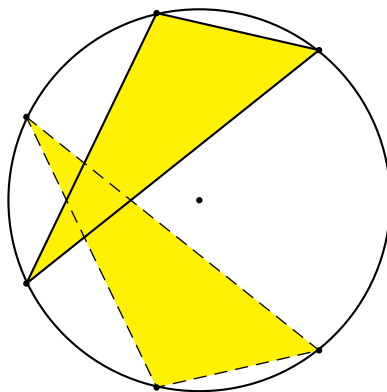


Heptagonal Triangles and Their Companions

Paul Yiu

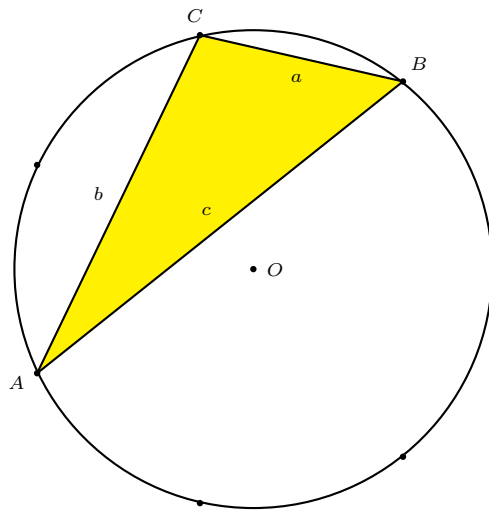
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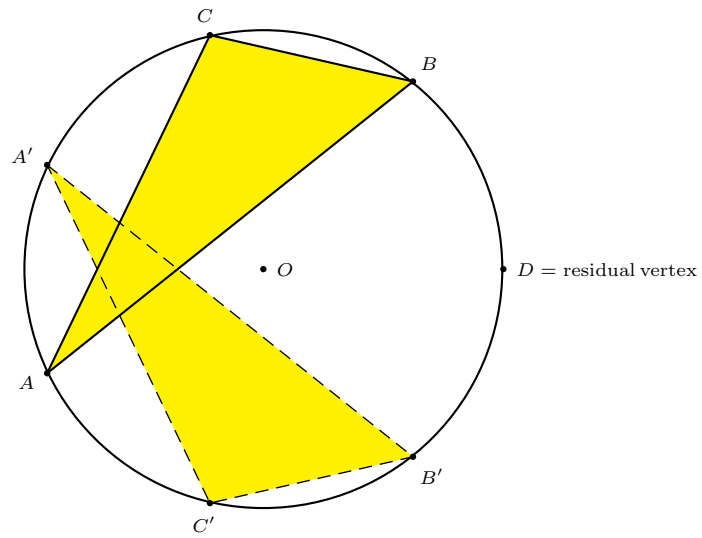
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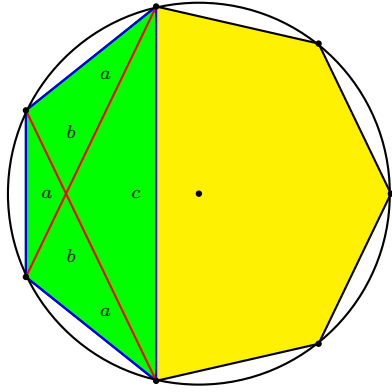
MAA Florida Sectional Meeting
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Florida Gulf Coast University

Abstract. A heptagonal triangle is a non-isosceles triangle formed by three vertices of a regular heptagon. Its angles are $\frac{\pi}{7}$, $\frac{2\pi}{7}$ and $\frac{4\pi}{7}$. As such, there is a unique choice of a companion heptagonal triangle formed by three of the remaining four vertices. Given a heptagonal triangle, we display a number of interesting companion pairs of heptagonal triangles on its nine-point circle and Brocard circles. Among other results on the geometry of the heptagonal triangle, we prove that the circumcenter and the Fermat points of a heptagonal triangle form an equilateral triangle. The proof is an interesting application of Lester's theorem that the Fermat points, the circumcenter and the nine-point center of a triangle are concyclic.

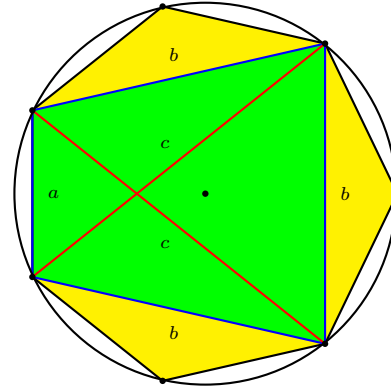
The heptagonal triangle T and its circumcircle

The companion of \mathbf{T} 

Relation between the sides of the heptagonal triangle

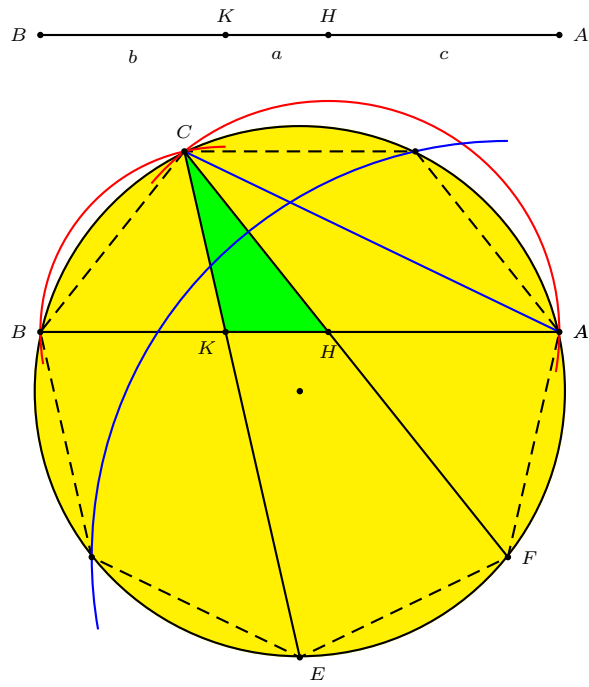


$$b^2 = a(c + a)$$



$$c^2 = a(a + b)$$

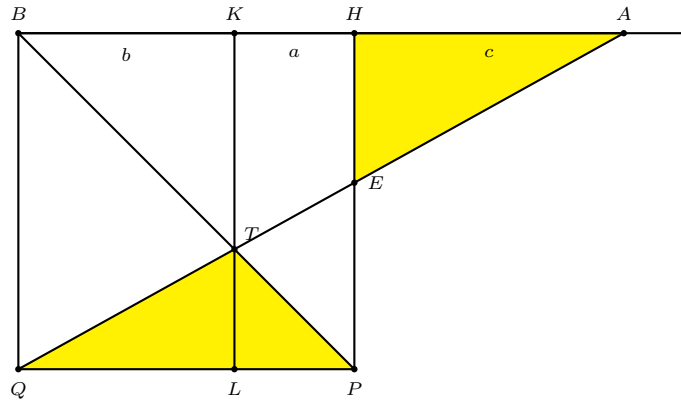
Archimedes' construction (Arabic tradition)



Division of segment: a neusis construction

Let $BHPQ$ be a square, with one side BH sufficiently extended.

Draw the diagonal BP . Place a ruler through Q , intersecting the diagonal BP at T , and the side HP at E , and the line BH at A such that the triangles AHE and TPQ have equal areas. Then,

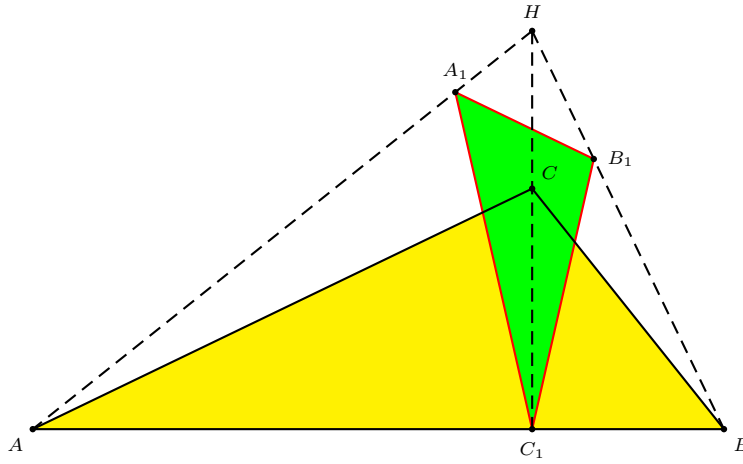


GSP

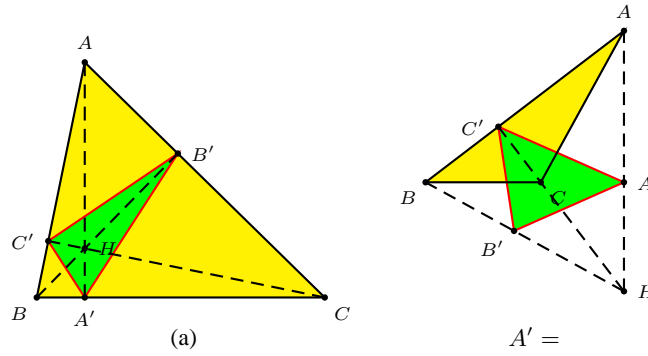
$$b^2 = a(c + a), \quad c^2 = (a + b)b.$$

The orthic triangle

The heptagonal triangle, apart from the equilateral triangle, is the only triangle similar its own orthic triangle.



Analysis:



Case (a): ABC acute:

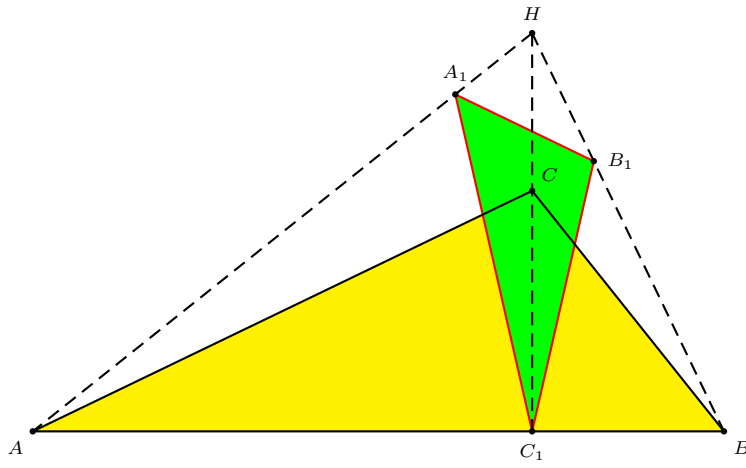
$$A' = \pi - 2A, \quad B' = \pi - 2B, \quad C' = \pi - 2C.$$

$$\{A', B', C'\} = \{A, B, C\} \implies A = B = C = \frac{\pi}{3}.$$

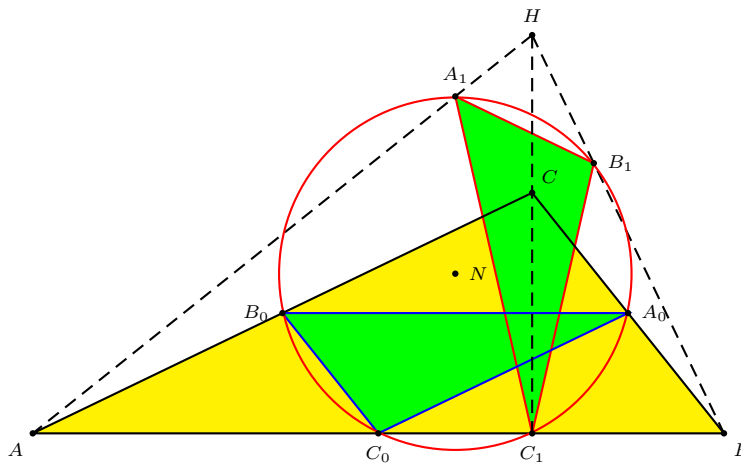
Case (b): Angle C obtuse:

$$A' = 2A, \quad B' = 2B, \quad C' = 2C - \pi.$$

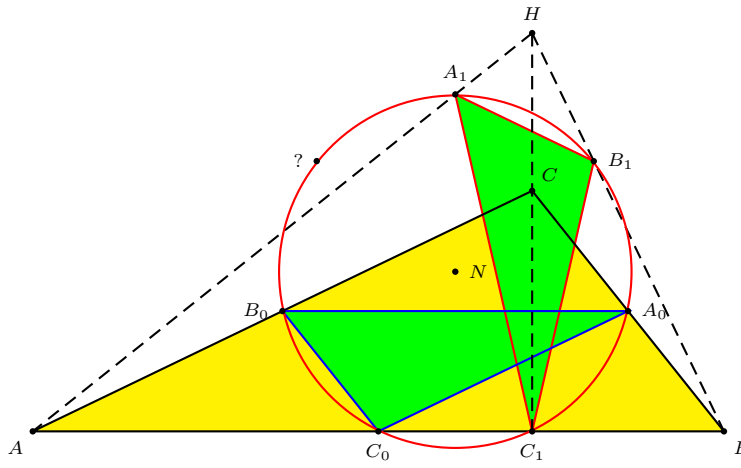
$$\{A', B', C'\} = \{A, B, C\} \implies \{A, B, C\} = \left\{ \frac{\pi}{7}, \frac{2\pi}{7}, \frac{4\pi}{7} \right\}.$$



The orthic triangle of the heptagonal triangle has similarity factor $\frac{1}{2}$, because ...



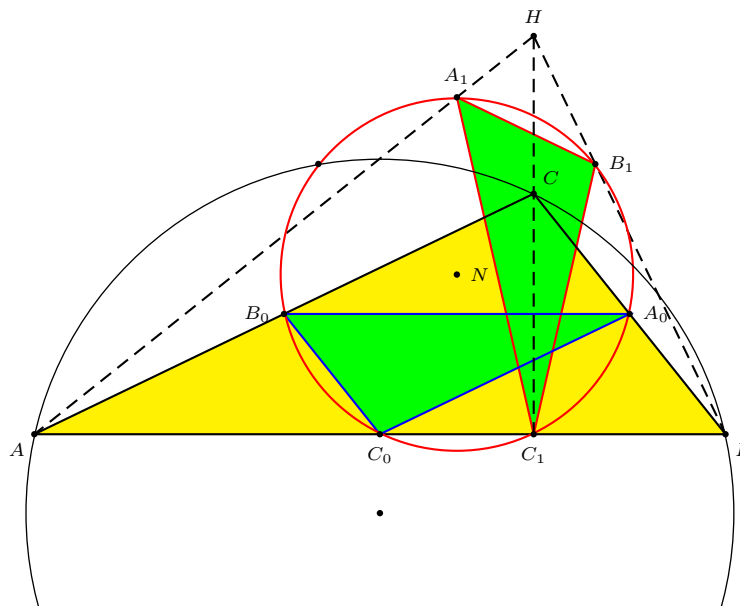
because its vertices are on the **nine-point circle** of T , which also contains the vertices of the **medial triangle**, *i.e.*, the midpoints of the sides of T .



The orthic triangle and the medial triangle are **companion heptagonal triangles**.

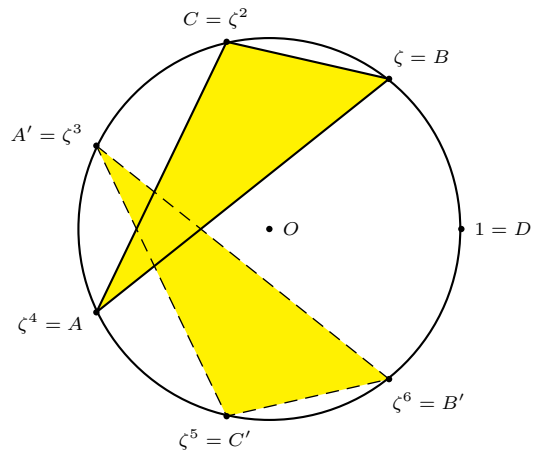
What can we say about the **residual vertex**?

This residual vertex also lies on the circumcircle of T .



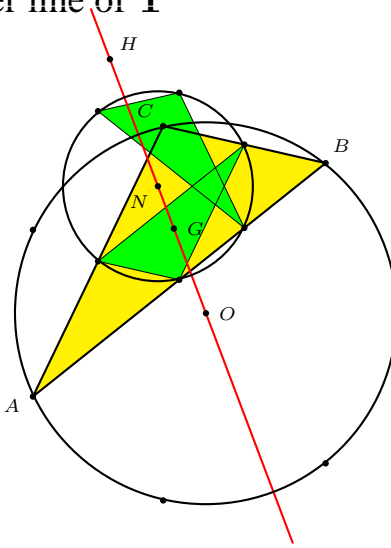
To justify this and probe more into the geometry of the heptagonal triangle, we make use of ...

Complex Coordinates



$\zeta =$ primitive 7-th root of unity

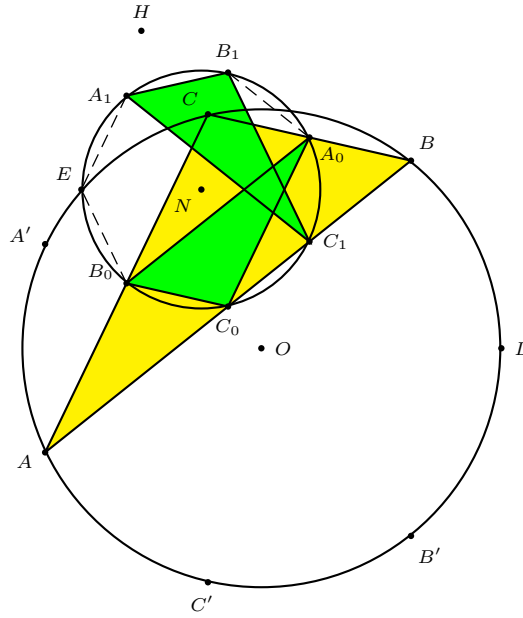
$$\zeta^6 + \zeta^5 + \zeta^4 + \zeta^3 + \zeta^2 + \zeta + 1 = 0.$$

Points on the Euler line of T 

$$OG : GN : NH = 2 : 1 : 3.$$

Center	Notation	Coordinates
circumcenter	O	0
centroid	G	$\frac{1}{3}(\zeta + \zeta^2 + \zeta^4)$
nine-point center	N	$\frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
orthocenter	H	$\zeta + \zeta^2 + \zeta^4$

Companionship of medial and orthic triangles



Center:	$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4)$
Residual vertex:	$E = \frac{1}{2}(-1 + \zeta + \zeta^2 + \zeta^4)$

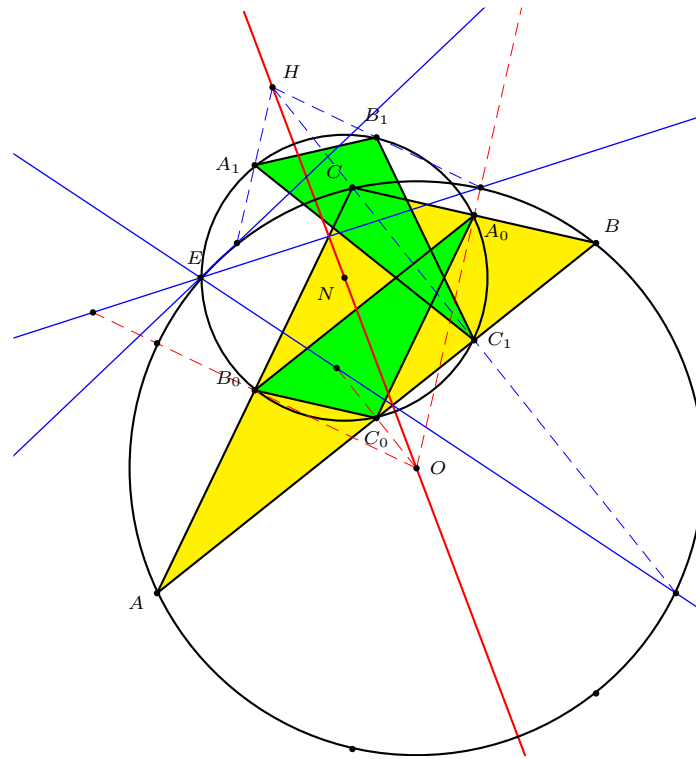
Rotation	Medial triangle	Rotation	Orthic triangle
ζ^4	$A_0 = \frac{1}{2}(\zeta + \zeta^2)$	ζ^3	$C_1 = \frac{1}{2}(\zeta + \zeta^2 - \zeta^3 + \zeta^4)$
ζ	$B_0 = \frac{1}{2}(\zeta^2 + \zeta^4)$	ζ^6	$A_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^6)$
ζ^2	$C_0 = \frac{1}{2}(\zeta + \zeta^4)$	ζ^5	$B_1 = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4 - \zeta^5)$

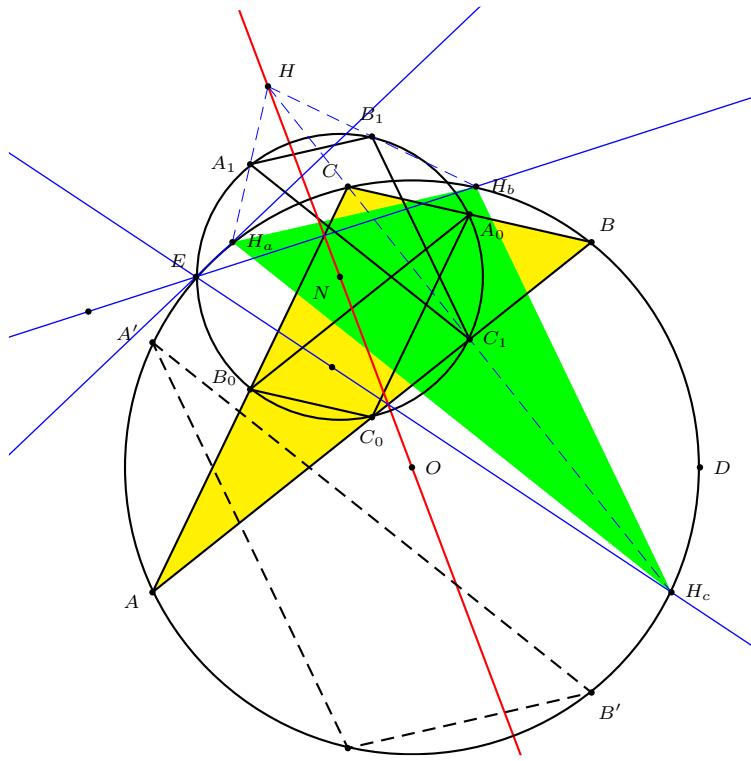
Why is E a point on the circumcircle of \mathbf{T} ?

Residual vertex E

= Euler reflection point of \mathbf{T}

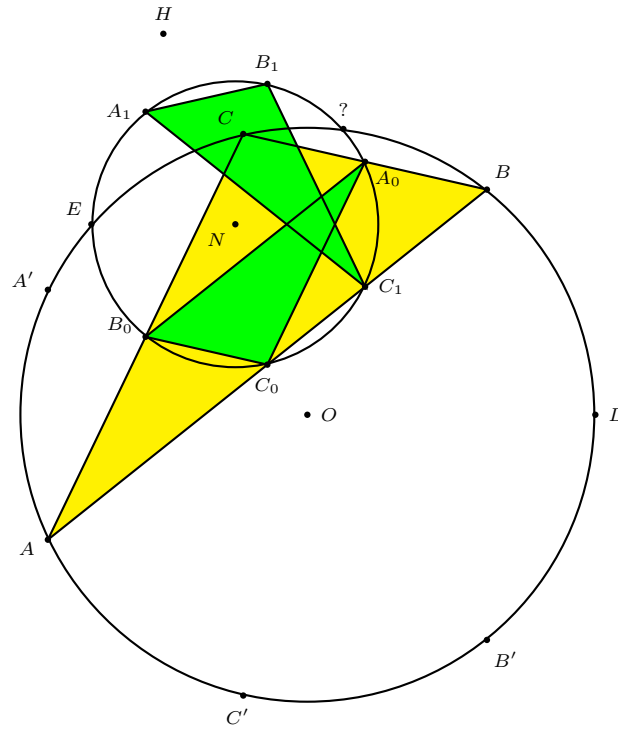
= Intersection of the reflections of the Euler line
in the three sidelines of \mathbf{T} .





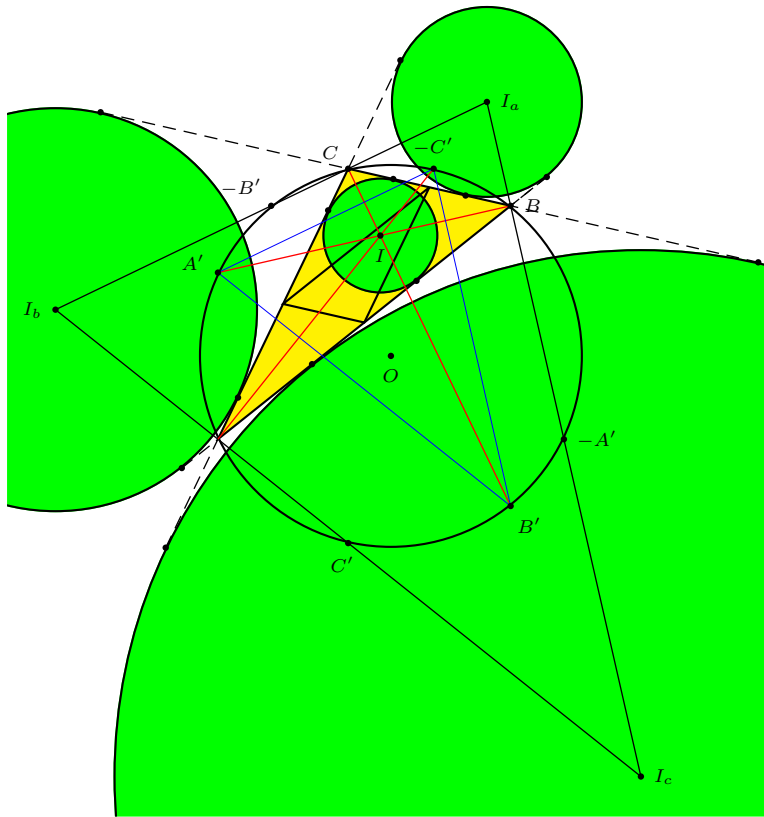
Remark. The reflections of H in the sides are the antipodes of the vertices of the companion of \mathbf{H}' .

The second intersection of the nine-point circle and the circum-circle

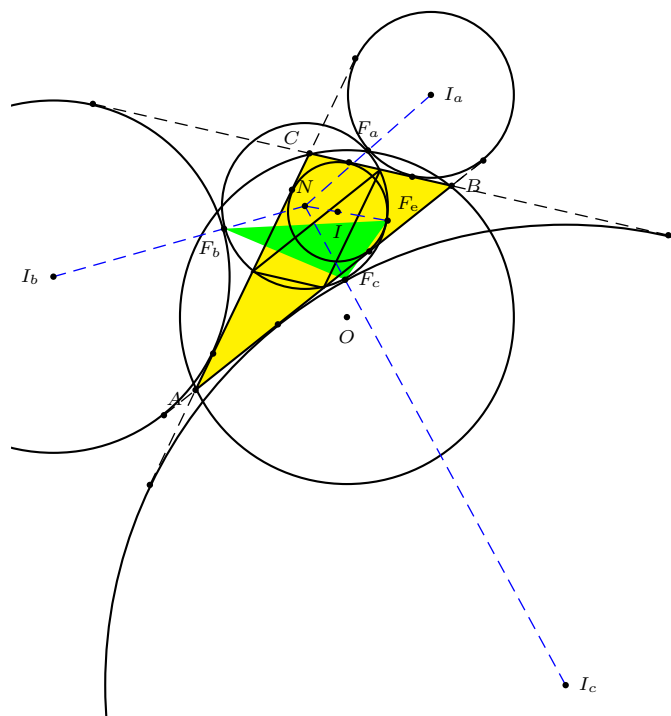


This is a point related to ...

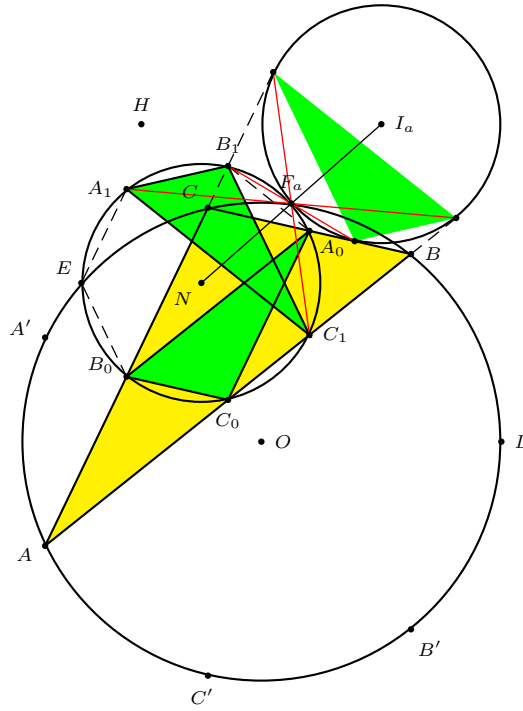
... the tritangent circles of T , *i.e.*, the **incircle** and **excircles**.



Feuerbach theorem: The nine-point circle is tangent to the incircle internally at F_e and to the excircles at F_a, F_b, F_c .

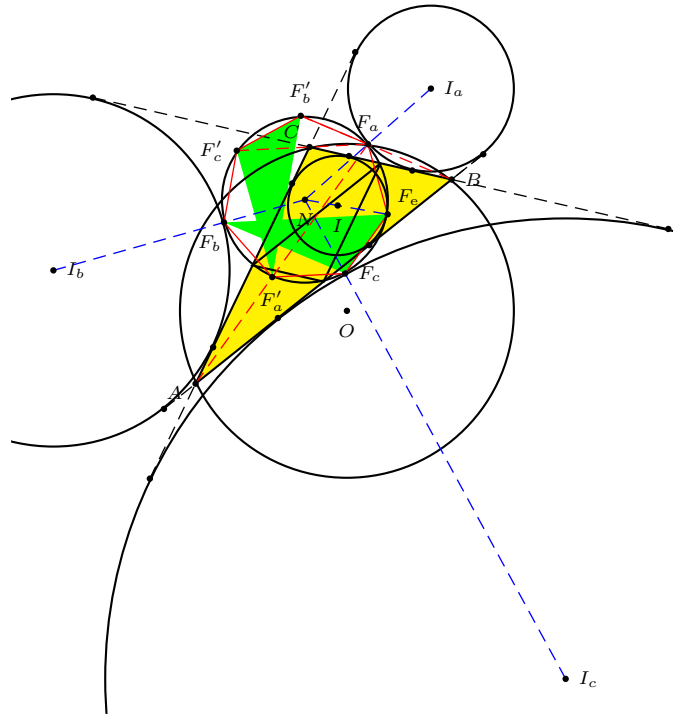


The second intersection of the circumcircle and the nine-point circle is F_a , the point of tangency of the nine-point circle with the A -excircle.



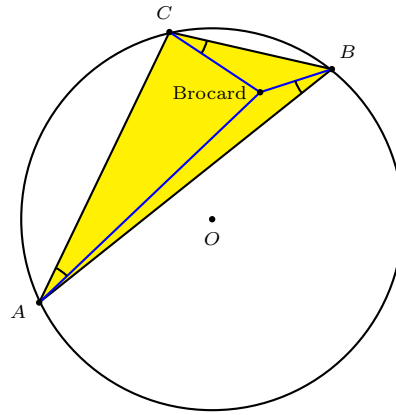
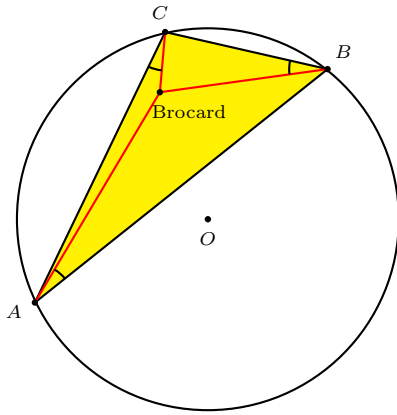
Another **companion pair** of heptagonal triangles

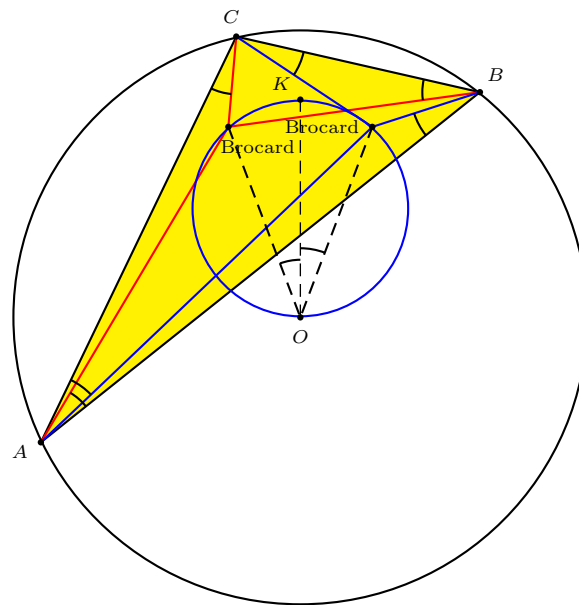
on the nine-point circle: triangle $F_b F_e F_c$ and triangle $F'_a F'_b F'_c$, where F'_a, F'_b, F'_c are the intersections of AF_a, BF_b, CF_c with the nine-point circle.



The Brocard points

Given a triangle there are two unique points (Brocard points) for each of which the three marked angles are equal (Brocard angle ω).

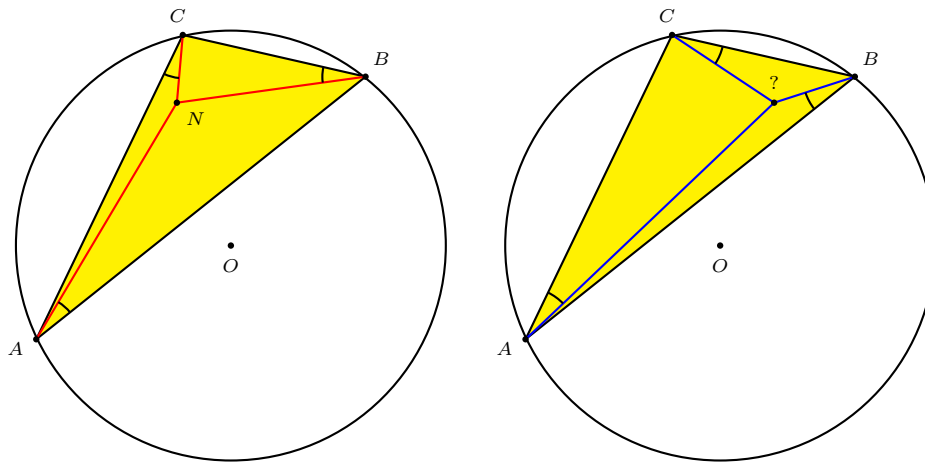




The two Brocard points are equidistant from O .

If K is the **symmedian point** of the triangle, then the circle with diameter OK contains the two Brocard points.

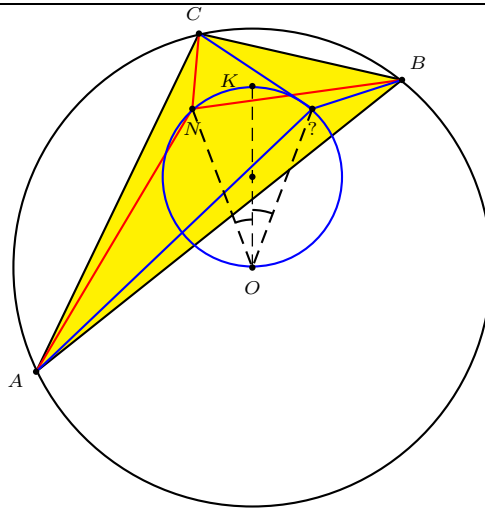
Each of the two chords makes an angle ω with OK .



For the heptagonal triangle \mathbf{T} ,
 (i) the Brocard angle ω satisfies

$$\cot \omega = \sqrt{7};$$

(ii) one of the Brocard points is the nine-point center N
 (Bankoff-Garfunkel, *Mathematics Magazine*, 46 (1973) 7–19).



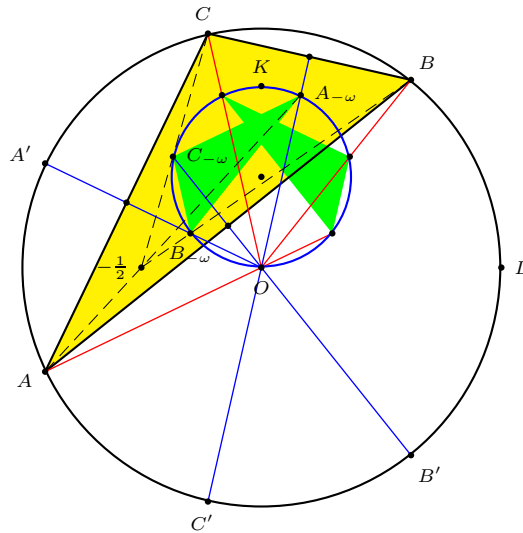
$$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4) \implies K = \frac{2}{\sqrt{7}} \cdot i$$

$$? = -\frac{1}{2}(\zeta^3 + \zeta^5 + \zeta^6).$$

Later, we shall identify ? as an interesting triangle center of **T**.

Meanwhile,

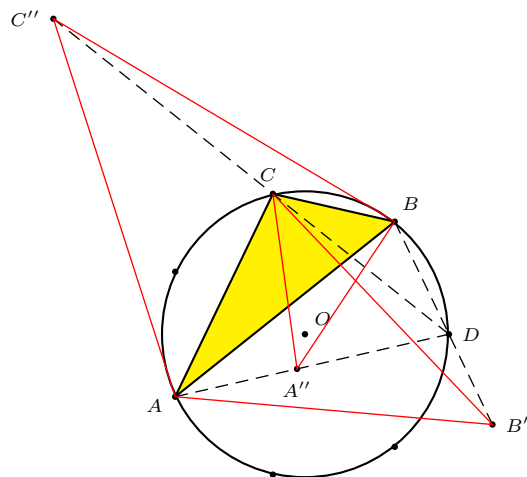
**a companion pair of heptagonal triangles
on the Brocard circle:**



The vertices are the second intersections of $OA, OB, OC, OA', OB', OC'$ with the nine-point circle.

$A_{-\omega}B_{-\omega}C_{-\omega}$ is called the first Brocard triangle of T . The Brocard circle is also called the **seven-point circle**, containing O, K , two Brocard points, three vertices of Brocard triangle.

The residual vertex of T



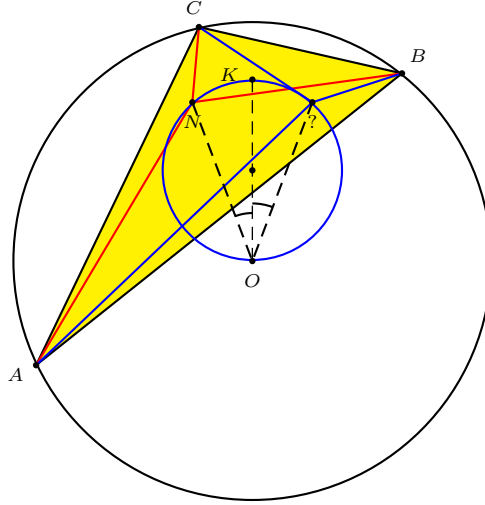
If similar isosceles triangles $A''BC$, $B''CA$, $C''AB$ of vertical angles 2ω are constructed on the sides, then AA'' , BB'' , CC'' intersect at D on the circumcircle.

What this means is that ...

the midpoint of DH

$$= \frac{1}{2}(1 + \zeta + \zeta^2 + \zeta^4) = -\frac{1}{2}(\zeta^3 + \zeta^5 + \zeta^6).$$

Note that this is the **second Brocard point** of \mathbf{T} we met before:

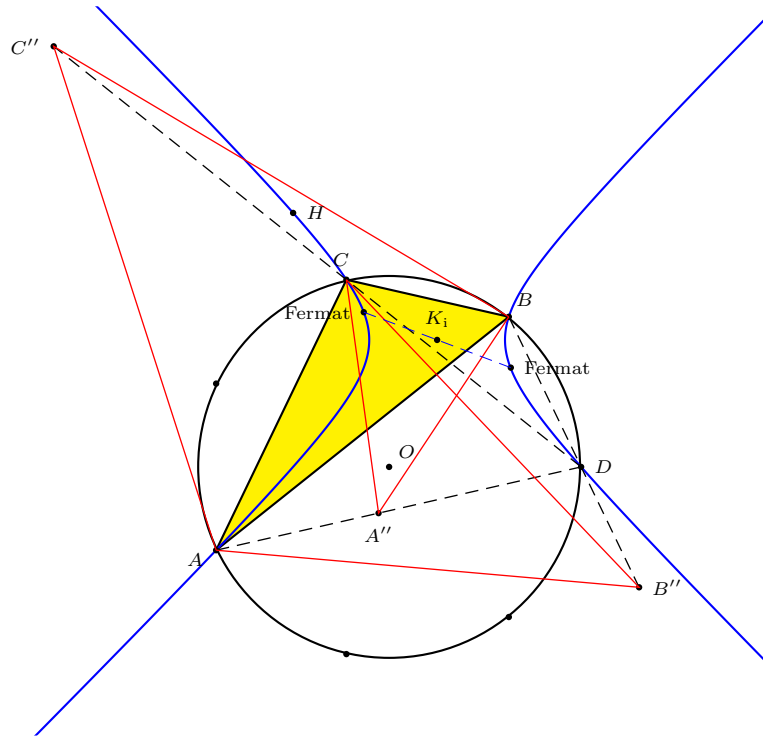


$$N = \frac{1}{2}(\zeta + \zeta^2 + \zeta^4) \implies K = \frac{2}{\sqrt{7}} \cdot i$$

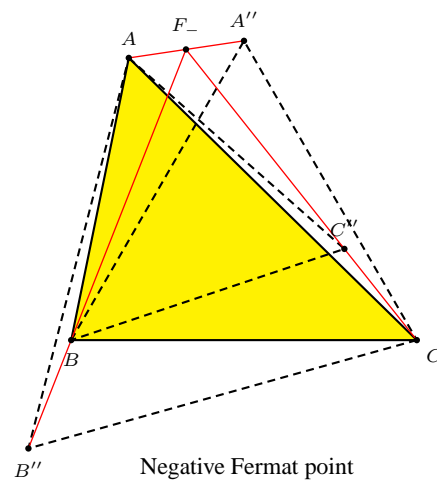
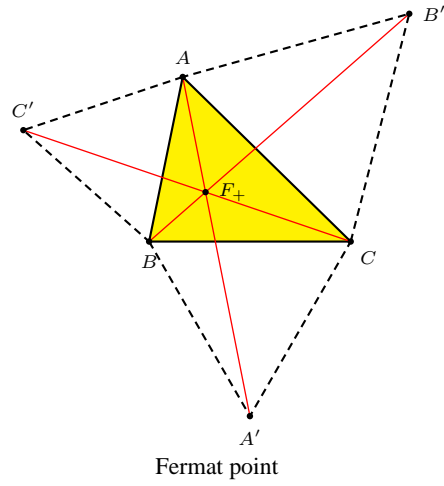
$$? = -\frac{1}{2}(\zeta^3 + \zeta^5 + \zeta^6).$$

But more importantly, it is

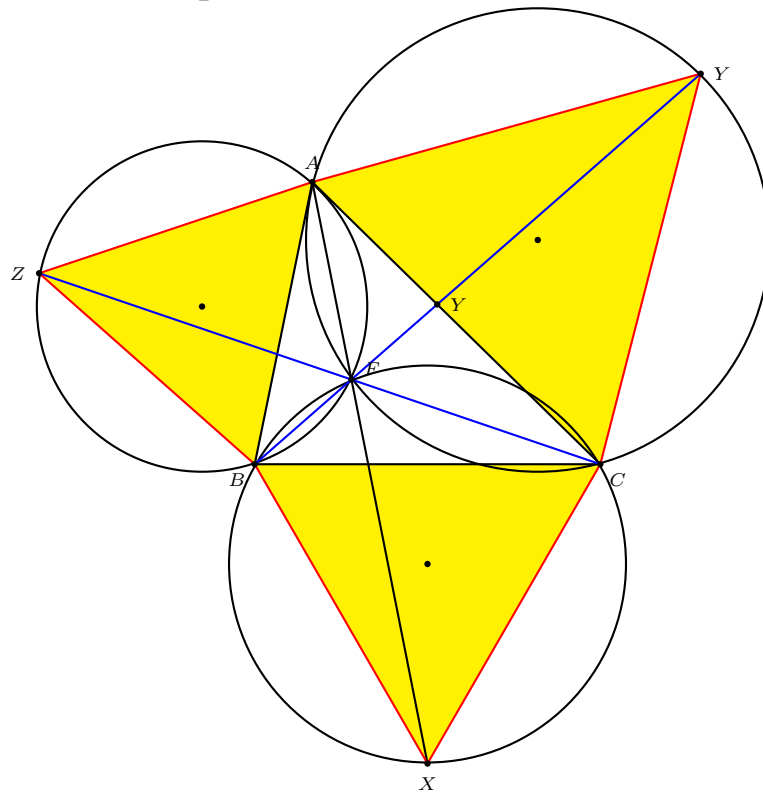
K_i is also the midpoint between the two **Fermat points** of T :



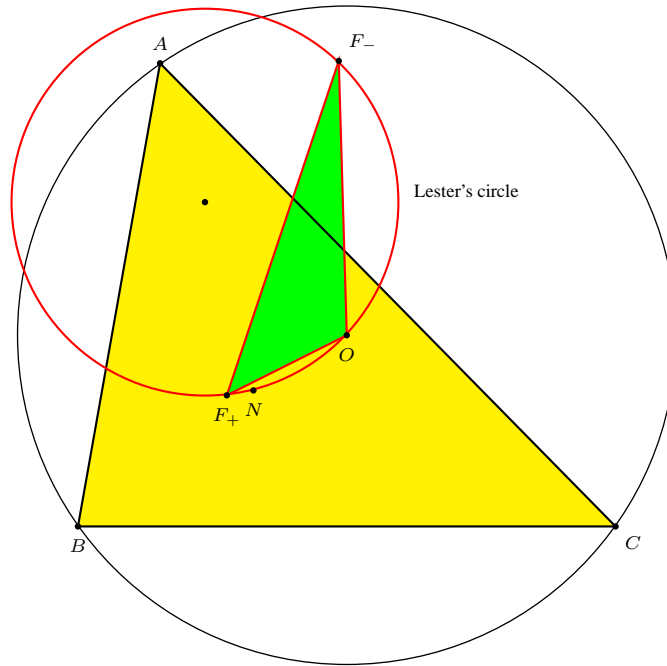
Recall the Fermat points



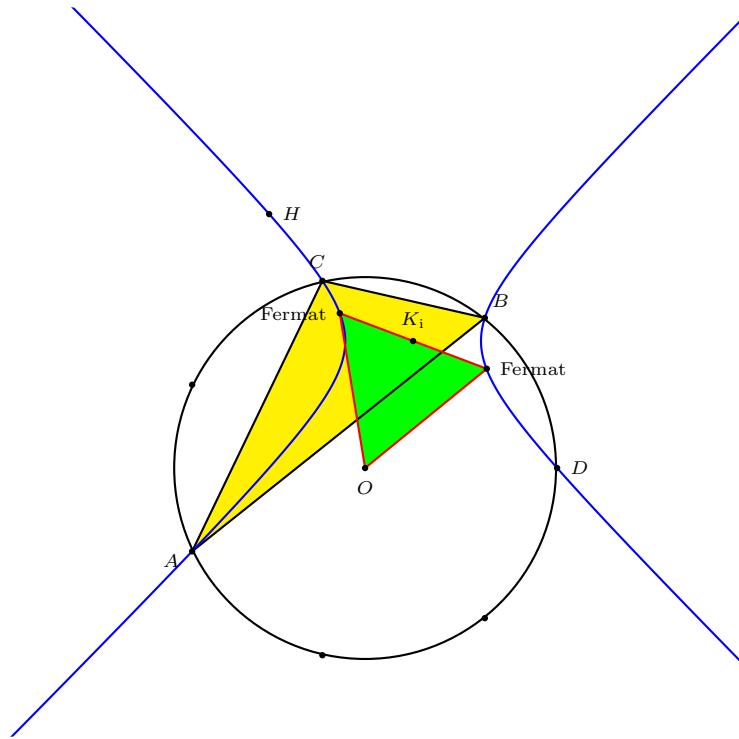
There are many wonderful properties of the Fermat points of a triangle. For example,



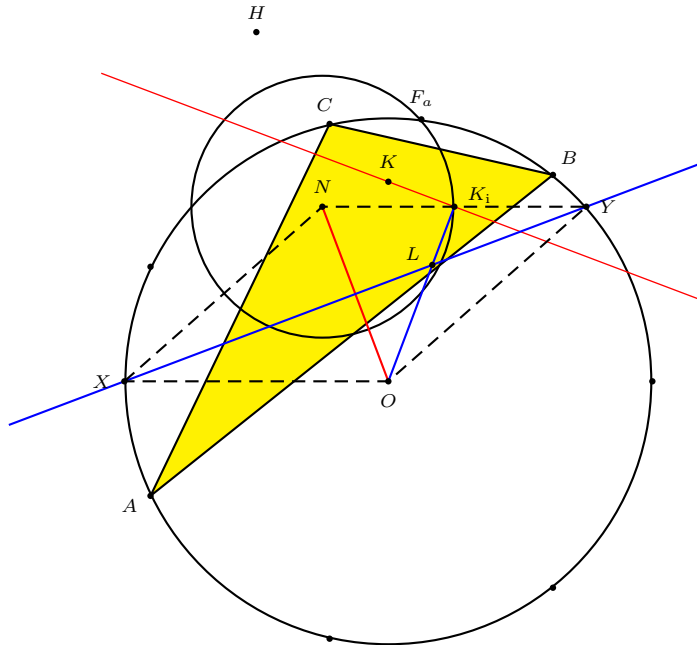
Lester Theorem: the circumcenter, the nine-point center, and the Fermat points are concyclic.



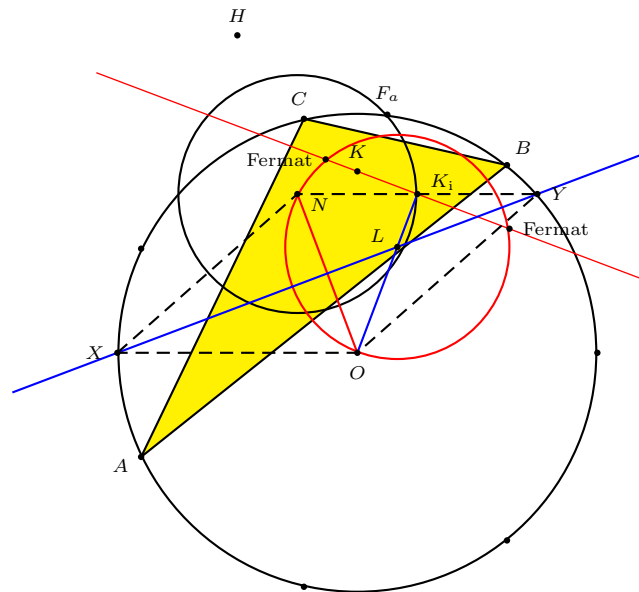
Theorem In the heptagonal triangle \mathbf{T} , the circumcenter and the Fermat points form an equilateral triangle.



- (2) The perpendicular bisector of ON intersects the circumcircle at $X = -1$ and $Y = \frac{1}{2}(1 - (\zeta^3 + \zeta^5 + \zeta^6))$ on the circumcircle.
- (3) This perpendicular bisector intersects OK_1 at $L = -\frac{1}{3}(\zeta^3 + \zeta^5 + \zeta^6)$.



(4) Consider the Lester circle. Since K_i is the midpoint between the Fermat points, the center of the Lester circle is the intersection of OK_i with the perpendicular bisector of ON . This is the point L .



(5) Since $OL = \frac{2}{3} \cdot OK_i$, the centroid and the circumcenter of the triangle OF_+F_- coincide.

This shows that triangle OF_+F_- is equilateral.

