2 = 3m^2(m-1)(m+1)\). This requires \(m^2 - 1 = 3k^2\) for an integer \(k\), and \(\triangle = 3km\). The solutions of \(m^2 - 3k^2 = 1\) can be arranged in a sequence

\[
\begin{pmatrix} m_{n+1} \\ k_{n+1} \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} m_n \\ k_n \end{pmatrix}, \quad \begin{pmatrix} m_1 \\ k_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.
\]

From these, we obtain the side lengths and the area.

The middle sides form a sequence \((b_n)\) given by

\[b_{n+2} = 4b_{n+1} - b_n, \quad b_0 = 2, \ b_1 = 4.\]

The areas of the triangles form a sequence

\[\triangle_{n+2} = 14\triangle_{n+1} - \triangle_n, \quad T_0 = 0, \ T_1 = 6.\]

<table>
<thead>
<tr>
<th>(n)</th>
<th>(b_n)</th>
<th>(T_n)</th>
<th>Heron triangle</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
<td>0</td>
<td>((1, 2, 3, 0))</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>6</td>
<td>((3, 4, 5, 6))</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
<td>84</td>
<td>((13, 14, 15, 84))</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>
9.4 Heron triangles with sides in arithmetic progression

We write \( s - a = u, \ s - b = v, \) and \( s - c = w. \)

\( a, b, c \) are in A.P. if and only if \( u, v, w \) are in A.P. Let \( u = v - d \) and \( w = v + d. \) Then we require \( 3v^2(v - d)(v + d) \) to be a square. This means \( v^2 - d^2 = 3t^2 \) for some integer \( t. \)

**Proposition 9.2.** Let \( d \) be a squarefree integer. If \( \gcd(x, y, z) = 1 \) and \( x^2 + dy^2 = z^2, \) then there are integers \( m \) and \( n \) satisfying \( \gcd(dm, n) = 1 \) such that (i)

\[
x = m^2 - dn^2, \quad y = 2mn, \quad z = m^2 + dn^2
\]

if \( m \) and \( dn \) are of different parity, or (ii)

\[
x = \frac{m^2 - dn^2}{2}, \quad y = mn, \quad z = \frac{m^2 + n^2}{2},
\]

if \( m \) and \( dn \) are both odd.

For the equation \( v^2 = d^2 + 3t^2, \) we take \( v = m^2 + 3n^2, \ d = m^2 - 3n^2, \) and obtain \( u = 6n^2, \ v = m^2 + 3n^2, \ w = 2m^2, \) leading to

\[
a = 3(m^2 + n^2), \quad b = 2(m^2 + 3n^2), \quad c = m^2 + 9n^2,
\]

for \( m, n \) of different parity and \( \gcd(m, 3n) = 1. \)

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( a - d )</th>
<th>( a )</th>
<th>( a + d )</th>
<th>area</th>
</tr>
</thead>
<tbody>
<tr>
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<td>2</td>
<td>15</td>
<td>14</td>
<td>13</td>
<td>84</td>
</tr>
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<td>4</td>
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<td>10920</td>
</tr>
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</table>

If \( m \) and \( n \) are both odd, we obtain

<table>
<thead>
<tr>
<th>( m )</th>
<th>( n )</th>
<th>( a - d )</th>
<th>( a )</th>
<th>( a + d )</th>
<th>area</th>
</tr>
</thead>
<tbody>
<tr>
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<td>1</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
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<td>5</td>
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<td>17</td>
<td>210</td>
</tr>
<tr>
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<td>1</td>
<td>15</td>
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<td>126</td>
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<td>53</td>
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<td>1</td>
<td>39</td>
<td>76</td>
<td>113</td>
<td>570</td>
</tr>
</tbody>
</table>
Example 9.2. Here is an example of a Heron triangle with subdivision into two Heron triangles with equal inradii.

\((a, b, c) = (15, 8, 17); CP = 6, PB = 9. AP = 10.\) The two small triangles have the same inradius 2.
Chapter 10

The integer equations

\[ x^2 - dy^2 = \pm 1 \]

10.1 Integer solutions of \( x^2 - dy^2 = 1 \)

Let \( d \) be a positive, non-square integer. The positive integer solutions of the Pell equation

\[ x^2 - dy^2 = 1 \]

form a sequence \((x_k, y_k)\) which can be obtained recursively by

\[
\begin{pmatrix}
x_{k+1} \\
y_{k+1}
\end{pmatrix} = \begin{pmatrix} a & db \\ b & a \end{pmatrix} \begin{pmatrix} x_k \\
y_k
\end{pmatrix}, \quad \begin{pmatrix} x_1 \\
y_1
\end{pmatrix} = \begin{pmatrix} a \\
b
\end{pmatrix},
\]

where \((a, b)\) is the smallest positive solution of the equation, usually called the fundamental solution.

Example 10.1. The fundamental solution of the Pell equation \( x^2 - 2y^2 = 1 \) is \((3, 2)\). This generates an infinite sequence of nonnegative solutions \((x_n, y_n)\) defined by

\[
x_{n+1} = 3x_n + 4y_n, \quad y_{n+1} = 2x_n + 3y_n; \quad x_0 = 1, \ y_0 = 0.
\]

The beginning terms are

\[
\begin{array}{cccccccccc}
 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
x_n & 3 & 17 & 99 & 577 & 3363 & 19601 & 114243 & 665857 & 3880899 & 22619537 \\
y_n & 2 & 12 & 70 & 408 & 2378 & 13860 & 80782 & 470832 & 2744210 & 15994428
\end{array}
\]
10.2 Integer solutions of \( x^2 - dy^2 = -1 \)

The equation \( x^2 - dy^2 = -1 \) may or may not be solvable in integers. If it has a solution \((x_1, y_1)\), then other solutions can be obtained recursively by

\[
\begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} a & db \\ b & a \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \quad \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} P_{l-1} \\ Q_{l-1} \end{pmatrix},
\]

making use of the fundamental solution of the positive Pell equation \( x^2 - dy^2 = 1 \).

10.3 Applications

10.3.1 Triangular numbers which are squares

Suppose the \( k \)-th triangular number \( T_k = \frac{1}{2}k(k + 1) \) is the square of \( n \). Then \( n^2 = \frac{1}{2}k(k + 1); 4k^2 + 4k + 1 = 8n^2 + 1; (2k + 1)^2 - 8n^2 = 1 \). The smallest positive solution of the Pell equation \( x^2 - 8y^2 = 1 \) being \((3,1)\), we have the solutions \((k_i, n_i)\) of the equation given by

\[
2k_{i+1} + 1 = 3(2k_i + 1) + 8n_i, \quad n_{i+1} = (2k_i + 1) + 3n_i, \quad k_0 = 1, \; n_0 = 1.
\]

This means

\[
k_{i+1} = 3k_i + 4n_i + 1, \quad n_{i+1} = 2k_i + 3n_i + 1, \quad k_0 = 1, \; n_0 = 1.
\]

The beginning values of \( k \) and \( n \) are as follows.

<table>
<thead>
<tr>
<th>( i )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k_i )</td>
<td>1</td>
<td>8</td>
<td>49</td>
<td>288</td>
<td>1681</td>
<td>9800</td>
<td>57121</td>
<td>332928</td>
<td>1940449</td>
<td>11309768</td>
<td>65911179</td>
<td>...</td>
</tr>
<tr>
<td>( n_i )</td>
<td>1</td>
<td>6</td>
<td>35</td>
<td>204</td>
<td>1189</td>
<td>6930</td>
<td>40391</td>
<td>235416</td>
<td>1372105</td>
<td>7997214</td>
<td>46611179</td>
<td>...</td>
</tr>
</tbody>
</table>

10.3.2 Pythagorean triangles with consecutive legs

Let \( x \) and \( x + 1 \) be the two shorter sides of a Pythagorean triangle, with hypotenuse \( y \). Then \( y^2 = x^2 + (x + 1)^2 = 2x^2 + 2x + 1 \). From this, \( 2y^2 = (2x + 1)^2 + 1 \). The equation With \( z = 2x + 1 \), this reduces to the Pell equation \( z^2 - 2y^2 = -1 \), which we know has solutions, with the of this equations are \((z_n, y_n)\) given recursively by smallest positive
one \((1, 1)\), and the equation \(z^2 - 2y^2 = 1\) has smallest positive solution \((3, 2)\). It follows that the solutions are given recursively by

\[
\begin{align*}
  z_{n+1} &= 3z_n + 4y_n, \\
  y_{n+1} &= 2z_n + 3y_n, \\
  z_0 &= 1, \quad y_0 = 1.
\end{align*}
\]

If we write \(z_n = 2x_n + 1\), these become

\[
\begin{align*}
  x_{n+1} &= 3x_n + 2y_n + 1, \\
  y_{n+1} &= 4x_n + 3y_n + 2, \quad x_0 = 0, \quad y_0 = 1.
\end{align*}
\]

The beginning values of \(x_n\) and \(y_n\) are as follows.

\[
\begin{array}{cccccccccccc}
\hline
n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \ldots \\
x_n & 3 & 20 & 119 & 696 & 4059 & 23660 & 137903 & 803760 & 4684659 & 27304196 & \ldots \\
y_n & 5 & 29 & 169 & 985 & 5741 & 33461 & 195025 & 1136689 & 6625109 & 38613965 & \ldots \\
\hline
\end{array}
\]

### 10.3.3

Find all integers \(n\) so that the mean and the standard deviation of \(n\) consecutive integers are both integers.

If the mean of \(n\) consecutive integers is an integer, \(n\) must be odd. We may therefore assume the numbers to be \(-m, -(m-1), \ldots, -1, 0, 1, \ldots, m-1, m\). The standard deviation of these number is \(\sqrt{\frac{1}{3}m(m+1)}\).

For this to be an integer, we must have \(\frac{1}{3}m(m+1) = k^2\) for some integer \(k\). \(m^2 = m = 3k^2; n^2 = (2m + 1)^2 = 12k^2 + 1\). The smallest positive solution of the Pell equation \(n^2 - 12k^2 = 1\) being \((7, 2)\), the solutions of this equations are given by \((n_i, k_i)\), where

\[
\begin{align*}
  n_{i+1} &= 7n_i + 24k_i, \\
  k_{i+1} &= 2n_i + 7k_i, \quad n_0 = 1, \quad k_0 = 0.
\end{align*}
\]

The beginning values of \(n\) and \(k\) are

\[
\begin{array}{cccccccccccc}
\hline
i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\
n_i & 7 & 97 & 1351 & 18817 & 262087 & 3850401 & 56043527 & 80843527 & \ldots \\
k_i & 2 & 28 & 390 & 5412 & 75658 & 1053780 & 14677262 & 204427888 & \ldots \\
\hline
\end{array}
\]