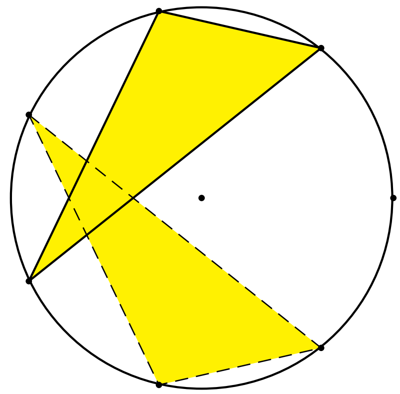
Heptagonal Triangles and Their Companions

Paul Yiu
Department of Mathematical Sciences,
Florida Atlantic University,

yiu@fau.edu



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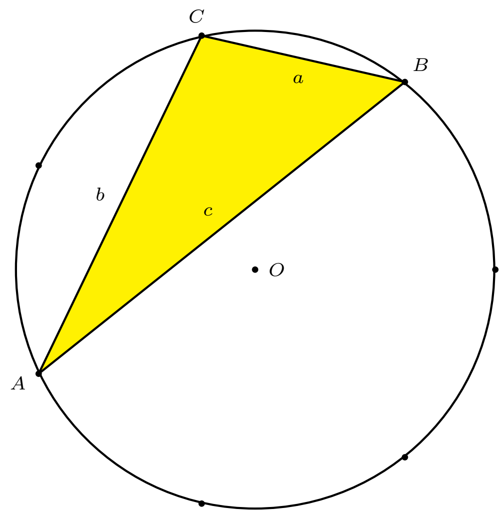
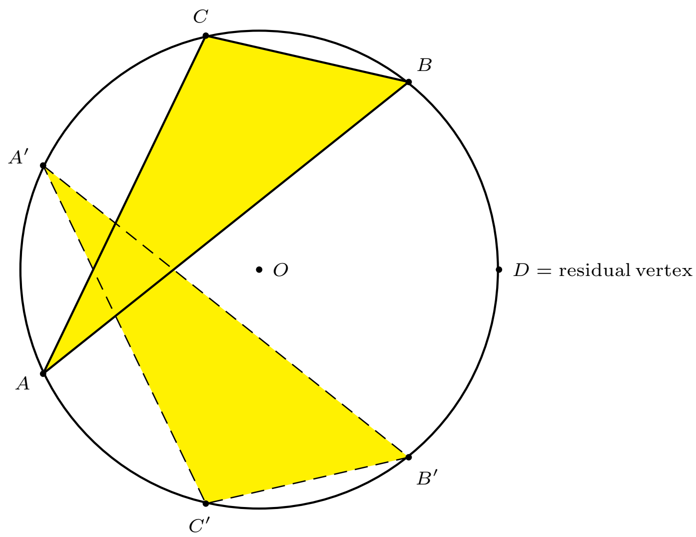
Abstract. A heptagonal triangle is a non-isosceles triangle formed by three vertices of a regular heptagon. Its angles are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. As such, there is a unique choice of a companion heptagonal triangle formed by three of the remaining four vertices. Given a heptagonal triangle, we display a number of interesting companion pairs of heptagonal triangles on its nine-point circle and Brocard circles. Among other results on the geometry of the heptagonal triangle, we prove that the circumcenter and the Fermat points of a heptagonal triangle form an equilateral triangle. The proof is an interesting application of Lester's theorem that the Fermat points, the circumcenter and the nine-point center of a triangle are concyclic.

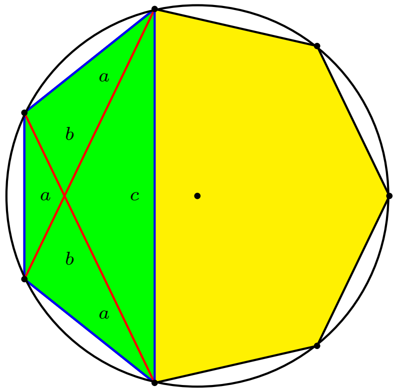
The heptagonal triangle $\mathbf{T}$ and its circumcircle



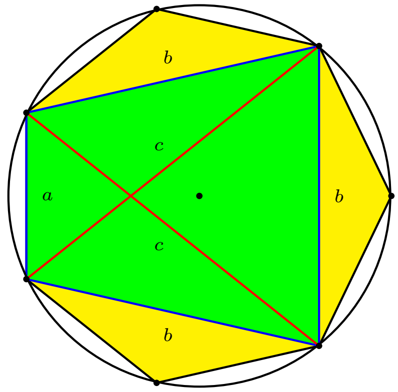
The companion of **T**



Relation between the sides of the heptagonal triangle

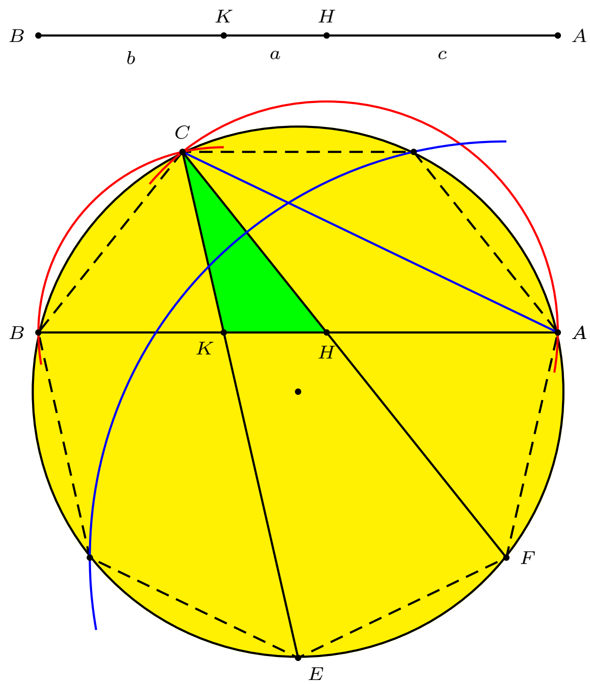


$b^2 = a(c + a)$



$c^2 = a(a + b)$

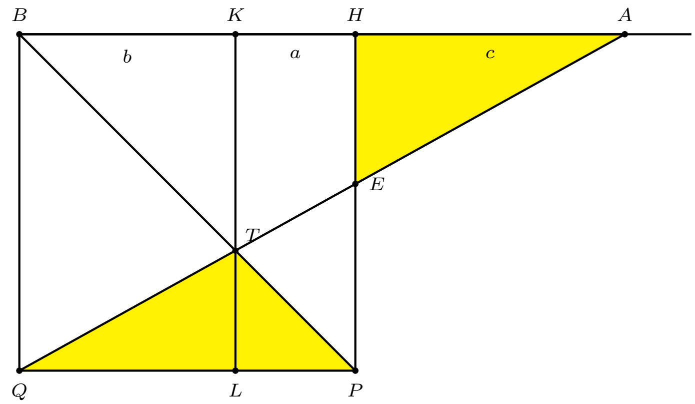
Archimedes' construction (Arabic tradition)



Division of segment: a neusis construction

Let $BHPQ$ be a square, with one side BH sufficiently extended.

Draw the diagonal BP. Place a ruler through Q, intersecting the diagonal BP at T, and the side HP at E, and the line BH at A such that the triangles AHE and TPQ have equal areas. Then,

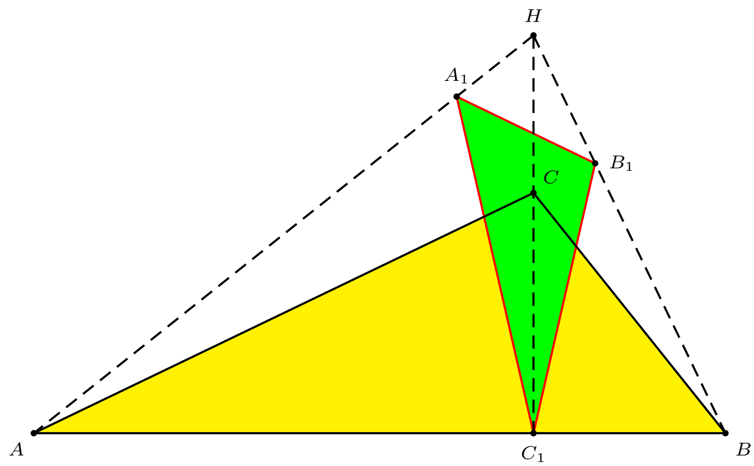


GSP

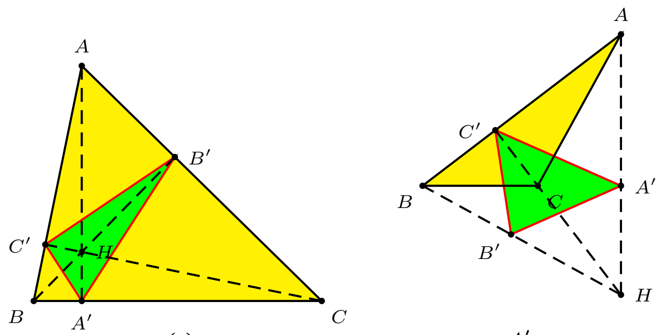
$$b^2 = a(c + a), \qquad c^2 = (a + b)b.$$

The orthic triangle

The heptagonal triangle, apart from the equilateral triangle, is the only triangle similar its own orthic triangle.



Analysis:



(a) $A' =$

Case (a): ABC acute:

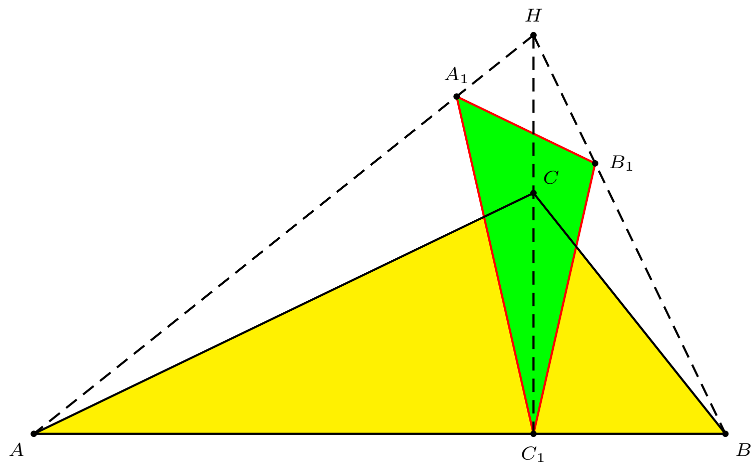
$$A' = \pi - 2A,\ B' = \pi - 2B,\ C' = \pi - 2C.$$

$$\{A', B', C'\} = \{A,\ B,\ C\} \implies A = B = C = \frac{\pi}{3}.$$

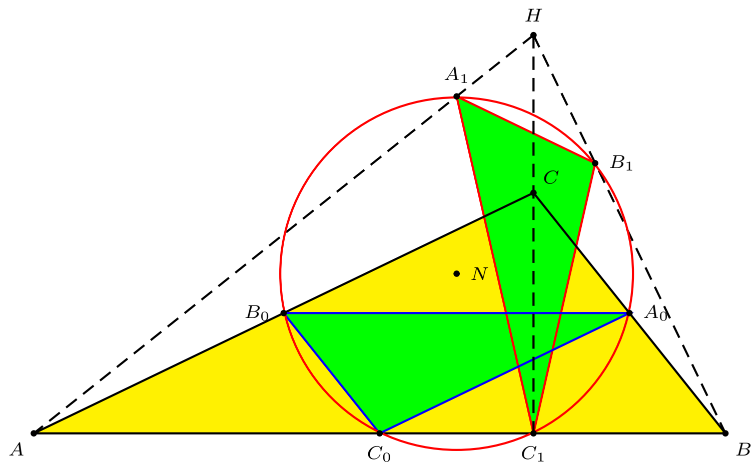
Case (b): Angle C obtuse:

$$A' = 2A,\ B' = 2B,\ C' = 2C - \pi.$$

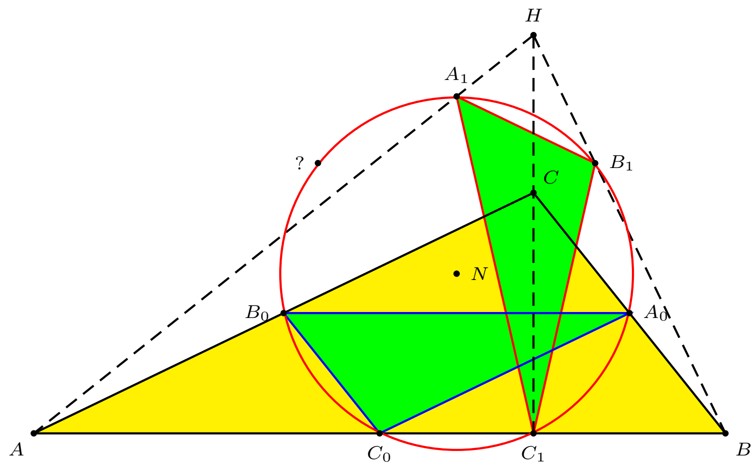
$$\{A', B', C'\} = \{A,\ B,\ C\} \implies \{A,\ B,\ C\} = \{\frac{\pi}{7},\ \frac{2\pi}{7},\ \frac{4\pi}{7}\}.$$



The orthic triangle of the heptagonal triangle has similarity factor $\frac{1}{2}$, because ...



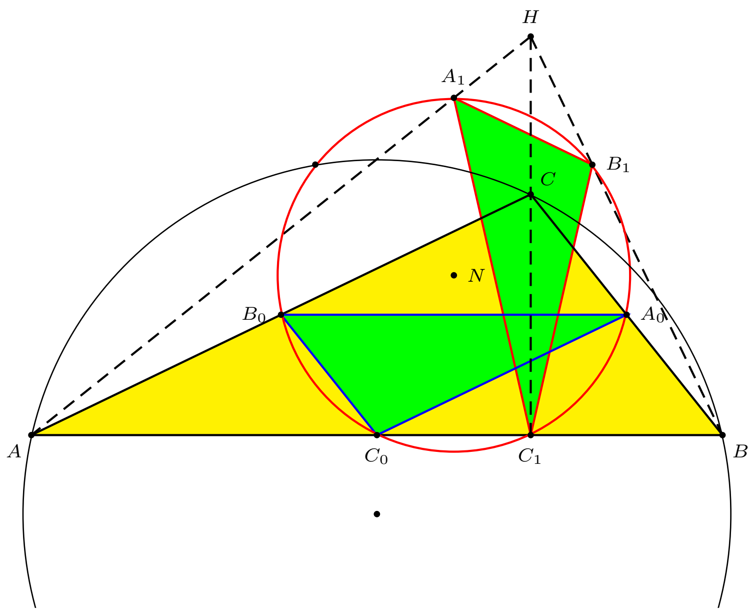
because its vertices are on the **nine-point circle** of $\mathbf{T}$, which also contains the vertices of the **medial triangle**, *i.e.*, the midpoints of the sides of $\mathbf{T}$.



The orthic triangle and the medial triangle are **companion heptagonal triangles**.

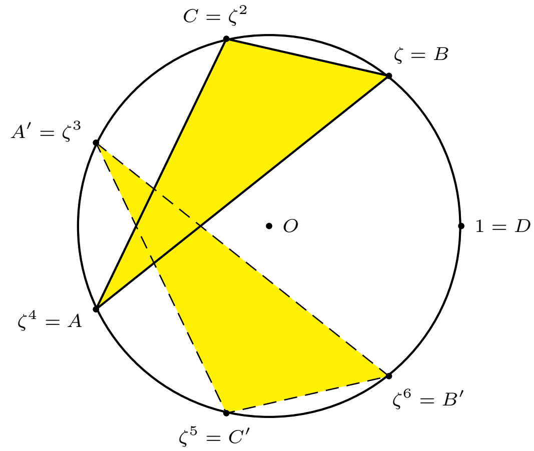
What can we say about the **residual vertex**?

This residual vertex also lies on the circumcircle of $\mathbf{T}$.



To justify this and probe more into the geometry of the heptagonal triangle, we make use of . . .

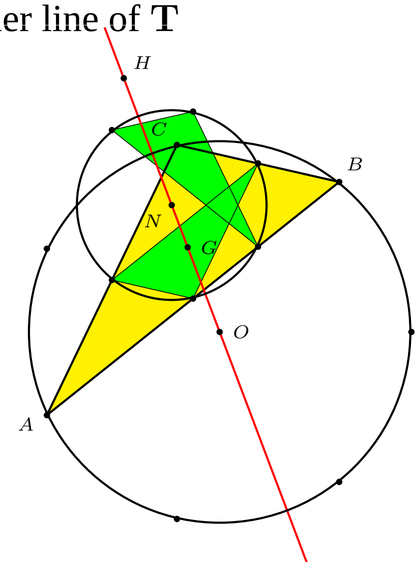
Complex Coordinates



$\zeta =$ primitive 7-th root of unity

$$\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0.$$

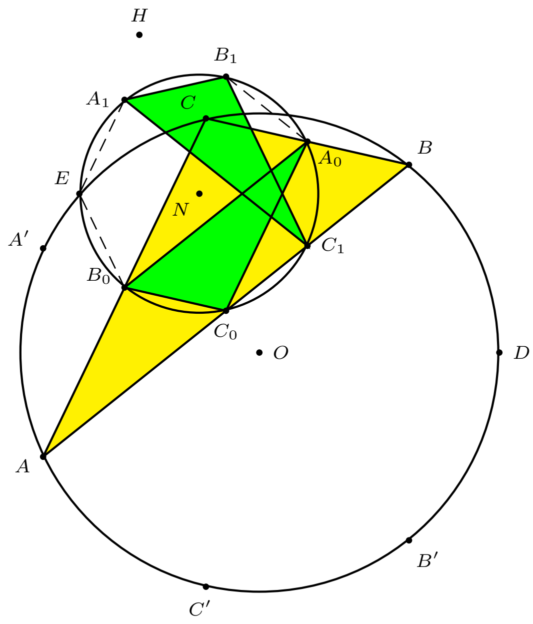
Points on the Euler line of $\mathbf{T}$



$$OG : GN : NH = 2 : 1 : 3.$$

Center	Notation	Coordinates
circumcenter	O	0
centroid	G	$\frac{1}{3}(\zeta + \zeta^2 + \zeta^4)$
nine-point center	N	$\frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
orthocenter	H	$\zeta + \zeta^2 + \zeta^4$

Companionship of medial and orthic triangles

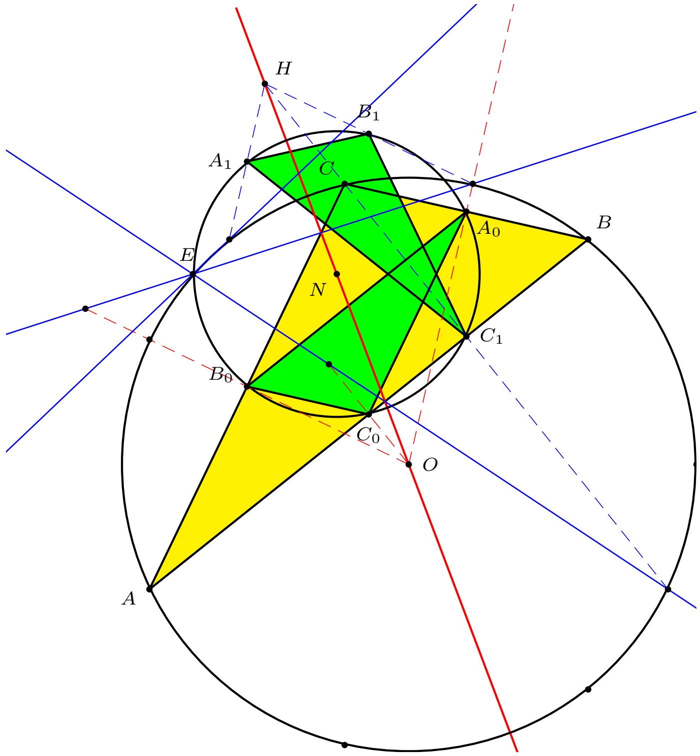


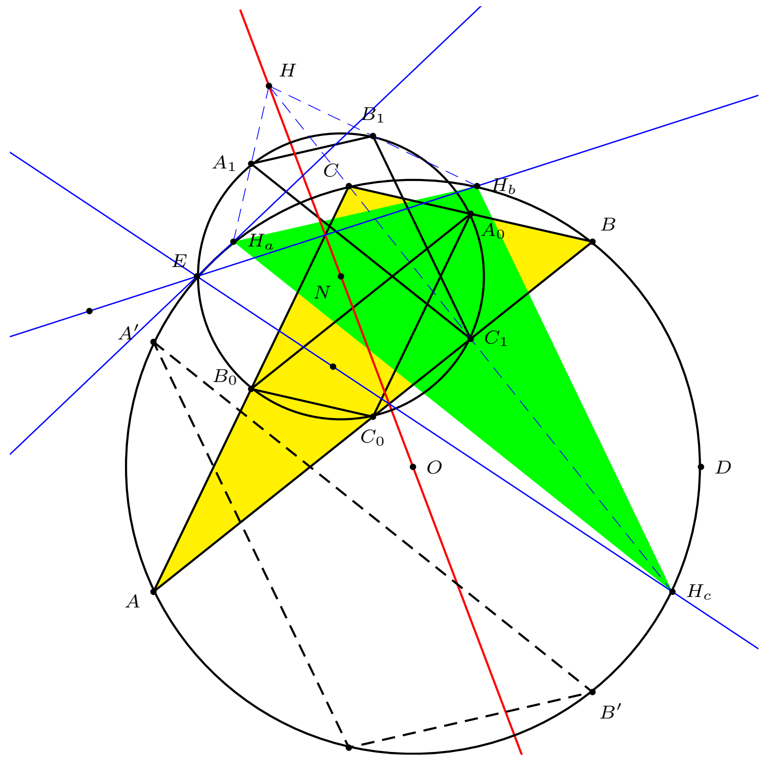
Center:	$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
Residual vertex:	$E = \frac{1}{2}(-1 + \zeta + \zeta^2 + \zeta^4)$

Rotation	Medial triangle	Rotation	Orthic triangle
ζ^4	$A_0 = \frac{1}{2}(\zeta + \zeta^2)$	ζ^3	$C_1 = \frac{1}{2}(\zeta + \zeta^2 - \zeta^3 + \zeta^4)$
ζ	$B_0 = \frac{1}{2}(\zeta^2 + \zeta^4)$	ζ^6	$A_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^6)$
ζ^2	$C_0 = \frac{1}{2}(\zeta + \zeta^4)$	ζ^5	$B_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^5)$

Why is E a point on the circumcircle of $\mathbf{T}$?

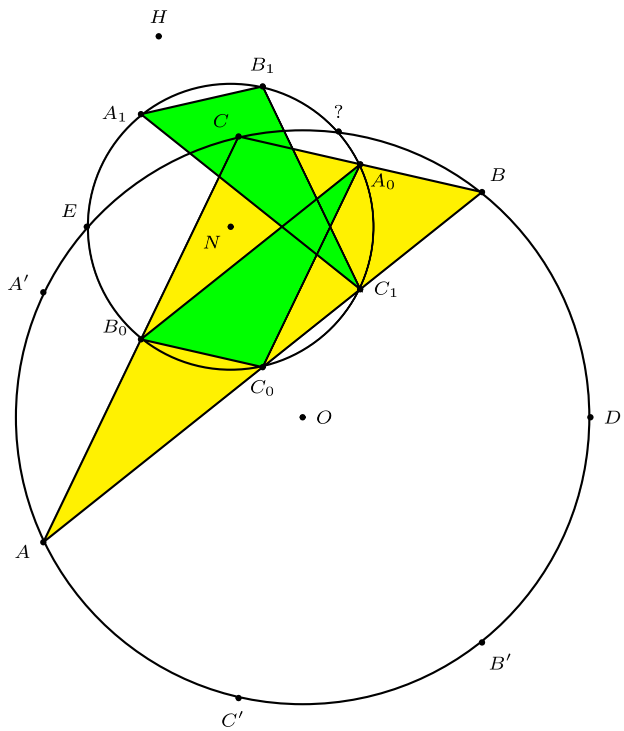
Residual vertex E
= Euler reflection point of $\mathbf{T}$
= Intersection of the reflections of the Euler line
in the three sidelines of $\mathbf{T}$.





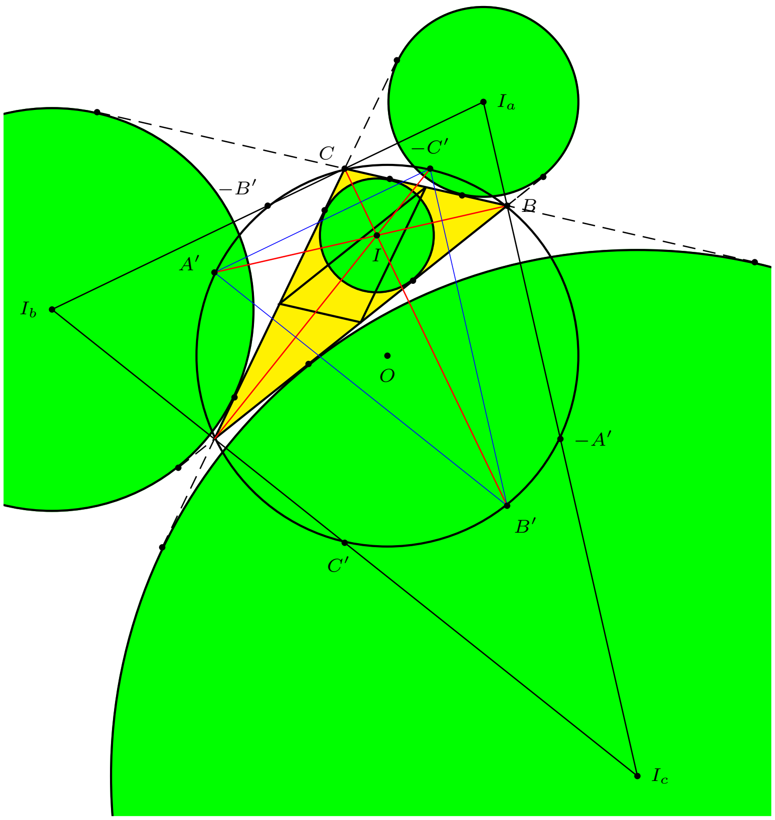
Remark. The reflections of H in the sides are the antipodes of the vertices of the companion of $\mathbf{H}'$.

The second intersection of the nine-point circle and the circumcircle

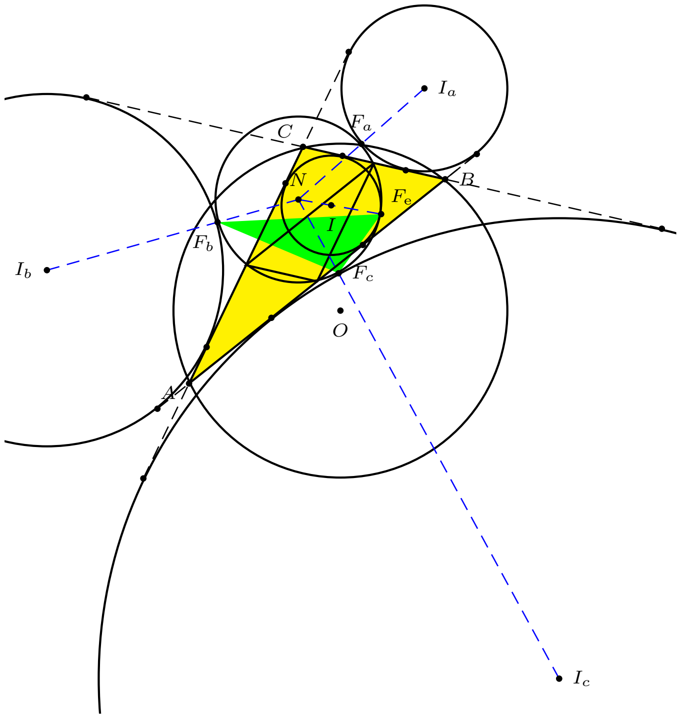


This is a point related to . . .

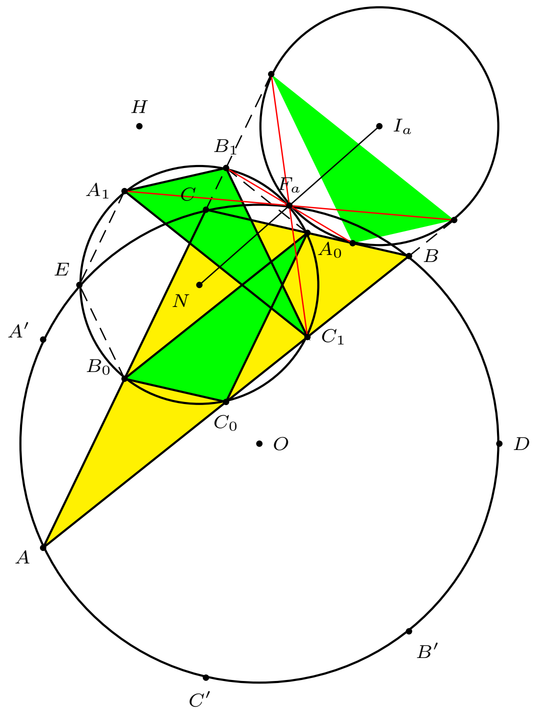
. . . the tritangent circles of **T**, *i.e.*, the **incircle** and **excircles**.



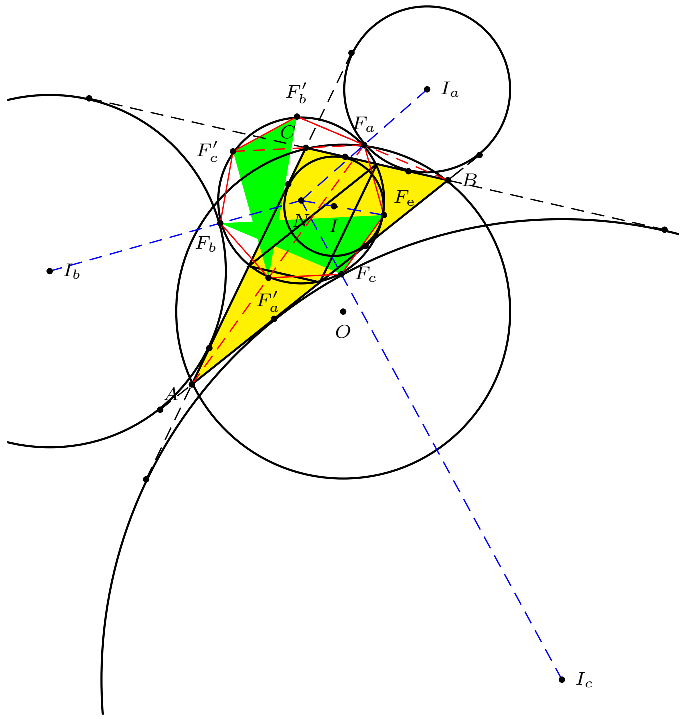
Feuerbach theorem: The nine-point circle is tangent to the incircle internally at F_e and to the excircles at F_a, F_b, F_c.



The second intersection of the circumcircle and the nine-point circle is F_a, the point of tangency of the nine-point circle with the A-excircle.

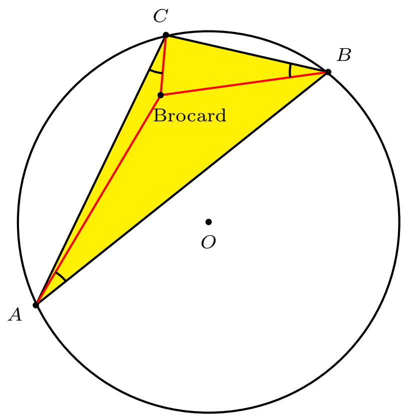


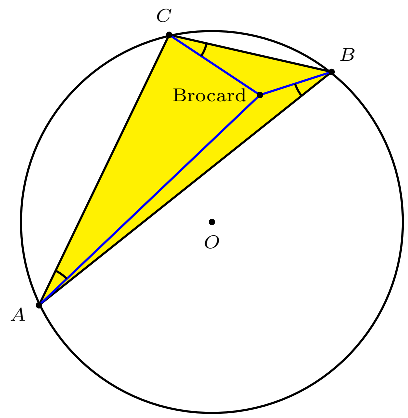
Another **companion pair** of heptagonal triangles **on the nine-point circle**: triangle $F_bF_eF_c$ and triangle $F'_aF'_bF'_c$, where F'_a, F'_b, F'_c are the intersections of AF_a, BF_a, CF_a with the nine-point circle.

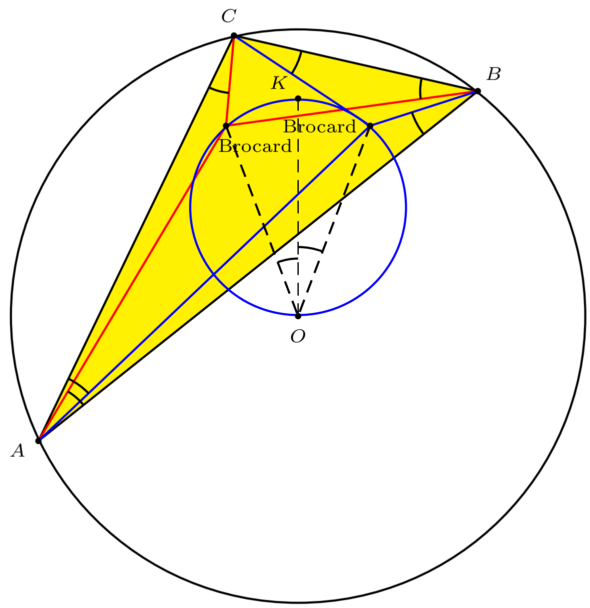


The Brocard points

Given a triangle there are two unique points (Brocard points) for each of which the three marked angles are equal (Brocard angle ω).



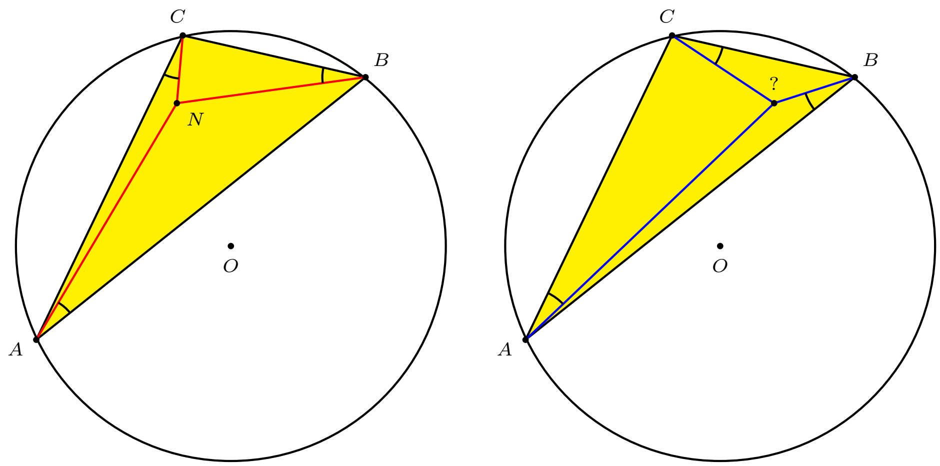




The two Brocard points are equidistant from O.

If K is the **symmedian point** of the triangle, then the circle with diameter OK contains the two Brocard points.

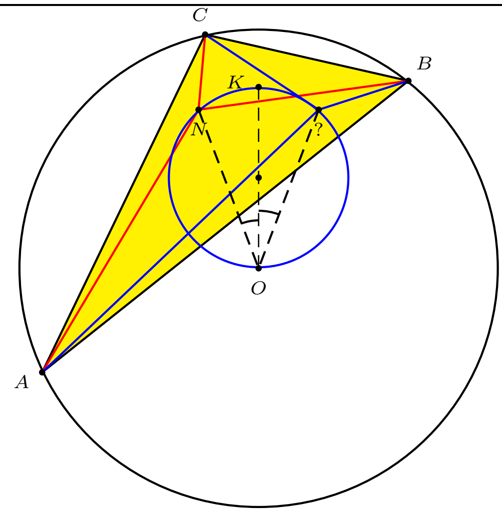
Each of the two chords makes an angle ω with OK.



For the heptagonal triangle $\mathbf{T}$,
(i) the Brocard angle ω satisfies

$$\cot\omega = \sqrt{7};$$

(ii) one of the Brocard points is the nine-point center N (Bankoff-Garfunkel, *Mathematics Magazine*, 46 (1973) 7–19).

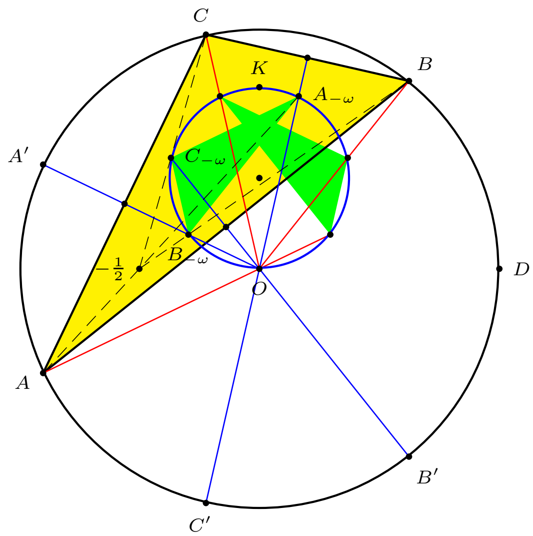


$$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4) \implies K = \frac{2}{\sqrt{7}} \cdot i$$
$$? = -\frac{1}{2}(\zeta^3 + \zeta^5 + \zeta^6).$$

Later, we shall identify ? as an interesting triangle center of **T**.

Meanwhile,

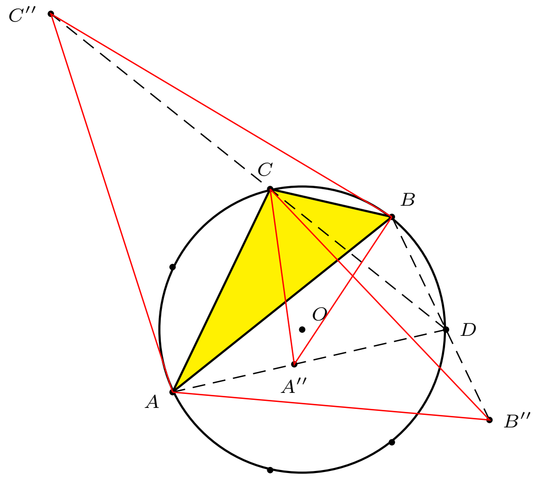
a **companion pair** of heptagonal triangles
on the Brocard circle:



The vertices are the second intersections of OA, OB, OC, OA', OB', OC' with the nine-point circle.

$A_{-\omega}B_{-\omega}C_{-\omega}$ is called the first Brocard triangle of **T**. The Brocard circle is also called the **seven-point circle**, containing O, K, two Brocard points, three vertices of Brocard triangle.

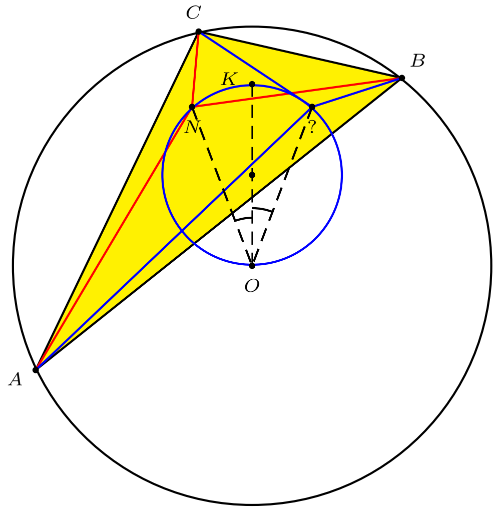
The **residual vertex** of **T**



If similar isosceles triangles $A''BC$, $B''CA$, $C''AB$ of vertical angles 2ω are constructed on the sides, then AA'', BB'', CC'' intersect at D on the circumcircle.

What this means is that ...

the midpoint of DH
$= \frac{1}{2}(1+\zeta+\zeta^2+\zeta^4) = -\frac{1}{2}(\zeta^3+\zeta^5+\zeta^6)$.
Note that this is the **second Brocard point** of **T** we met before:

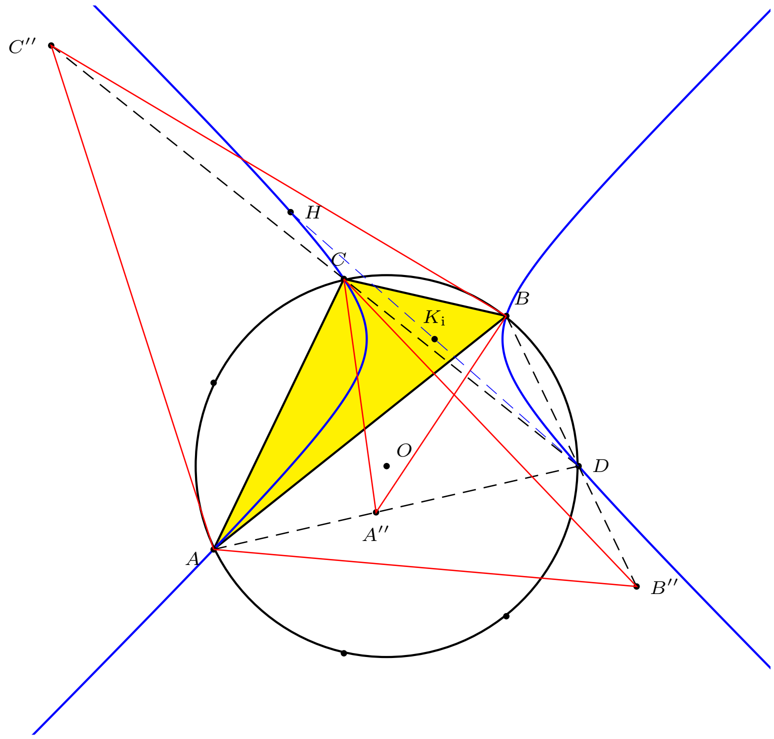


$$N = \frac{1}{2}(\zeta+\zeta^2+\zeta^4) \implies K = \frac{2}{\sqrt{7}}\cdot i$$

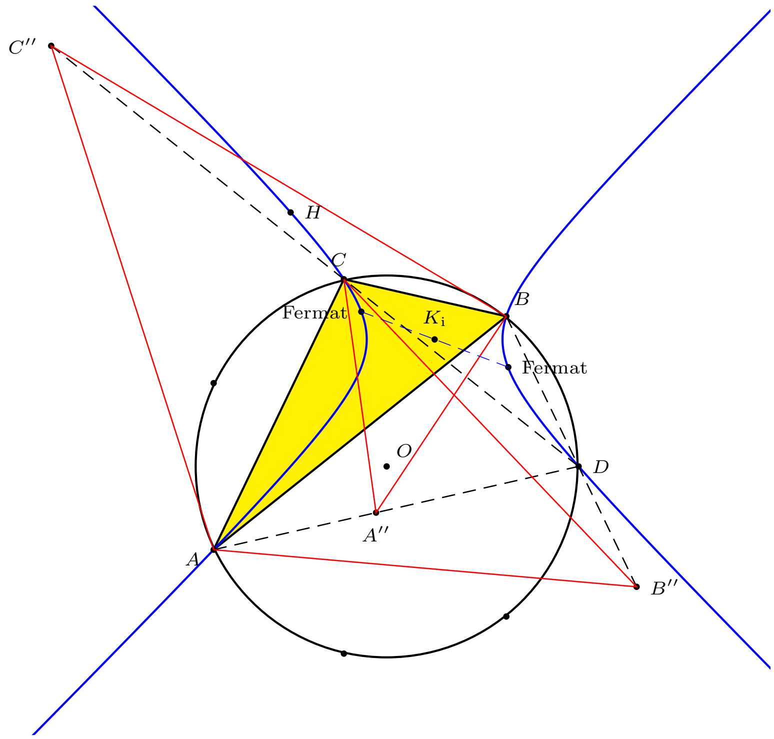
$$? = -\frac{1}{2}(\zeta^3+\zeta^5+\zeta^6).$$

But more importantly, it is

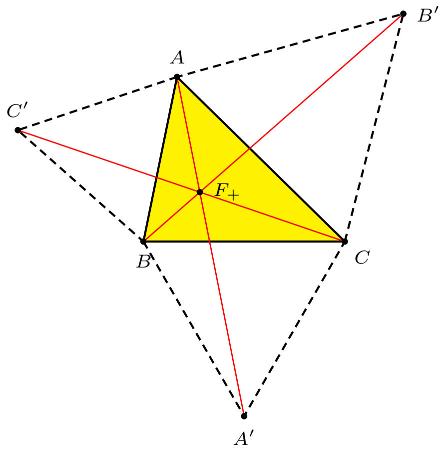
the center $K_{\rm i}$ of the **Kiepert hyperbola**.



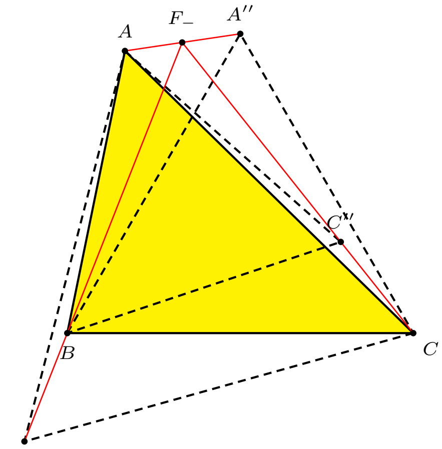
K_{i} is also the midpoint between the two **Fermat points** of **T**:



Recall the Fermat points

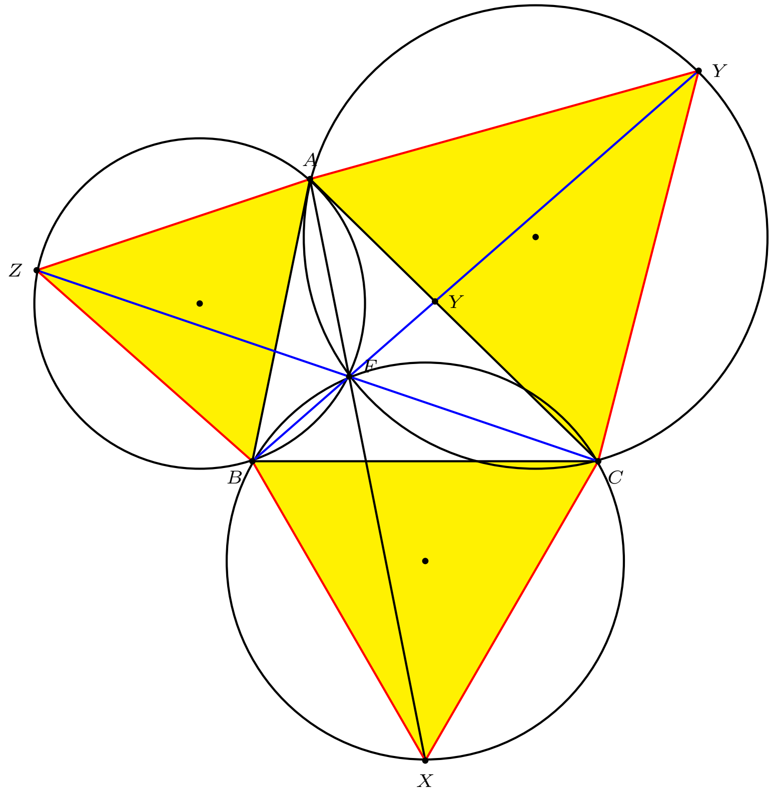


Fermat point

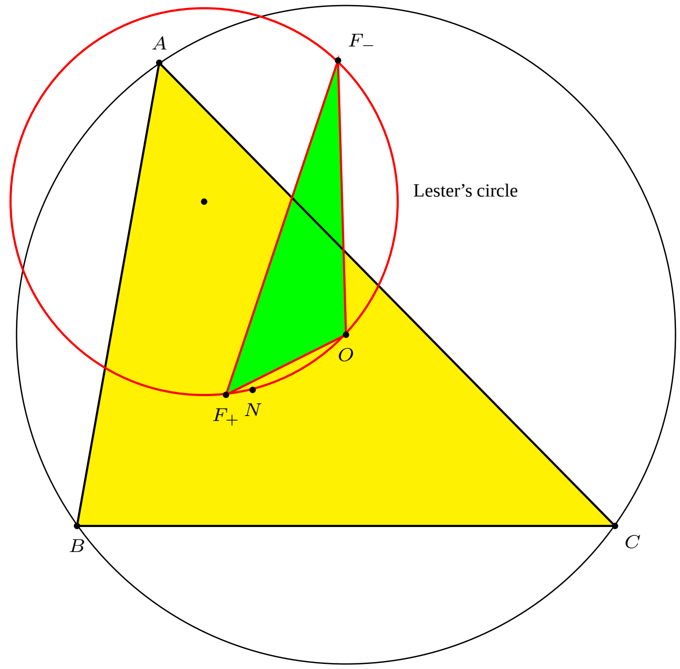


Negative Fermat point

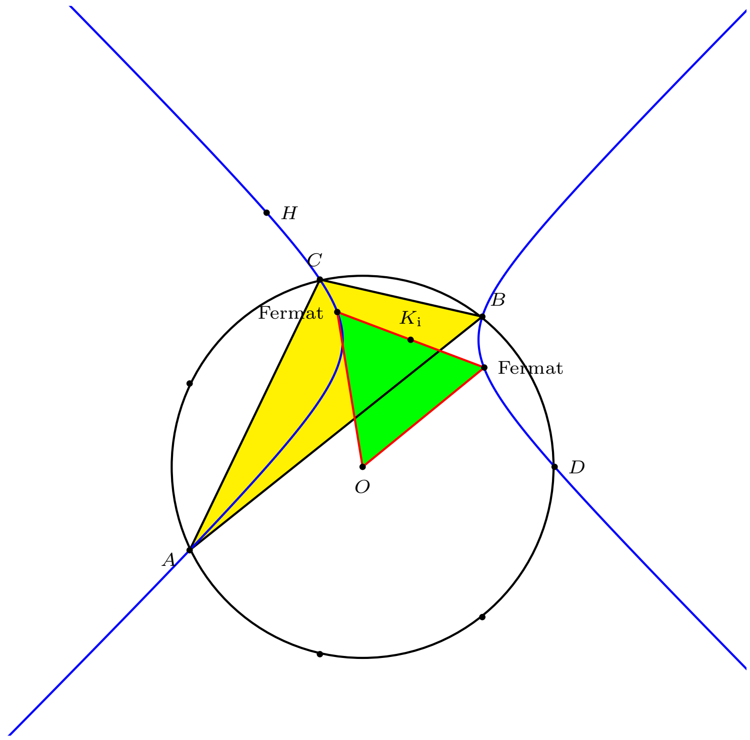
There are many wonderful properties of the Fermat points of a triangle. For example,



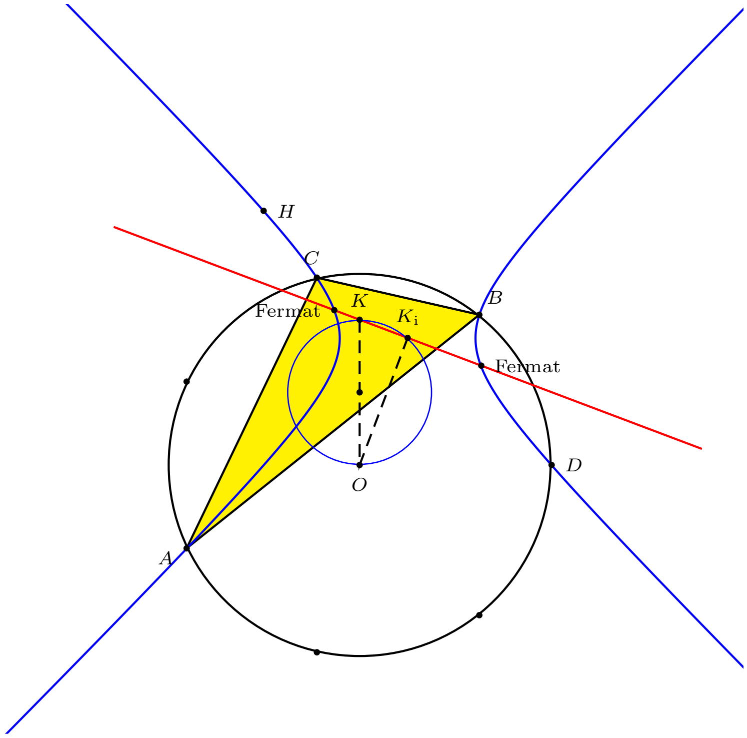
Lester Theorem: the circumcenter, the nine-point center, and the Fermat points are concyclic.



Theorem In the heptagonal triangle **T**, the circumcenter and the Fermat points form an equilateral triangle.

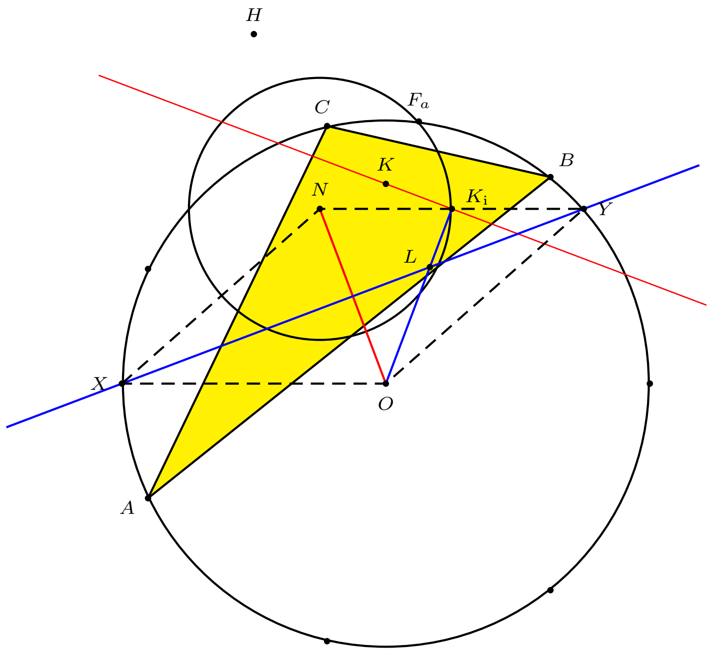


Proof. (1) The line joining the Fermat points also contains the symmedian point K, and is perpendicular to OK_{i}.

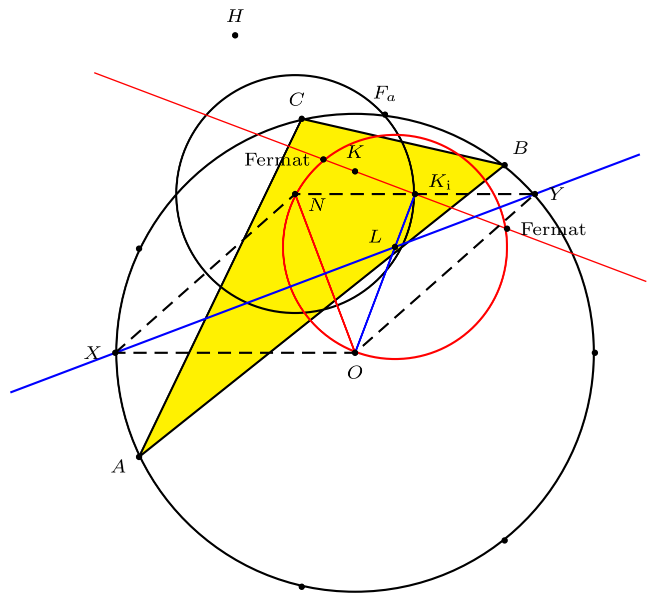


(2) The perpendicular bisector of ON intersects the circumcircle at
$X = -1$ and $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$ on the circumcircle.

(3) This perpendicular bisector intersects $OK_{\rm i}$ at
$L = -\frac{1}{3}(\zeta^3 + \zeta^5 + \zeta^6)$.



(4) Consider the Lester circle. Since K_i is the midpoint between the Fermat points, the center of the Lester circle is the intersection of OK_i with the perpendicular bisector of ON. This is the point L.



(5) Since $OL = \frac{2}{3} \cdot OK_{\rm i}$, the centroid and the circumcenter of the triangle OF_+F_- coincide.

This shows that triangle OF_+F_- is equilateral.