Lines Simultaneously Bisecting the Perimeter and Area of a Triangle

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1. Introduction

The problem of enumeration of lines simultaneously bisecting the perimeter and area of a given triangle has been studied in several recent articles. In this note we give a ruler and compass construction and a simple enumeration of such lines (Theorem 3 below). These are called equalizers in [2] etc and B-lines in [4]. We shall use the term perimeter-area bisectors.

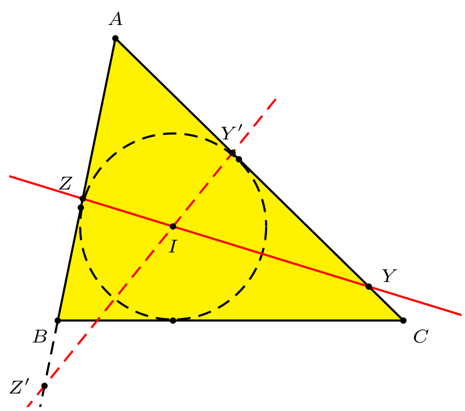

Figure 1.

Given triangle ABC with sidelengths $BC = a$, $CA = b$, and $AB = c$, let s denote the semiperimeter. Let Y and Z be points on the sides AC and AB respectively. We call the line YZ
(i) a perimeter bisector in angle A if $AY + AZ = ZB + BC + CY = s$,
(ii) an area bisector in angle A if the areas of triangle AZY and convex quadrilateral $ZBCY$ are each one half of the area of triangle ABC,
(iii) a perimeter-area bisector in angle A if it is both a perimeter bisector and an area bisector in angle A.

A perimeter-area bisector YZ in angle A of triangle ABC depends on the lengths of $AY = y$ and $AZ = z$. We require $y + z = s$ and $yz = \frac{1}{2}bc$. These two lengths are the roots of the quadratic polynomial

$$\begin{aligned} Q_a(t) :=&\ t^2 - st + \frac{1}{2}bc \\ =&\ \left(t - \frac{s}{2}\right)^2 - \frac{D_a}{4}, \end{aligned}$$

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where

$$\begin{aligned} D_a &= s^2 - 2bc \\ &= ((s-a)+(s-b)+(s-c))^2 - 2((s-c)+(s-a))((s-a)+(s-b)) \\ &= (s-b)^2 + (s-c)^2 - (s-a)^2. \end{aligned}$$

It is interesting to note that a perimeter bisector necessarily passes through the incenter of the triangle ([2, 4]). We shall make use of this to simplify a construction in §5 below. Clearly, if the line YZ is a perimeter-area bisector in angle A, then so is its reflection in the bisector of angle A, provided that the reflections Z' of Y, and Y' of Z are on the sides AB and AC respectively. The line $Y'Z'$ in Figure 6, for example, is not a perimeter-area bisector in angle A.

2. Construction of perimeter-area bisectors in angle A

Figure 2 shows a simple geometric construction of these two lengths.

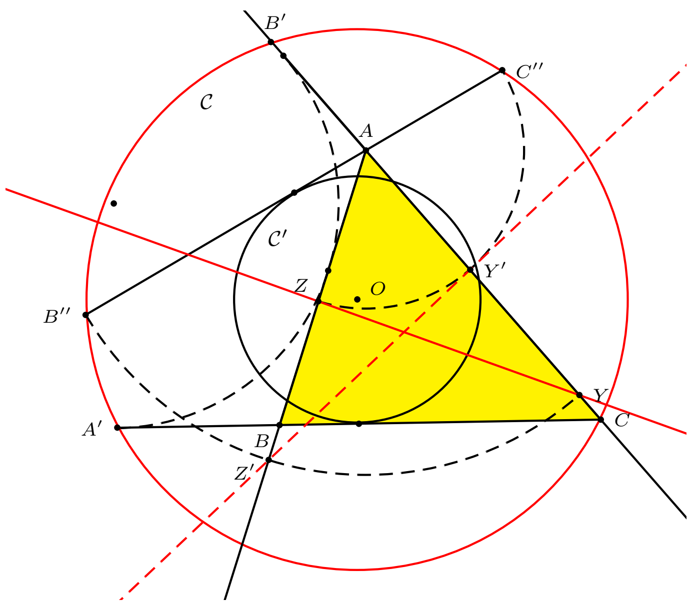

FIGURE 2.

Suppose there is a circle $\mathcal{C}$ which has
(i) a chord through A divided into two segments of lengths b and $\frac{1}{2}c$, and
(ii) another chord of length s.
Then a tangent through A to the concentric circle $\mathcal{C}'$ tangent to the chord in (ii) solves our problem, because it clearly has the same length s, and by the intersecting chords theorem, the product of the two segments divided by A is the same as the product of the segments of the chord in (i).

Now, given triangle ABC, we take $\mathcal{C}$ to be the circumcircle of $A'B'C$, where A' is a point on the extension of CB such that $CA' = s$, and B' is a point on the extension of CA such that $AB' = \frac{c}{2}$. (The point A' is the point of tangency of the line BC with the excircle on the opposite side of C.) The circle $\mathcal{C}'$ is concentric with $\mathcal{C}$, and tangent to $A'C$ at its midpoint.

If the vertex A is on or outside the circle $\mathcal{C}'$, construct a tangent from A to $\mathcal{C}'$ to intersect $\mathcal{C}$ the circle $\mathcal{C}$ at B'' and C''. With these, construct points Y on AC, Z' on AB such that $AY = AZ' = AB''$, and Y' on AC, Z on AB such that $AY' = AZ = AC''$. Each of the lines YZ and $Y'Z'$ is a perimeter-area bisector in angle A, provided both of its endpoints are on the sides AC and AB respectively. For example, in Figure 2, the line YZ is a perimeter-area bisector, but $Y'Z'$ is not.

Clearly, if the vertex A is inside the circle $\mathcal{C}'$, there is no perimeter-area bisector in angle A.

We also speak of perimeter-area bisectors in angles B and C. In §§3, 4 below, we determine the number of perimeter-area bisector in terms of the lengths a, b, c.

3. Isosceles triangles

We begin with the equilateral triangle. Clearly each angle bisector is a perimeter-area bisector. There are no others since for the perimeter-area bisectors in angle A, the polynomial $Q_a(t) = (t-a)\left(t - \frac{a}{2}\right)$ leads to the bisectors of angle B and C.

Lemma 1. *A perimeter-area bisector passes through a vertex of the triangle if and only if it is isosceles.*

Proof. If $AB = AC$, then the bisector of angle A is clearly a perimeter-area bisector in angle B and C. Conversely, let a perimeter-area bisector YZ in angle A pass through a vertex, say $Y = C$. Then, b is a root of the polynomial $Q_a(t)$. Since $Q_a(b) = \frac{1}{2}b(b-a)$, we must have $b = a$, and the triangle is isosceles. □

Now, let ABC be isosceles with $AB = AC$. We determine the perimeter-area bisectors in angle A. Since $b = c$, in this case,

$$Q_a(t) = \left(t - \frac{a+2b}{4}\right)^2 - \frac{D_a}{4},$$

where $D_a = \frac{(a+2b)^2 - 8b^2}{4}$.

If $D_a < 0$, there is no perimeter-area bisector in angle A.

If $D_a = 0$, then $Q_a(t)$ has a unique positive root $\frac{a+2b}{4} < b$. In this case, there is a unique perimeter-area bisector in angle A.

Suppose $D_a > 0$ so that $Q_a(t)$ has two positive roots.

(i) If $a > b$, then $Q_a(b) = \frac{1}{2}b(b-a) < 0$, and $Q_a(t)$ has a root greater than b. In this case, there is no perimeter-area bisector in angle A.

(ii) If $a < b$, then since $0 < \frac{s}{2} < b$, and

$$Q_a(0) > 0, \qquad Q_a\left(\frac{s}{2}\right) = -\frac{D_a}{4} < 0, \qquad Q_a(b) > 0,$$

both roots of Q_a are smaller than b. In this case, there are two perimeter-area bisectors in angle A.

Counting also the bisector of angle A, we obtain the following enumeration of perimeter-area bisectors of an isosceles triangle.

Proposition 2. *Let ABC be an isosceles triangle with $b = c$, the number of perimeter-area bisectors is*
1 *if* $D_a < 0$ *or* $a > b$,
2 *if* $D_a = 0$,
3 *if* $D_a > 0$ *and* $a \leq b$.

Remark. In terms of a and $b = c$, D_a is negative, zero, or positive according as a is greater than, equal to, or less than $2(\sqrt{2}-1)b$.

4. SCALENE TRIANGLES

Since we have dealt with the isosceles case, we shall assume $a < b < c$.

If $D_a < 0$, then $Q_a(t)$ has no real root, and there is no perimeter-area bisector in angle A.

If $D_a = 0$, then $Q_a(t)$ has a double root $\frac{s}{2}$, which is smaller than b. (Proof: $\frac{s}{2} = \frac{a+b+c}{4} < \frac{a+b}{2} < b$). Therefore, there is a unique perimeter-area bisector in angle A. This is the perpendicular to the bisector of angle A at the incenter.

If $D_a > 0$, the two positive roots of $Q_a(t)$ are smaller than b (and c) since

$$Q_a(0) = \frac{1}{2}bc > 0, \qquad Q_a\left(\frac{s}{2}\right) = -\frac{D_a}{4} < 0, \qquad Q_a(b) = \frac{1}{2}b(b-a) > 0.$$

In this case, there are two perimeter-area bisectors.

Now consider the perimeter-area bisector in angle B. These are given by the roots of the quadratic polynomial

$$Q_b(t) = t^2 - st + \frac{1}{2}ca = \left(t - \frac{s}{2}\right)^2 - \frac{D_b}{4},$$

where

$$D_b = (s-c)^2 + (s-a)^2 - (s-b)^2 > 0.$$

In this case,

$$Q_b(0) = \frac{1}{2}ca > 0, \qquad Q_b(a) = -\frac{1}{2}a(b-a) < 0, \qquad Q_b(c) = \frac{1}{2}c(c-b) > 0.$$

The two roots are smaller than c, but are separated by a. There is only one perimeter-area bisector in the “middle” angle B.

Finally, the perimeter-area bisectors in angle C are given by the roots of the quadratic polynomial

$$Q_c(t) = t^2 - st + \frac{1}{2}ab = \left(t - \frac{s}{2}\right)^2 - \frac{D_c}{4},$$

where

$$D_c = (s-a)^2 + (s-b)^2 - (s-c)^2 > 0.$$

In this case,

$$Q_c(0) = \frac{1}{2}ab > 0, \quad Q_c(a) = -\frac{1}{2}a(c-a) < 0, \quad Q_c(b) = -\frac{1}{2}b(c-b) < 0.$$

The larger root is greater than b (and a). There is no perimeter-area bisector in the largest angle.

Combining these results with Proposition 2, we obtain the following enumeration result on the number of perimeter-area bisectors.

Theorem 3. *Let ABC be a triangle with $a \le b \le c$. The number of perimeter-area bisectors is* 1, 2, *or* 3 *according as*

$$D_a = (s-b)^2 + (s-c)^2 - (s-a)^2$$

is negative, zero, or positive.

5. Triangles with exactly two perimeter-area bisectors

Here is a construction of triangles satisfying $D_a = 0$. Let X be a point on a given a segment BC. Construct
(i) a point PY on the perpendicular to BC at B such that $BP = XC$,
(ii) the circle, center X, radius XP, to intersect the line BC at B' and C' (so that B and B' are on the same side of X, as are C and C'),
(iii) the circles, center B, radius BC', and center C, radius CB', to intersect at A.

ABC is a triangle whose incircle touches BC at X, and satisfies

$$(s-b)^2 + (s-c)^2 = (s-a)^2.$$

The perimeter-area bisector in angle A is the perpendicular to the bisector at the incenter I (see [2, Lemma 1] and Figure 3).

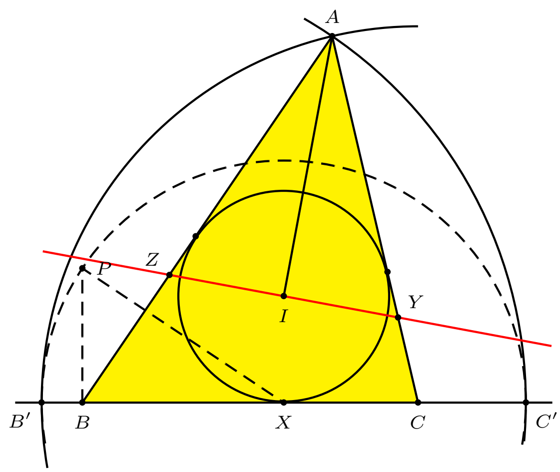

FIGURE 3.

Remark. If in (ii) above, the radius of the circle is taken to be shorter (respectively longer) than XP, then with A constructed in (iii), the number of perimeter-area bisectors in angle A is 2 (respectively 0).

There is a unique right triangle satisfying $D_a = 0$. If we put $t = \tan\frac{A}{2}$, then $a : b : c = 2t : 1 - t^2 : 1 + t^2$, and $D_a = (t+1)(2t^3 - 2t^2 + 3t - 1)$. This has a unique real root, which is approximately $0.396608\cdots$. Correspondingly, $A \approx 43.2674$ degrees.

6. INTEGER TRIANGLES

We determine scalene triangles with *integer* sides whose perimeter-area bisectors are given by points on the sides with rational (integer or half-integer) distances from the vertices.

6.1. Triangles with $D_a < 0$. We require $D_b = (s-c)^2 + (s-a)^2 - (s-b)^2 = v^2$ for an integer v.

Let $d_1 = h_1^2 + k_1^2$ with $h_1 > k_1$ and $d_2 = h_2^2 + k_2^2$ with $h_2 > k_2$. We further assume $k_1h_2 < h_1k_2 < 3k_1h_2$. The product d_1d_2 is a sum of two squares in two ways, as $d_1d_2 = p^2 + q^2 = u^2 + v^2$ with

$$\begin{aligned} p &= h_1h_2 + k_1k_2, \quad q = h_1k_2 - k_1h_2; \\ u &= h_1h_2 - k_1k_2, \quad v = h_1k_2 + k_1h_2. \end{aligned}$$

Note that $p > u > q$. Also,

$$\begin{aligned} u^2 + q^2 - p^2 &= h_1^2k_2^2 + h_2^2k_1^2 - 6h_1h_2k_1k_2 \\ &< 2h_1^2k_2^2 - 6h_1h_2k_1k_2 \\ &= 2h_1k_2(h_1k_2 - 3h_2k_1) \\ &< 0. \end{aligned}$$

Therefore, by setting $s - a = p$, $s - b = u$, $s - c = q$, we have $D_a < 0$ and $D_b = (s-c)^2 + (s-a)^2 - (s-b)^2 = v^2$.

For example, by using the five smallest Pythagorean triples, we obtain the following integer triangles (a, b, c) with only one perimeter-area bisectors X_bZ_b in angle B with (BX_b, BZ_b) given in the rightmost column in the table below.

(d_1, d_2)	(h_1, k_1)	(h_2, k_2)	(p, q)	(u, v)	(a, b, c, s)	(BX_b, BZ_b)
$(5, 29)$	$(4, 3)$	$(21, 20)$	$(144, 17)$	$(24, 143)$	$(41, 161, 168, 185)$	$(21, 164)$
$(13, 5)$	$(12, 5)$	$(4, 3)$	$(63, 16)$	$(33, 56)$	$(49, 79, 96, 112)$	$(28, 84)$
$(13, 17)$	$(12, 5)$	$(15, 8)$	$(220, 21)$	$(140, 171)$	$(161, 241, 360, 381)$	$(105, 276)$
$(13, 29)$	$(12, 5)$	$(21, 20)$	$(352, 135)$	$(152, 345)$	$(287, 487, 504, 639)$	$(147, 492)$
$(17, 5)$	$(15, 8)$	$(4, 3)$	$(84, 13)$	$(36, 77)$	$(49, 97, 120, 133)$	$(28, 105)$
$(17, 29)$	$(15, 8)$	$(21, 20)$	$(475, 132)$	$(155, 468)$	$(287, 607, 630, 762)$	$(147, 615)$
$(25, 5)$	$(24, 7)$	$(4, 3)$	$(117, 44)$	$(75, 100)$	$(119, 161, 192, 236)$	$(68, 168)$
$(25, 13)$	$(24, 7)$	$(12, 5)$	$(323, 36)$	$(253, 204)$	$(289, 359, 576, 612)$	$(204, 408)$
$(25, 17)$	$(24, 7)$	$(15, 8)$	$(416, 87)$	$(304, 297)$	$(391, 503, 720, 807)$	$(255, 552)$

6.2. Triangles with $D_a = 0$. In this case, $(s-b)^2 + (s-c)^2 = (s-a)^2$. It is enough to determine (a, b, c) from a Pythagorean triple. In this case, $AY_a = AZ_a = s$. Note that $D_b = 2(s-c)^2$ and the perimeter-area bisector in angle B cannot intersect the sides BC and BA with rational length BX_b and BZ_b. Here are some small examples.

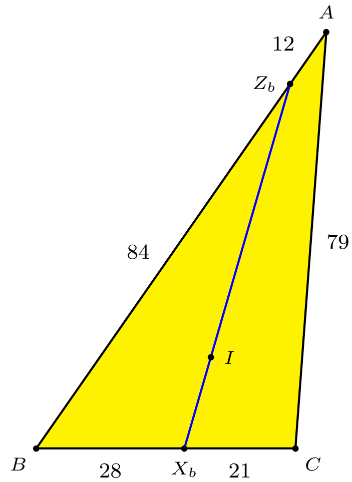

Figure 4

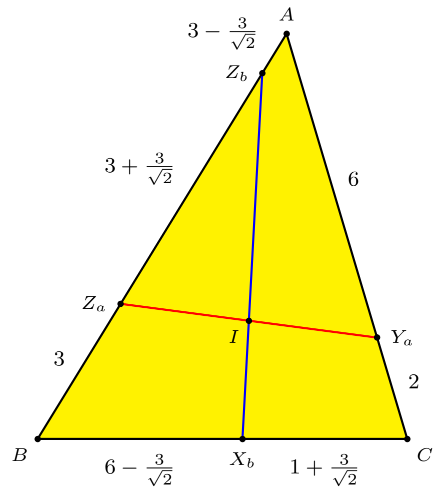

Figure 5

$(s-a, s-b, s-c)$	(a, b, c)	s
$(5, 4, 3)$	$(7, 8, 9)$	12
$(13, 12, 5)$	$(17, 18, 25)$	30
$(17, 15, 8)$	$(23, 25, 32)$	40
$(25, 24, 7)$	$(31, 32, 49)$	56
$(29, 21, 20)$	$(41, 49, 50)$	70

6.3. **Triangles with $D_a > 0$.** We require D_a and D_b to be squares of integers u and v. Let $D := (s-a)^2 + (s-b)^2 + (s-c)^2$. Then $D = u^2 + 2(s-a)^2 = v^2 + 2(s-b)^2$. A number D can be written as $u^2 + 2x^2$ in two different ways if and only if it is divisible by two prime numbers congruent to 1 or 3 $\pmod 8$. Let D be one such number with

$$D = u_1^2 + 2x^2 = v^2 + 2y^2, \quad x > y.$$

If, in addition, $D - x^2 - y^2 = z^2$ for an integer $z < y$, then by setting $s - a = x$, $s - b = y$, $s - c = z$, we obtain $(a, b, c) = (y+z, z+x, x+y)$ for which $D_a = u^2$ and $D_b = v^2$. The table below shows all possibilities with squarefree $D < 10000$, where we reduce (a, b, c) be a factor $\frac{1}{2}$ when x, y, z are all odd.

D	(x,y,z)	(a,b,c)	$(\frac{s-u}{2},\frac{s+u}{2})$	$(\frac{s-v}{2},\frac{s+v}{2})$
1254	$(25,23,10)$	$(33,35,48)$	$(28,30)$	$(22,36)$
1691	$(29,25,15)$	$(20,22,27)$	$(\frac{33}{2},18)$	$(12,\frac{45}{2})$
1971	$(31,29,13)$	$(21,22,30)$	$(\frac{33}{2},20)$	$(14,\frac{45}{2})$
2097	$(32,28,17)$	$(45,49,60)$	$(35,42)$	$(27,50)$
2466	$(35,29,20)$	$(49,55,64)$	$(40,44)$	$(28,56)$
3894	$(43,37,26)$	$(63,69,80)$	$(46,60)$	$(36,70)$
4161	$(44,40,25)$	$(65,69,84)$	$(46,63)$	$(39,70)$
4419	$(47,37,29)$	$(33,38,42)$	$(28,\frac{57}{2})$	$(18,\frac{77}{2})$
5643	$(53,47,25)$	$(36,39,50)$	$(30,\frac{65}{2})$	$(\frac{45}{2},40)$
5814	$(53,43,34)$	$(77,87,96)$	$(58,72)$	$(42,88)$
6059	$(55,53,15)$	$(34,35,54)$	$(30,\frac{63}{2})$	$(\frac{51}{2},36)$
6099	$(55,43,35)$	$(39,45,49)$	$(\frac{63}{2},35)$	$(21,\frac{91}{2})$
7403	$(59,49,39)$	$(44,49,54)$	$(\frac{63}{2},42)$	$(24,\frac{99}{2})$
7491	$(61,59,17)$	$(38,39,60)$	$(\frac{65}{2},36)$	$(\frac{57}{2},40)$
8899	$(63,57,41)$	$(49,52,60)$	$(\frac{65}{2},48)$	$(28,\frac{105}{2})$

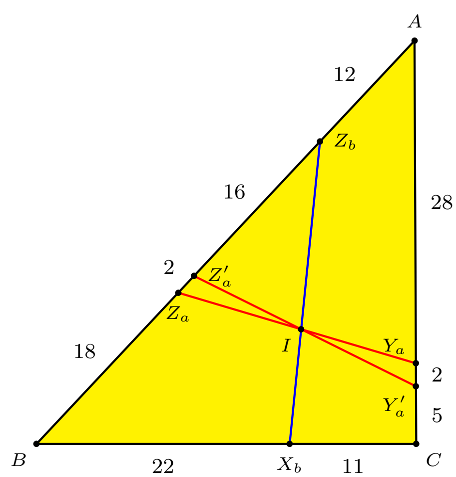

FIGURE 6.

REFERENCES

[1] J. A. Dunn and J. E. Pretty, Halving a triangle, *Math. Gazette*, Vol. 56 (1972) 105 – 108.

[2] D. Kodokostas, Triangle equalizers, *Math. Mag.*, 83 (2010) 141–146.

[3] I. Sfikas, Equalizers in right triangles, *Global Journal of Adv. Research on Classical and Modern Geometry*, 3 (2014) 38–43.

[4] A. Todd, Bisecting a triangle, *Pi Mu Epsilon Journal*, 11:1 (1999) 31 – 37.

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