Chapter 1

Pythagoras Theorem and Its Applications

1.1 Pythagoras Theorem and its converse

1.1.1 Pythagoras Theorem

The lengths $a \leq b < c$ of the sides of a right triangle satisfy the relation

$$a^2 + b^2 = c^2.$$

1.1.2 Converse Theorem

If the lengths of the sides of a triangle satisfy the relation $a^2 + b^2 = c^2$, then the triangle contains a right angle.
Proof. Let $ABC$ be a triangle with $BC = a$, $CA = b$, and $AB = c$ satisfying $a^2 + b^2 = c^2$. Consider another triangle $XYZ$ with

$$YZ = a, \quad XZ = b, \quad \angle XZY = 90^\circ.$$  

By the Pythagorean theorem, $XY^2 = a^2 + b^2 = c^2$, so that $XY = c$. Thus the triangles $\triangle ABC \equiv \triangle XYZ$ by the SSS test. This means that $\angle ACB = \angle XZY$ is a right angle.

Exercise

1. Dissect two given squares into triangles and quadrilaterals and rearrange the pieces into a square.

2. $BCX$ and $CDY$ are equilateral triangles inside a rectangle $ABCD$. The lines $AX$ and $AY$ are extended to intersect $BC$ and $CD$ respectively at $P$ and $Q$. Show that

(a) $\triangle APQ$ is an equilateral triangle;

(b) $\triangle APB + \triangle ADQ = \triangle CPQ$. 

3. $ABC$ is a triangle with a right angle at $C$. If the median on the side $a$ is the geometric mean of the sides $b$ and $c$, show that $c = 3b$.

4. (a) Suppose $c = a + kb$ for a right triangle with legs $a$, $b$, and hypotenuse $c$. Show that $0 < k < 1$, and

$$a : b : c = 1 - k^2 : 2k : 1 + k^2.$$  

(b) Find two right triangles which are not similar, each satisfying $c = \frac{3}{4}a + \frac{4}{5}b$. 

5. $ABC$ is a triangle with a right angle at $C$. If the median on the side $c$ is the geometric mean of the sides $a$ and $b$, show that one of the acute angles is $15^\circ$.

6. Let $ABC$ be a right triangle with a right angle at vertex $C$. Let $CXPY$ be a square with $P$ on the hypotenuse, and $X$, $Y$ on the sides. Show that the length $t$ of a side of this square is given by

$$\frac{1}{t} = \frac{1}{a} + \frac{1}{b}.$$ 

$$1/a + 1/b = 1/t.$$ 

\[ 1/a^2 + 1/b^2 = 1/d^2. \]

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\[ a : b : c = 12 : 35 : 37 \text{ or } 12 : 5 : 13. \] More generally, for $h \leq k$, there is, up to similarity, a unique right triangle satisfying $c = ha + kb$ provided

(i) $h < 1 \leq k$, or 
(ii) $\sqrt{2} \leq h = k < 1$, or 
(iii) $h, k > 0, h^2 + k^2 = 1$.

There are two such right triangles if 

$$0 < h < k < 1, \quad h^2 + k^2 > 1.$$
7. Let $ABC$ be a right triangle with sides $a$, $b$ and hypotenuse $c$. If $d$ is the height of on the hypotenuse, show that
\[
\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{d^2}.
\]

8. (Construction of integer right triangles) It is known that every right triangle of integer sides (without common divisor) can be obtained by choosing two relatively prime positive integers $m$ and $n$, one odd, one even, and setting
\[
a = m^2 - n^2, \quad b = 2mn, \quad c = m^2 + n^2.
\]
(a) Verify that $a^2 + b^2 = c^2$.
(b) Complete the following table to find all such right triangles with sides $< 100$:

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1.2 Euclid’s Proof of Pythagoras Theorem

1.2.1 Euclid’s proof

1.2.2 Application: construction of geometric mean

Construction 1

Given two segments of length \(a < b\), mark three points \(P, A, B\) on a line such that \(PA = a\), \(PB = b\), and \(A, B\) are on the same side of \(P\). Describe a semicircle with \(PB\) as diameter, and let the perpendicular through \(A\) intersect the semicircle at \(Q\). Then \(PQ^2 = PA \cdot PB\), so that the length of \(PQ\) is the geometric mean of \(a\) and \(b\).
Construction 2

Given two segments of length $a$, $b$, mark three points $A$, $P$, $B$ on a line ($P$ between $A$ and $B$) such that $PA = a$, $PB = b$. Describe a semicircle with $AB$ as diameter, and let the perpendicular through $P$ intersect the semicircle at $Q$. Then $PQ^2 = PA \cdot PB$, so that the length of $PQ$ is the geometric mean of $a$ and $b$.

\[ y^2 = a(a+b) = a^2 + ab, \]
\[ y^2 = a^2 + x^2. \]
Therefore, $ab = x^2$.

Example

To cut a given rectangle of sides $a < b$ into three pieces that can be rearranged into a square.  

\[ \text{This construction is valid as long as } a \geq \frac{1}{4}b. \]

\[ \text{Phillips and Fisher, p.465.} \]
Exercise

1. The midpoint of a chord of length $2a$ is at a distance $d$ from the midpoint of the minor arc it cuts out from the circle. Show that the diameter of the circle is $\frac{a^2 + d^2}{d}$.

2. Two parallel chords of a circle has lengths 168 and 72, and are at a distance 64 apart. Find the radius of the circle.  

3. A crescent is formed by intersecting two circular arcs of qual radius. The distance between the two endpoints $A$ and $B$ is $a$. The central line intersects the arcs at two points $P$ and $Q$ at a distance $d$ apart. Find the radius of the circles.

4. $ABPQ$ is a rectangle constructed on the hypotenuse of a right triangle $ABC$. $X$ and $Y$ are the intersections of $AB$ with $CP$ and $CQ$ respectively.

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3Answer: The distance from the center to the longer chord is 13. From this, the radius of the circle is 85. More generally, if these chords has lengths $2a$ and $2b$, and the distance between them is $d$, the radius $r$ of the circle is given by

$$r^2 = \frac{[d^2 + (a - b)^2][d^2 + (a + b)^2]}{4d^2}.$$
(a) If $ABPQ$ is a square, show that $XY^2 = BX \cdot AY$.
(b) If $AB = \sqrt{2} \cdot AQ$, show that $AX^2 + BY^2 = AB^2$.

1.3 Construction of regular polygons

1.3.1 Equilateral triangle, regular hexagon, and square

Given a circle of radius $a$, we denote by

\[
\begin{align*}
z_n & \quad \text{the length of a side of an inscribed regular } n-\text{gon,} \\
Z_n & \quad \text{the length of a side of a circumscribed regular } n-\text{gon.}
\end{align*}
\]

\[
\begin{align*}
z_3 &= \sqrt{3}a, \quad Z_3 = 2\sqrt{3}a; \\
z_4 &= \sqrt{2}a, \quad Z_4 = 2a; \\
z_6 &= 1, \quad Z_6 = \frac{2}{3}\sqrt{3}a.
\end{align*}
\]
Exercise

1. $AB$ is a chord of length 2 in a circle $O(2)$. $C$ is the midpoint of the minor arc $AB$ and $M$ the midpoint of the chord $AB$.

Show that (i) $CM = 2 - \sqrt{3}$; (ii) $BC = \sqrt{6} - \sqrt{2}$.

Deduce that

$$\tan 15^\circ = 2 - \sqrt{3}, \quad \sin 15^\circ = \frac{1}{4}(\sqrt{6} - \sqrt{2}), \quad \cos 15^\circ = \frac{1}{4}(\sqrt{6} + \sqrt{2}).$$
1.4 The regular pentagon and its construction

1.4.1 The regular pentagon

Since $XB = XC$ by symmetry, the isosceles triangles $CAB$ and $XCB$ are similar. From this,

$$\frac{AC}{AB} = \frac{CX}{CB},$$

and $AC \cdot CB = AB \cdot CX$. It follows that

$$AX^2 = AB \cdot XB.$$

1.4.2 Division of a segment into the golden ratio

Such a point $X$ is said to divide the segment $AB$ in the golden ratio, and can be constructed as follows.

1. Draw a right triangle $ABP$ with $BP$ perpendicular to $AB$ and half in length.
2. Mark a point $Q$ on the hypotenuse $AP$ such that $PQ = PB$.
3. Mark a point $X$ on the segment $AB$ such that $AX = AQ$.

Then $X$ divides $AB$ into the golden ratio, namely,

$$AX : AB = XB : AX.$$
Exercise

1. If \( X \) divides \( AB \) into the golden ratio, then \( AX : XB = \phi : 1 \), where
\[
\phi = \frac{1}{2}(\sqrt{5} + 1) \approx 1.618 \ldots
\]
Show also that \( \frac{AX}{AB} = \frac{1}{2}(\sqrt{5} - 1) = \phi - 1 = \frac{1}{\phi} \).

2. If the legs and the altitude of a right triangle form the sides of another right triangle, show that the altitude divides the hypotenuse into the golden ratio.

3. \( ABC \) is an isosceles triangle with a point \( X \) on \( AB \) such that \( AX = CX = BC \). Show that
   (i) \( \angle BAC = 36^\circ \);
   (ii) \( AX : XB = \phi : 1 \).
Suppose \( XB = 1 \). Let \( E \) be the midpoint of the side \( AC \). Show that
\[
XE = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}.
\]
Deduce that
\[
\cos 36^\circ = \frac{\sqrt{5} + 1}{4}, \quad \sin 36^\circ = \frac{1}{2}\sqrt{10 - 2\sqrt{5}}, \quad \tan 36^\circ = \sqrt{5 - 2\sqrt{5}}.
\]

4. \( ABC \) is an isosceles triangle with \( AB = AC = 4 \). \( X \) is a point on \( AB \) such that \( AX = CX = BC \). Let \( D \) be the midpoint of \( BC \). Calculate the length of \( AD \), and deduce that
\[
\sin 18^\circ = \frac{\sqrt{5} - 1}{4}, \quad \cos 18^\circ = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}, \quad \tan 18^\circ = \frac{1}{5}\sqrt{25 - 10\sqrt{5}}.
\]
1.4.3 Construction of a regular pentagon

1. Divide a segment $AB$ into the golden ratio at $X$.
2. Construct the circles $A(X)$ and $X(B)$ to intersect at $C$.
3. Construct a circle center $C$, radius $AB$, to meet the two circles $A(X)$ and $B(AX)$ at $D$ and $E$ respectively.

Then, $ACBED$ is a regular pentagon.

Exercise

1. Justify the following construction of an inscribed regular pentagon.

1.5 The cosine formula and its applications

1.5.1 The cosine formula

$$c^2 = a^2 + b^2 - 2ab \cos \gamma.$$
Exercise

1. Show that the (4,5,6) triangle has one angle equal to twice of another.

2. If $\gamma = 2\beta$, show that $c^2 = (a + b)b$.

3. Find a simple relation between the sum of the areas of the three squares $S_1, S_2, S_3$, and that of the squares $T_1, T_2, T_3$.

4. $ABC$ is a triangle with $a = 12$, $b + c = 18$, and $\cos \alpha = \frac{7}{38}$. Show that $a^3 = b^3 + c^3$.

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$^4$AMM E688, P.A. Pizá. Here, $b = 9 - \sqrt{5}$, and $c = 9 + \sqrt{5}$. 
1.5.2 Stewart’s Theorem

If $X$ is a point on the side $BC$ (or its extension) such that $BX : XC = \lambda : \mu$, then

$$AX^2 = \frac{\lambda b^2 + \mu c^2}{\lambda + \mu} - \frac{\lambda\mu a^2}{(\lambda + \mu)^2}.$$  

*Proof.* Use the cosine formula to compute the cosines of the angles $AXB$ and $AXC$, and note that $\cos ABC = -\cos AXB$.

1.5.3 Apollonius Theorem

The length $m_a$ of the median $AD$ is given by

$$m_a^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2).$$

*Proof.* Apply Stewart’s Theorem with $\lambda = \mu = 1$.

**Exercise**

1. $m_b = m_c$ if and only if $b = c$.

2. $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$.

3. The lengths of the sides of a triangle are 136, 170, and 174. Calculate the lengths of its medians.

4. Suppose $c^2 = \frac{a^2 + b^2}{2}$. Show that $m_c = \frac{\sqrt{2}}{2}c$. Give a euclidean construction of triangles satisfying this condition.

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Answers: 158, 131, 127.
5. If \( m_a : m_b : m_c = a : b : c \), show that the triangle is equilateral.

6. Suppose \( m_b : m_c = c : b \). Show that either
   (i) \( b = c \), or
   (ii) the quadrilateral \( AEGF \) is cyclic.
   Show that the triangle is equilateral if both (i) and (ii) hold. 6

7. Show that the median \( m_a \) can never be equal to the arithmetic mean of \( b \) and \( c \). 7

8. The median \( m_a \) is the geometric mean of \( b \) and \( c \) if and only if \( a = \sqrt{2}|b - c| \).

1.5.4 Length of angle bisector

The length \( w_a \) of the (internal) bisector of angle \( A \) is given by

\[
w_a^2 = bc[1 - (\frac{a}{b + c})^2].
\]

Proof. Apply Stewart’s Theorem with \( \lambda = c \) and \( \mu = b \).

Exercise

1. \( w_a^2 = \frac{4bcs(s-a)}{(b+c)^2} \).

2. The lengths of the sides of a triangle are 84, 125, 169. Calculate the lengths of its internal bisectors. 8

3. (Steiner - Lehmus Theorem) If \( w_a = w_b \), then \( a = b \). 9

4. Suppose \( w_a : w_b = b : a \). Show that the triangle is either isosceles, or \( \gamma = 60^\circ \). 10

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6 Crux 383. In fact, \( b^2m_b^2 - c^2m_c^2 = \frac{1}{4}(c - b)(c + b)(b^2 + c^2 - 2a^2) \).
7 Complete the triangle \( ABC \) to a parallelogram \( ABA'C \).
8 Answers: 275, 2635, 12600.
9 Hint: Show that

\[
\frac{a}{(b + c)^2} - \frac{b}{(c + a)^2} = \frac{(a - b)[(a + b + c)^2 - ab]}{(b + c)^2(c + a)^2}.
\]

10 \( a^2w_a^2 - b^2w_b^2 = \frac{abc(b-a)(a+b+c)^2}{(a+c)^2(b+c)^2} [a^2 - ab + b^2 - c^2] \).
5. Show that the length of the external angle bisector is given by

\[ w'_a^2 = bc\left(\frac{a}{b-c}\right)^2 - 1 \]

\[ = \frac{4bc(s-b)(s-c)}{(b-c)^2}. \]

6. In triangle \(ABC\), \(\alpha = 12^\circ\), and \(\beta = 36^\circ\). Calculate the ratio of the lengths of the external angle bisectors \(w'_a\) and \(w'_b\).  

1.6 Appendix: Synthetic proofs of Steiner - Lehmus Theorem

1.6.1 First proof.  

Suppose \(\beta < \gamma\) in triangle \(ABC\). We show that the bisector \(BM\) is longer than the bisector \(CN\).

Choose a point \(L\) on \(BM\) such that \(\angle NCL = \frac{1}{2}\beta\). Then \(B, N, L, C\) are concyclic since \(\angle NBL = \angle NCL\). Note that

\[ \angle NBC = \beta < \frac{1}{2}(\beta + \gamma) = \angle LCB, \]

and both are acute angles. Since smaller chords of a circle subtend smaller acute angles, we have \(CN < BL\). It follows that \(CN < BM\).

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11Answer: 1:1. The counterpart of the Steiner - Lehmus theorem does not hold. See Crux Math. 2 (1976) pp. 22 – 24. D.L.MacKay (AMM E312): if the external angle bisectors of \(B\) and \(C\) of a scalene triangle \(ABC\) are equal, then \(\frac{a+b}{b}\) is the geometric mean of \(\frac{a+b}{b}\) and \(\frac{a+c}{c}\). See also Crux 1607 for examples of triangles with one internal bisector equal to one external bisector.

1.6.2 Second proof. 13

Suppose the bisectors $BM$ and $CN$ in triangle $ABC$ are equal. We shall show that $\beta = \gamma$. If not, assume $\beta < \gamma$. Compare the triangles $CBM$ and $BCN$. These have two pairs of equal sides with included angles $\angle CBM = \frac{1}{2}\beta < \frac{1}{2}\gamma = \angle BCN$, both of which are acute. Their opposite sides therefore satisfy the relation $CM < BN$.

![Diagram](image)

Complete the parallelogram $BMGN$, and consider the triangle $CNG$. This is isosceles since $CN = BM = NG$. Note that

$$\angle CGN = \frac{1}{2}\beta + \angle CGM,$$

$$\angle GCN = \frac{1}{2}\gamma + \angle GMC.$$

Since $\beta < \gamma$, we conclude that $\angle CGM > \angle GMC$. From this, $CM > GM = BN$. This contradicts the relation $CM < BN$ obtained above.

Exercise

1. The bisectors of angles $B$ and $C$ of triangle $ABC$ intersect the median $AD$ at $E$ and $F$ respectively. Suppose $BE = CF$. Show that triangle $ABC$ is isosceles. 14

13M. Descube, 1880.
14Crux 1897; also CMJ 629.
Chapter 2

The circumcircle and the incircle

2.1 The circumcircle

2.1.1 The circumcenter

The perpendicular bisectors of the three sides of a triangle are concurrent at the circumcenter of the triangle. This is the center of the circumcircle, the circle passing through the three vertices of the triangle.

2.1.2 The sine formula

Let $R$ denote the circumradius of a triangle $ABC$ with sides $a, b, c$ opposite to the angles $\alpha, \beta, \gamma$ respectively.
\[
\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = 2R.
\]

**Exercise**

1. The internal bisectors of angles $B$ and $C$ intersect the circumcircle of $\triangle ABC$ at $B'$ and $C'$.
   (i) Show that if $\beta = \gamma$, then $BB' = CC'$.
   (ii) If $BB' = CC'$, does it follow that $\beta = \gamma$?

2. If $H$ is the orthocenter of triangle $ABC$, then the triangles $HAB$, $HBC$, $HCA$ and $ABC$ have the same circumradius.

3. Given three angles $\alpha$, $\beta$, $\gamma$ such that $\theta + \phi + \psi = 60^\circ$, and an equilateral triangle $XYZ$, construct outwardly triangles $AYZ$ and $BZX$ such that $\angle AYZ = 60^\circ + \psi$, $\angle AZY = 60^\circ + \phi$ such that $\angle BZX = 60^\circ + \theta$, $\angle BXZ = 60^\circ + \psi$. Suppose the sides of $XYZ$ have unit length.
   (a) Show that
   \[
   AZ = \frac{\sin(60^\circ + \psi)}{\sin \theta}, \quad \text{and} \quad BZ = \frac{\sin(60^\circ + \psi)}{\sin \phi}.
   \]
   (b) In triangle $ABZ$, show that $\angle ZAB = \theta$ and $\angle ZBA = \phi$.  

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\(^1\)(ii) No. $BB' = CC'$ if and only if $\beta = \gamma$ or $\alpha = \frac{2\pi}{3}$.  

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(c) Suppose a third triangle \( XYC \) is constructed outside \( XYZ \) such that \( \angle CYX = 60^\circ + \theta \) and \( \angle CYX = 60^\circ + \phi \). Show that \( BX, BZ \) are the trisectors of the angles of triangle \( ABC \).

(d) Show that \( AY \cdot BZ \cdot CX = AZ \cdot BX \cdot CY \).

(e) Suppose the extensions of \( BX \) and \( AY \) intersect at \( P \). Show that the triangles \( PXZ \) and \( PYZ \) are congruent.

### 2.1.3 Johnson’s Theorem

Suppose three circles \( A(r), B(r), \) and \( C(r) \) have a common point \( P \). If the \( (B), (C) \) circles \( (C) \) and \( (A) \) intersect again at \( Y \), then the circle through \( X, Y, Z \) also has radius \( r \).
Proof. (1) $BPCX$, $APCY$ and $APBZ$ are all rhombi. Thus, $AY$ and $BX$ are parallel, each being parallel to $PC$. Since $AY = BX$, $ABXY$ is a parallelogram, and $XY = AB$.

(2) Similarly, $YZ = BC$ and $ZX = CA$. It follows that the triangles $XYZ$ and $ABC$ are congruent.

(3) Since triangle $ABC$ has circumradius $r$, the circumcenter being $P$, the circumradius of $XYZ$ is also $r$.

Exercise

1. Show that $AX$, $BY$ and $CZ$ have a common midpoint.

2.2 The incircle

2.2.1 The incenter

The internal angle bisectors of a triangle are concurrent at the incenter of the triangle. This is the center of the incircle, the circle tangent to the three sides of the triangle.

If the incircle touches the sides $BC$, $CA$ and $AB$ respectively at $X$, $Y$, and $Z$,

$$AY = AZ = s - a, \quad BX = BZ = s - b, \quad CX = CY = s - c.$$
Exercise

1. Show that the three small circles are equal.

2. The incenter of a right triangle is equidistant from the midpoint of the hypotenuse and the vertex of the right angle. Show that the triangle contains a 30° angle.

3. Show that $XYZ$ is an acute angle triangle.

4. Let $P$ be a point on the side $BC$ of triangle $ABC$ with incenter $I$. Mark the point $Q$ on the side $AB$ such that $BQ = BP$. Show that $IP = IQ$. 
Continue to mark $R$ on $AC$ such that $AR = AQ$, $P'$ on $BC$ such that $CP' = CR$, $Q'$ on $AB$ such that $BQ' = BP'$, $R'$ on $AC$ such that $AR' = AQ'$. Show that $CP = CR'$, and that the six points $P, Q, R, P', Q', R'$ lie on a circle, center $I$.

5. The inradius of a right triangle is $r = s - c$.

6. The incircle of triangle $ABC$ touches the sides $AC$ and $AB$ at $Y$ and $Z$ respectively. Suppose $BY = CZ$. Show that the triangle is isosceles.

7. A line parallel to hypotenuse $AB$ of a right triangle $ABC$ passes through the incenter $I$. The segments included between $I$ and the sides $AC$ and $BC$ have lengths 3 and 4.

8. $Z$ is a point on a segment $AB$ such that $AZ = u$ and $ZB = v$. Suppose the incircle of a right triangle with $AB$ as hypotenuse touches $AB$ at $Z$. Show that the area of the triangle is equal to $uv$. Make use of this to give a euclidean construction of the triangle. 

\[ \text{Solution. Let } r \text{ be the inradius. Since } r = s - c \text{ for a right triangle, } a = r + u \text{ and} \]

\[ \]
9. \(AB\) is an arc of a circle \(O(r)\), with \(\angle AOB = \alpha\). Find the radius of the circle tangent to the arc and the radii through \(A\) and \(B\). 

![Diagram](image)

10. A semicircle with diameter \(BC\) is constructed outside an equilateral triangle \(ABC\). \(X\) and \(Y\) are points dividing the semicircle into three equal parts. Show that the lines \(AX\) and \(AY\) divide the side \(BC\) into three equal parts.

![Diagram](image)

11. Suppose each side of equilateral triangle has length \(2a\). Calculate the radius of the circle tangent to the semicircle and the sides \(AB\) and \(AC\). 

12. \(AB\) is a diameter of a circle \(O(\sqrt{5}a)\). \(PXYQ\) is a square inscribed in the semicircle. Let \(C\) a point on the semicircle such that \(BC = 2a\).

\[
b = r + v. \quad \text{From} \quad (r + u)^2 + (r + v)^2 = (u + v)^2, \quad \text{we obtain} \quad (r + u)(r + v) = 2uv \quad \text{so that the area is} \quad \frac{1}{2}(r + u)(r + v) = uv. \quad \text{If} \quad h \quad \text{is the height on the hypotenuse, then} \quad \frac{1}{2}(u + v)h = uv. \quad \text{This leads to a simple construction of the triangle.}
\]

\[
3\text{Hint: The circle is tangent to the arc at its midpoint.}
\]

\[
4\text{\(\frac{1}{2}(1 + \sqrt{3})a.\)}
\]
(a) Show that the right triangle $ABC$ has the same area as the square $PXYQ$.

(b) Find the inradius of the triangle $ABC$.\(^5\)

(c) Show that the incenter of $\triangle ABC$ is the intersection of $PX$ and $BY$.

13. A square of side $a$ is partitioned into 4 congruent right triangles and a small square, all with equal inradii $r$. Calculate $r$.

14. An equilateral triangle of side $2a$ is partitioned symmetrically into a quadrilateral, an isosceles triangle, and two other congruent triangles. If the inradii of the quadrilateral and the isosceles triangle are equal,\(^5\)

\[^5\] $r = (3 - \sqrt{5})a$. 
find this radius. What is the inradius of each of the remaining two triangles? 6

15. Let the incircle \( I(r) \) of a right triangle \( \triangle ABC \) (with hypotenuse \( AB \)) touch its sides \( BC, CA, AB \) at \( X, Y, Z \) respectively. The bisectors \( AI \) and \( BI \) intersect the circle \( Z(I) \) at the points \( M \) and \( N \). Let \( CR \) be the altitude on the hypotenuse \( AB \).

Show that
(i) \( XN = YM = r \);
(ii) \( M \) and \( N \) are the incenters of the right triangles \( ABR \) and \( BCR \) respectively.

16. \( CR \) is the altitude on the hypotenuse \( AB \) of a right triangle \( ABC \).

Show that the area of the triangle determined by the incenters of triangles \( ABC, ACR \), and \( BCR \) is \( \frac{(s-c)^3}{c} \). 7

17. The triangle is isosceles and the three small circles have equal radii. Suppose the large circle has radius \( R \). Find the radius of the small circles. 8

---

6(\sqrt{3} - \sqrt{2})a.

7Make use of similarity of triangles.

8Let \( \theta \) be the semi-vertical angle of the isosceles triangle. The inradius of the triangle is \( \frac{2R\sin \theta \cos \theta}{1 + \sin \theta} = 2R \sin \theta (1 - \sin \theta) \). If this is equal to \( \frac{R}{c}(1 - \sin \theta) \), then \( \sin \theta = \frac{1}{c} \). From this, the inradius is \( \frac{1}{c} R \).
18. The large circle has radius $R$. The four small circles have equal radii. Find this common radius.\footnote{Let $\theta$ be the smaller acute angle of one of the right triangles. The inradius of the right triangle is $\frac{2R\cos\theta\sin\theta}{1 + \sin\theta + \cos\theta}$. If this is equal to $\frac{5}{2}(1 - \sin\theta)$, then $5\sin\theta - \cos\theta = 1$. From this, $\sin\theta = \frac{5}{13}$, and the inradius is $\frac{4}{13}R$.}

2.3 The excircles

2.3.1 The excenter

The internal bisector of each angle and the external bisectors of the remaining two angles are concurrent at an excenter of the triangle. An excircle can be constructed with this as center, tangent to the lines containing the three sides of the triangle.
2.3.2 The exradii

The exradii of a triangle with sides $a$, $b$, $c$ are given by

$$ r_a = \frac{\Delta}{s-a}, \quad r_b = \frac{\Delta}{s-b}, \quad r_c = \frac{\Delta}{s-c}. $$

**Proof.** The areas of the triangles $I_A BC$, $I_A CA$, and $I_A AB$ are $\frac{1}{2}ar_a$, $\frac{1}{2}br_a$, and $\frac{1}{2}cr_a$ respectively. Since

$$ \Delta = -\Delta I_A BC + \Delta I_A CA + \Delta I_A AB, $$

we have

$$ \Delta = \frac{1}{2}r_a(-a+b+c) = r_a(s-a), $$

from which $r_a = \frac{\Delta}{s-a}$.

**Exercise**

1. If the incenter is equidistant from the three excenters, show that the triangle is equilateral.

2. Show that the circumradius of $\triangle I_AI_B I_C$ is $2R$, and the area is $\frac{abc}{2r}$.

3. Show that for triangle $ABC$, if any two of the points $O$, $I$, $H$ are concyclic with the vertices $B$ and $C$, then the five points are concyclic. In this case, $\alpha = 60^\circ$.

4. Suppose $\alpha = 60^\circ$. Show that $IO = IH$.

5. Suppose $\alpha = 60^\circ$. If the bisectors of angles $B$ and $C$ meet their opposite sides at $E$ and $F$, then $IE = IF$.

6. Show that $\frac{r}{r_a} = \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$. 
7. Let $P$ be a point on the side $BC$. Denote by $r'$, $r''$, $\rho'$, $\rho''$ the inradius and exradius of triangle $\triangle ABP$ and $\triangle APC$. Show that $\frac{r' r''}{\rho' \rho''}$ is independent of the position of $P$.

8. Let $M$ be the midpoint of the arc $BC$ of the circumcircle not containing the vertex $A$. Show that $M$ is also the midpoint of the segment $I_I_A$.

9. Let $M'$ be the midpoint of the arc $BAC$ of the circumcircle of triangle $\triangle ABC$. Show that each of $M'BIC$ and $M'CIB$ is an isosceles triangle. Deduce that $M'$ is indeed the midpoint of the segment $I_BI_C$.

10. The circle $BIC$ intersects the sides $AC$, $AB$ at $E$ and $F$ respectively. Show that $EF$ is tangent to the incircle of $\triangle ABC$. \footnote{Hint: Show that $IF$ bisects angle $AFE$.}
11. The incircle of triangle $ABC$ touches the side $BC$ at $X$. The line $AX$ intersects the perpendicular bisector of $BC$ at $K$. If $D$ is the midpoint of $BC$, show that $DK = r_C$.

### 2.4 Heron’s formula for the area of a triangle

Consider a triangle $ABC$ with area $\triangle$. Denote by $r$ the inradius, and $r_a$ the radius of the excircle on the side $BC$ of triangle $ABC$. It is convenient to introduce the semiperimeter $s = \frac{1}{2}(a + b + c)$.

- $\triangle = rs$. 
From the similarity of triangles $AIZ$ and $AI'Z'$,
\[
\frac{r}{r_a} = \frac{s - a}{s}.
\]

From the similarity of triangles $CIY$ and $I'CY'$,
\[
r \cdot r_a = (s - b)(s - c).
\]

From these,
\[
r = \frac{(s - a)(s - b)(s - c)}{s},
\]
\[
\Delta = \sqrt{s(s - a)(s - b)(s - c)}.
\]

This latter is the famous Heron formula.

Exercise

1. The altitudes a triangle are 12, 15 and 20. What is the area of the triangle? \(^{11}\)

2. Find the inradius and the exradii of the (13,14,15) triangle.

3. The length of each side of the square is $6a$, and the radius of each of the top and bottom circles is $a$. Calculate the radii of the other two circles.

\(^{11}\) $\Delta = 150$. The lengths of the sides are 25, 20 and 15.
4. If one of the ex-radii of a triangle is equal to its semiperimeter, then the triangle contains a right angle.

5. \( \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} \).

6. \( r_ar_br_c = r^2s \).

7. Show that
   (i) \( r_a + r_b + r_c = \frac{-s^2 + (ab + bc + ca)s}{\Delta} \);
   (ii) \( (s - a)(s - b)(s - c) = -s^3 + (ab + bc + ca)s \).

   Deduce that \( r_a + r_b + r_c = 4R + r \).

2.4.1 Appendix: A synthetic proof of \( r_a + r_b + r_c = 4R + r \)
**Proof.**  
(1) The midpoint $M$ of the segment $II_A$ is on the circumcircle.  
(2) The midpoint $M'$ of $IBIC$ is also on the circumcircle.  
(3) $MM'$ is indeed a diameter of the circumcircle, so that $MM' = 2R$.  
(4) If $D$ is the midpoint of $BC$, then $DM' = \frac{1}{2}(r_b + r_c)$.  
(5) Since $D$ is the midpoint of $XX'$, $QX' = IX = r$, and $IAQ = r_a - r$.  
(6) Since $M$ is the midpoint of $II_A$, $MD$ is parallel to $IAQ$ and is half in length. Thus, $MD = \frac{1}{2}(r_a - r)$.  
(7) It now follows from $MM' = 2R$ that $r_a + r_b + r_c - r = 4R$. 
Chapter 3

The Euler line and
the nine-point circle

3.1 The orthocenter

3.1.1

The three altitudes of a triangle are concurrent. The intersection is the orthocenter of the triangle.

The orthocenter is a triangle is the circumcenter of the triangle bounded by the lines through the vertices parallel to their opposite sides.

3.1.2

The orthocenter of a right triangle is the vertex of the right angle.
If the triangle is obtuse, say, $\alpha > 90^\circ$, then the orthocenter $H$ is outside the triangle. In this case, $C$ is the orthocenter of the acute triangle $ABH$.

### 3.1.3 Orthocentric quadrangle

More generally, if $A$, $B$, $C$, $D$ are four points one of which is the orthocenter of the triangle formed by the other three, then each of these points is the orthocenter of the triangle whose vertices are the remaining three points. In this case, we call $ABCD$ an orthocentric quadrangle.

### 3.1.4 Orthic triangle

The orthic triangle of $ABC$ has as vertices the traces of the orthocenter $H$ on the sides. If $ABC$ is an acute triangle, then the angles of the orthic triangle are $180^\circ - 2\alpha$, $180^\circ - 2\beta$, and $180^\circ - 2\gamma$.

If $ABC$ is an obtuse triangle, with $\gamma > 90^\circ$, then $ABH$ is acute, with angles $90^\circ - \beta$, $90^\circ - \alpha$, and $180^\circ - \gamma$. The triangles $ABC$ and $ABH$ have the same orthic triangle, whose angles are then $2\beta$, $2\alpha$, and $2\gamma - 180^\circ$.

**Exercise**

1. If $ABC$ is an acute triangle, then $YZ = a \cos \alpha$. How should this be modified if $\alpha > 90^\circ$?

2. If an acute triangle is similar to its orthic triangle, then the triangle must be equilateral.
3. Let $H$ be the orthocenter of an acute triangle. $AH = 2R \cdot \cos \alpha$, and $HX = 2R \cdot \cos \beta \cos \gamma$, where $R$ is the circumradius.

4. If an obtuse triangle is similar to its orthic triangle, find the angles of the triangle. \(^1\)

### 3.2 The Euler line

#### 3.2.1 Theorem

The circumcenter $O$, the orthocenter $H$ and the median point $M$ of a non-equilateral triangle are always collinear. Furthermore, $OG : GH = 1 : 2$.

**Proof.** Let $Y$ be the projection of the orthocenter $H$ on the side $AC$.

The line $OGH$ is called the **Euler line** of the triangle.

1. $AH = AY / \sin \gamma = c \cos \alpha / \sin \gamma = 2R \cos \alpha$.

2. $OD = R \cos \alpha$.

3. If $OH$ and $AD$ intersect at $G'$, then $\triangle AG'H \sim \triangle DG'O$, and $AG' = 2G'D$.

4. Consequently, $G' = G$, the centroid of $\triangle ABC$.

\(^1\) $180^\circ$, $360^\circ$, and $720^\circ$. 

---

\(\begin{align*}
A & \quad Y \\
B & \quad H \\
C & \quad O \\
G' & \\
\end{align*}\)
Exercise

1. Show that a triangle is equilateral if and only if any two of the points coincide.

   circumcenter, incenter, centroid, orthocenter.

2. Show that the incenter $I$ of a non-equilateral triangle lies on the Euler line if and only if the triangle is isosceles.

3. Let $O$ be the circumcenter of $\triangle ABC$. Denote by $D$, $E$, $F$ the projections of $O$ on the sides $BC$, $CA$, $AB$ respectively. $DEF$ is called the medial triangle of $ABC$.
   (a) Show that the orthocenter of $DEF$ is the circumcenter $O$ of $\triangle ABC$.
   (b) Show that the centroid of $DEF$ is the centroid of $\triangle ABC$.
   (c) Show that the circumcenter $N$ of $DEF$ also lies on the Euler line of $\triangle ABC$. Furthermore,

   $OG : GN : NH = 2 : 1 : 3$.

4. Let $H$ be the orthocenter of triangle $ABC$. Show that the Euler lines of $\triangle ABC$, $\triangle HBC$, $\triangle HCA$ and $\triangle HAB$ are concurrent.  

5. Show that the Euler line is parallel (respectively perpendicular) to the internal bisector of angle $C$ if and only if $\gamma = \frac{2\pi}{3}$ (respectively $\frac{\pi}{3}$).

6. A diameter $d$ of the circumcircle of an equilateral triangle $ABC$ intersects the sides $BC$, $CA$ and $AB$ at $D$, $E$ and $F$ respectively. Show that the Euler lines of the triangles $AEF$, $BFD$ and $CDE$ form an equilateral triangle symmetrically congruent to $ABC$, the center of symmetry lying on the diameter $d$.  

\[^2\text{Hint: find a point common to them all.}\]
\[^3\text{Thébault, AMM E547.}\]
7. The Euler lines of triangles $IBC$, $ICA$, $IAB$ are concurrent.  

### 3.3 The nine-point circle

Let $ABC$ be a given triangle, with

(i) $D$, $E$, $F$ the midpoints of the sides $BC$, $CA$, $AB$,

(ii) $P$, $Q$, $R$ the projections of the vertices $A$, $B$, $C$ on their opposite sides, the altitudes $AP$, $BQ$, $CR$ concurring at the orthocenter $H$,

(iii) $X$, $Y$, $Z$ the midpoints of the segments $AH$, $BH$, $CH$.

The nine points $D$, $E$, $F$, $P$, $Q$, $R$, $X$, $Y$, $Z$ are concyclic.

This is called the *nine-point circle* of $\triangle ABC$. The center of this circle is the nine-point center $F$. It is indeed the circumcircle of the *medial* triangle $DEF$.

The center $F$ of the nine-point circle lies on the Euler line, and is the midway between the circumcenter $O$ and the orthocenter $H$.

---

*Crux 1018. Schliffer-Veldkamp.*
Exercise

1. $P$ and $Q$ are two points on a semicircle with diameter $AB$. $AP$ and $BQ$ intersect at $C$, and the tangents at $P$ and $Q$ intersect at $X$. Show that $CX$ is perpendicular to $AB$.

2. Let $P$ be a point on the circumcircle of triangle $ABC$, with orthocenter $H$. The midpoint of $PH$ lies on the nine-point circle of the triangle.
3. (a) Let $ABC$ be an isosceles triangle with $a = 2$ and $b = c = 9$. Show that there is a circle with center $I$ tangent to each of the excircles of triangle $ABC$.

(b) Suppose there is a circle with center $I$ tangent externally to each of the excircles. Show that the triangle is equilateral.

(c) Suppose there is a circle with center $I$ tangent internally to each of the excircles. Show that the triangle is equilateral.

4. Prove that the nine-point circle of a triangle trisects a median if and only if the side lengths are proportional to its medians lengths in some order.

3.4  **Power of a point with respect to a circle**

The *power* of a point $P$ with respect to a circle $O(r)$ is defined as

$$O(r)_P := OP^2 - r^2.$$  

This number is positive, zero, or negative according as $P$ is outside, on, or inside the circle.

3.4.1

For any line $\ell$ through $P$ intersecting a circle $(O)$ at $A$ and $B$, the signed product $PA \cdot PB$ is equal to $(O)_P$, the power of $P$ with respect to the circle $(O)$.

If $P$ is outside the circle, $(O)_P$ is the square of the tangent from $P$ to $(O)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{circle_power.png}
\caption{Illustration of power of a point with respect to a circle.}
\end{figure}
3.4.2 Theorem on intersecting chords

If two lines containing two chords $AB$ and $CD$ of a circle $(O)$ intersect at $P$, the signed products $PA \cdot PB$ and $PC \cdot PD$ are equal.

**Proof.** Each of these products is equal to the power $(O)_P = OP^2 - r^2$.

**Exercise**

1. If two circles intersect, the common chord, when extended, bisects the common tangents.

2. $E$ and $F$ are the midpoints of two opposite sides of a square $ABCD$. $P$ is a point on $CE$, and $FQ$ is parallel to $AE$. Show that $PQ$ is tangent to the incircle of the square.
3. (The butterfly theorem) Let $M$ be the midpoint of a chord $AB$ of a circle $(O)$. $PY$ and $QX$ are two chords through $M$. $PX$ and $QY$ intersect the chord $AB$ at $H$ and $K$ respectively.

(i) Use the sine formula to show that

$$\frac{HX \cdot HP}{HM^2} = \frac{KY \cdot KQ}{KM^2}.$$

(ii) Use the intersecting chords theorem to deduce that $HM = KM$.

4. $P$ and $Q$ are two points on the diameter $AB$ of a semicircle. $K(T)$ is the circle tangent to the semicircle and the perpendiculars to $AB$ at $P$ and $Q$. Show that the distance from $K$ to $AB$ is the geometric mean of the lengths of $AP$ and $BQ$. 
3.5 Distance between O and I

3.5.1 Theorem

The distance $d$ between the circumcenter $O$ and the incenter $I$ of $\triangle ABC$ is given by

$$R^2 - d^2 = 2Rr.$$ 

Proof. Join $AI$ to cut the circumcircle at $X$. Note that $X$ is the midpoint of the arc $BC$. Furthermore,

1. $IX = XB = XC = 2R\sin \frac{\alpha}{2},$
2. $IA = r/\sin \frac{\alpha}{2},$ and
3. $R^2 - d^2 =$ power of $I$ with respect to the circumcircle $= IA \cdot IX = 2Rr.$

3.5.2 Corollary

$r = 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}.$
Proof. Note that triangle \( XIC \) is isosceles with \( \angle XIC = \beta \). This means \( IC = 2XC \cdot \sin \frac{\beta}{2} = 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \). It follows that
\[
r = IC \cdot \sin \frac{\gamma}{2} = 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}.
\]

### 3.5.3 Distance between \( O \) and excenters

\[
OI_A^2 = R^2 + 2Rr_a.
\]

**Exercise**

1. Given the circumcenter, the incenter, and a vertex of a triangle, to construct the triangle.

2. Given a circle \( O(R) \) and \( r < \frac{1}{2}R \), construct a point \( I \) inside \( O(R) \) so that \( O(R) \) and \( I(r) \) are the circumcircle and incircle of a triangle?

3. Given a point \( I \) inside a circle \( O(R) \), construct a circle \( I(r) \) so that \( O(R) \) and \( I(r) \) are the circumcircle and incircle of a triangle?

4. Given a circle \( I(r) \) and a point \( O \), construct a circle \( O(R) \) so that \( O(R) \) and \( I(r) \) are the circumcircle and incircle of a triangle?

5. Show that the line joining the circumcenter and the incenter is parallel to a side of the triangle if and only if one of the following condition holds.
   (a) One of the angles has cosine \( \frac{r}{R} \);
   (b) \( s^2 = \frac{(2R-r)^2(R+r)}{R-r} \).

6. The power of \( I \) with respect to the circumcircle is \( \frac{abc}{a+b+c} \).

7. \( AIO \leq 90^\circ \) if and only if \( 2a \leq b + c \).

8. Make use of the relation
\[
a = r(\cot \frac{\beta}{2} + \cot \frac{\gamma}{2})
\]
to give an alternative proof of the formula \( r = 4R \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \).

9. Show that \( XI_A = XI \).

---

\(^6\)Johnson, §298(i). This power is \( OI^2 - R^2 = 2Rr = \frac{abc}{2s} \cdot \frac{s}{2} = \frac{abc}{2s} \).
Chapter 4

Circles

4.1 Tests for concyclic points

4.1.1

Let $A$, $B$, $C$, $D$ be four points such that the lines $AB$ and $CD$ intersect (extended if necessary) at $P$. If $AP \cdot BP = CP \cdot DP$, then the points $A$, $B$, $C$, $D$ are concyclic.

4.1.2

Let $P$ be a point on the line containing the side $AB$ of triangle $ABC$ such that $AP \cdot BP = CP^2$. Then the line $CP$ touches the circumcircle of triangle $ABC$ at the point $C$.

Exercise

1. Let $ABC$ be a triangle satisfying $\gamma = 90^\circ + \frac{1}{2} \beta$. If $Z$ is the point on the side $AB$ such that $BZ = BC = a$, then the circumcircle of triangle
2. Let \(\triangle ABC\) be a triangle satisfying \(\gamma = 90^\circ + \frac{1}{2} \beta\). Suppose that \(M\) is the midpoint of \(BC\), and that the circle with center \(A\) and radius \(AM\) meets \(BC\) again at \(D\). Prove that \(MD = AB\).

3. Suppose that \(\triangle ABC\) is a triangle satisfying \(\gamma = 90^\circ + \frac{1}{2} \beta\), that the exterior bisector of angle \(A\) intersects \(BC\) at \(D\), and that the side \(AB\) touches the incircle of triangle \(\triangle ABC\) at \(F\). Prove that \(CD = 2AF\).

### 4.2 Tangents to circles

The centers of the two circles \(A(a)\) and \(A(b)\) are at a distance \(d\) apart. Suppose \(d > a + b\) so that the two circles do not intersect. The internal common tangent \(PQ\) has length

\[
\sqrt{d^2 - (a + b)^2}.
\]
Suppose $d > |a - b|$ so that none of the circle contains the other. The external common tangent $XY$ has length $\sqrt{d^2 - (a - b)^2}$.

Exercise

1. In each of the following cases, find the ratio $AB : BC$.  

2. Two circles $A(a)$ and $B(b)$ are tangent externally at a point $P$. The common tangent at $P$ intersects the two external common tangents $XY, X'Y'$ at $K, K'$ respectively.
   
   (a) Show that $\angle AKB$ is a right angle.
   
   (b) What is the length $PK$?
   
   (c) Find the lengths of the common tangents $XY$ and $KK'$.

$^1\sqrt{3} : \sqrt{3} + 2$ in the case of 4 circles.
3. *A(a)* and *B(b)* are two circles with their centers at a distance *d* apart. *AP* and *AQ* are the tangents from *A* to circle *B(b)*. These tangents intersect the circle *A(a)* at *H* and *K*. Calculate the length of *HK* in terms of *d*, *a*, and *b*. \(^2\)

4. Tangents are drawn from the center of two given circles to the other circles. Show that the chords *HK* and *H'K'* intercepted by the tangents are equal.

5. *A(a)* and *B(b)* are two circles with their centers at a distance *d* apart. From the extremity *A'P* of the diameter of *A(a)* on the line *AB*, tangents are constructed to the circle *B(b)*. Calculate the radius of the circle tangent internally to *A(a)* and to these tangent lines. \(^3\)

\(^2\)Answer: \(\frac{2ab}{d+\sqrt{a^2+b^2}}\).
\(^3\)Answer: \(\frac{2ab}{d+\sqrt{a^2+b^2}}\).
6. Show that the two incircles have equal radii.

7. $ABCD$ is a square of unit side. $P$ is a point on $BC$ so that the incircle of triangle $ABP$ and the circle tangent to the lines $AP$, $PC$ and $CD$ have equal radii. Show that the length of $BP$ satisfies the equation

$$2x^3 - 2x^2 + 2x - 1 = 0.$$ 

8. $ABCD$ is a square of unit side. $Q$ is a point on $BC$ so that the incircle of triangle $ABQ$ and the circle tangent to $AQ$, $QC$, $CD$ touch each other at a point on $AQ$. Show that the radii $x$ and $y$ of the circles satisfy the equations

$$y = \frac{x(3 - 6x + 2x^2)}{1 - 2x^2}, \quad \sqrt{x} + \sqrt{y} = 1.$$ 

Deduce that $x$ is the root of

$$4x^3 - 12x^2 + 8x - 1 = 0.$$
4.3 Tangent circles

4.3.1 A basic formula

Let $AB$ be a chord of a circle $O(R)$ at a distance $h$ from the center $O$, and $P$ a point on $AB$. The radii of the circles $K(r)$ tangent to $AB$ at $P$ and also to the minor arc $AB$ are

$$r = \frac{AP \cdot PB}{2(R + h)}$$

respectively.

The case for the major arc is similar.

Proof. Let $M$ be the midpoint of $AB$ and $MP = x$. Let $K(r)$ be the circle tangent to $AB$ at $P$ and to the minor arc $AB$. We have

$$(R - r)^2 = x^2 + (h + r)^2,$$

from which

$$r = \frac{R^2 - x^2 - h^2}{2(R + h)} = \frac{R^2 - OP^2}{2(R + h)} = \frac{AP \cdot PB}{2(R + h)}.$$

The case for the major arc is similar.

4.3.2 Construction

Let $C$ be the midpoint of arc $AB$. Mark a point $Q$ on the circle so that $PQ = CM$. Extend $QP$ to meet the circle again at $H$. Then $r = \frac{1}{2}PH$,

from this the center $K$ can be located easily.
Remarks

(1) If the chord $AB$ is a diameter, these two circles both have radius

$$\frac{AP \cdot PB}{2R}.$$ 

(2) Note that the ratio $r : r' = R - h : R + h$ is independent of the position of $P$ on the chord $AB$.

4.3.3

Let $\theta$ be the angle between an external common tangent of the circles $K(r)$, $K'(r')$ and the center line $KK'$. Clearly,

$$\sin \theta = \frac{r' - r}{r' + r} = \frac{1 - \frac{r}{r'}}{1 + \frac{r}{r'}} = \frac{1 - \frac{R - h}{R + h}}{1 + \frac{R - h}{R + h}} = \frac{h}{R}.$$ 

This is the same angle between the radius $OA$ and the chord $AB$. Since the center line $KK'$ is perpendicular to the chord $AB$, the common tangent is perpendicular to the radius $OA$. This means that $A$ is the midpoint of the minor arc cut out by an external common tangent of the circles $(K)$ and $(K')$.

4.3.4

Let $P$ and $Q$ be points on a chord $AB$ such that the circles $(K_P)$ and $(K_Q)$, each being tangent to the chord and the minor arc $AB$, are also tangent to...
each other externally. Then the internal common tangent of the two circles passes through the midpoint of the major arc $AB$.

**Proof.** Let $T$ be the point of contact, and $CD$ the chord of $(O)$ which is the internal common tangent of the circles $K(P)$ and $K(Q)$. Regarding these two circles are tangent to the chord $CD$, and $AB$ as an external common tangent, we conclude that $C$ is the midpoint of the arc $AB$.

### 4.3.5

This leads to a simple construction of the two neighbors of $(K_P)$, each tangent to $(K_P)$, to the chord $AB$, and to the arc $AB$ containing $K_P$.

Given a circle $(K_P)$ tangent to $(O)$ and a chord $AB$, let $C$ be the midpoint of the arc not containing $K_P$.

1. Construct the tangents from $CT$ and $CT'$ to the circle $(K_P)$.
2. Construct the bisector of the angle between $\frac{CT}{CT'}$ and $AB$ to intersect the ray $\frac{K_P T}{K_P T'}$ at $\frac{K_Q}{K_Q'}$.

Then, $K_Q$ and $K_Q'$ are the centers of the two neighbors of $(K_P)$. $AB$, and to the arc $AB$ containing $K_P$. 
Exercise

1. Let $C$ be the midpoint of the major arc $AB$. If two neighbor circles $(K_P)$ and $(K_Q)$ are congruent, then they touch each other at a point $T$ on the diameter $CM$ such that $CT = CA$.

2. The curvilinear triangle is bounded by two circular arcs $A(B)$ and $B(A)$, and a common radius $AB$. $CD$ is parallel to $AB$, and is at a distance $b$. Denote the length of $AB$ by $a$. Calculate the radius of the inscribed circle.

3. If each side of the square has length $a$, calculate the radii of the two small circles.

4. Given a chord $AB$ of a circle $(O)$ which is not a diameter, locate the points $P$ on $AB$ such that the radius of $(K_P)$ is equal to $\frac{1}{2}(R - h)$. \(^4\)

\(^4\)Answer: $x = \pm \sqrt{2h(R - h)}$. 

\[\text{Answer: } x = \pm \sqrt{2h(R - h)}.\]
5. \(A(B)\) and \(B(A)\) are two circles each with center on the circumference of the other. Find the radius of the circle tangent to one of the circles internally, the other externally, and the line \(AB\).

6. \(A(a)\) and \(B(b)\) are two semicircles tangent internally to each other at \(O\). A circle \(K(r)\) is constructed tangent externally to \(A(a)\), internally to \(B(b)\), and to the line \(AB\) at a point \(X\). Show that

\[
BX = \frac{b(3a - b)}{a + b}, \quad \text{and} \quad r = \frac{4ab(b - a)}{(a + b)^2}.
\]

### 4.3.6

Here is an alternative for the construction of the neighbors of a circle \((K_P)\) tangent to a chord \(AB\) at \(P\), and to the circle \((O)\). Let \(M\) be the midpoint of the chord \(AB\), at a distance \(h\) from the center \(O\). At the point \(P\) on \(AB\) with \(MP = x\), the circle \(K_P(r_P)\) tangent to \(AB\) at \(P\) and to the minor arc \(AB\) has radius

\[
r_P = \frac{R^2 - h^2 - x^2}{2(R + h)}.
\]
To construct the two circles tangent to the minor arc, the chord \( AB \), and the circle \((K_P)\), we proceed as follows.

1. Let \( C \) be the midpoint of the major arc \( AB \). Complete the rectangle \( BMCD \), and mark on the line \( AB \) points \( A' \), \( B' \) such that \( A'M = MB' = MD \).

2. Let the perpendicular to \( AB \) through \( P \) intersect the circle \((O)\) at \( P_1 \) and \( P_2 \).

3. Let the circle passing through \( P_1 \), \( P_2 \), and \( A'B' \) intersect the chord \( AB \) at \( Q \), \( Q' \).

Then the circles tangent to the minor arc and to the chord \( AB \) at \( Q \) and \( Q' \) are also tangent to the circle \((K_P)\).

**Proof.** Let \((K_P)\) and \((K_Q)\) be two circles each tangent to the minor arc and the chord \( AB \), and are tangent to each other externally. If their points of contact have coordinates \( x \) and \( y \) on \( AB \) (with midpoint \( M \) as origin), then

\[
(x - y)^2 = 4r_{P}r_{Q} = \frac{(R^2 - h^2 - x^2)(R^2 - h^2 - y^2)}{(R + h)^2}.
\]

Solving this equation for \( y \) in terms of \( x \), we have

\[
y - x = \frac{R^2 - h^2 - x^2}{\sqrt{2R^2 + 2Rh}} \pm \sqrt{2R^2 + 2Rh} \pm x.
\]
Now, \( R^2 - h^2 - x^2 = AM^2 - MP^2 = AP \cdot PB = P_1P \cdot PP_2 \), and \( 2R^2 + 2Rh = (R + h)^2 + (R^2 - h^2) = MC^2 + MB^2 = MD^2 \). This justifies the above construction.

### 4.4 Mixtilinear incircles

L. Bankoff \(^6\) has coined the term *mixtilinear incircle* of a triangle for a circle tangent to two sides and the circumcircle internally. Let \( K(\rho) \) be the circle tangent to the sides \( AB, AC \), and the circumcircle at \( X_3, X_2 \), and \( A' \) respectively. If \( E \) is the midpoint of \( AC \), then Then \( KX_2 = \rho \) and \( OE = R \cos \beta \). Also, \( AX_2 = \rho \cot \frac{\alpha}{2} \), and \( AE = \frac{1}{2}b = R \sin \beta \).

![Diagram of mixtilinear incircles](image)

Since \( OK = R - \rho \), it follows that

\[
(R - \rho)^2 = (\rho - R \cos \beta)^2 + (\rho \cot \frac{\alpha}{2} - R \sin \beta)^2.
\]

Solving this equation, we obtain

\[
\rho = 2R \tan^2 \frac{\alpha}{2} \left[ \cot \frac{\alpha}{2} \sin \beta - 1 + \cos \beta \right].
\]

By writing \( \sin \beta = 2 \sin^2 \frac{\beta}{2} \cos \frac{\beta}{2} \), and \( 1 - \cos \beta = 2 \sin^2 \frac{\beta}{2} \), we have

\[
\rho = 4R \tan^2 \frac{\alpha}{2} \sin \frac{\beta}{2} \left[ \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \sin \frac{\alpha}{2} - \sin \frac{\beta}{2} \right]
\]

\[= 4R \frac{\sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\cos^2 \frac{\alpha}{2}} \left[ \frac{\cos \frac{\alpha}{2} \cos \frac{\beta}{2} - \sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{2} \right] \]
\[= 4R \frac{\sin \frac{\alpha}{2} \sin \frac{\beta}{2}}{\cos^2 \frac{\alpha}{2}} \cos \frac{\alpha + \beta}{2} \]
\[= 4R \frac{\sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}}{\cos^2 \frac{\alpha}{2}} \]
\[= \frac{r}{\cos^2 \frac{\alpha}{2}} \]

We summarize this with a slight change of notation.

4.4.1

The radius of the mixtilinear incircle in the angle \( A \) is given by

\[ \rho_1 = r \cdot \sec^2 \frac{\alpha}{2}. \]

This formula enables one to locate the mixtilinear incenter \( K_1 \) very easily. Note that the segment \( X_2X_3 \) contains the incenter \( I \) as its midpoint, and the mixtilinear incenter \( K_1 \) is the intersection of the perpendiculars to \( AB \) and \( AC \) at \( X_3 \) and \( X_2 \) respectively.

Exercise

1. In each of the following cases, the largest circle is the circumcircle of the triangle (respectively equilateral and right). The smallest circle is the incircle of the triangle, and the other circle touches two sides of the triangle and the circumcircle. Compute the ratio of the radii of the two smaller circles.
2. \(ABC\) is a right triangle for which the mixtilinear incircle \((K)\) of the right angle touches the circumcircle at a point \(P\) such that \(KP\) is parallel to a leg of the triangle. Find the ratio of the sides of the triangle.  

\[
\text{
\begin{center}
\includegraphics[width=0.3\textwidth]{triangle.png}
\end{center}
}
\]

3. \(ABC\) is an isosceles triangle with \(AB = AC = 2\) and \(BC = 3\). Show that the \(\rho_1 = 2\rho_2\).

4. \(ABC\) is an isosceles triangle with \(AB = AC\). If \(\rho_1 = k\rho_2\), show that \(k < 2\), and the sides are in the ratio \(1 : 1 : 2 - k\).  

5. The large circle has radius \(R\). The three small circles have equal radii. Find this common radius.

4.4.2

Consider also the mixtilinear incircles in the angles \(B\) and \(C\). Suppose the mixtilinear incircle in angle \(B\) touch the sides \(BC\) and \(AB\) at the points \(Y_1\)

\[7.3.4.5.\]

\[8\text{If } \rho_2 = k\rho_1, \text{ then } \tan \frac{\beta}{2} = \sqrt{\frac{k}{4-k}}, \text{ so that } \cos \beta = \frac{1 - \frac{k}{2}}{1 + \frac{k}{2}} = \frac{2-k}{2}.\]

\[9\text{Answer: } \frac{3\sqrt{5}}{4} R.\]
and $Y_3$ respectively, and that in angle $C$ touch the sides $BC$ and $AC$ at $Z_1$ and $Z_2$ respectively.

Each of the segments $X_2X_3$, $Y_3Y_1$, and $Z_1Z_2$ has the incenter $I$ as midpoint. It follows that the triangles $IY_1Z_1$ and $IY_3Z_2$ are congruent, and the segment $Y_3Z_2$ is parallel to the side $BC$ containing the segment $Y_1Z_1$, and is tangent to the incircle. Therefore, the triangles $AY_3Z_2$ and $ABC$ are similar, the ratio of similarity being

$$\frac{Y_3Z_2}{a} = \frac{h_a - 2r}{h_a},$$

with $h_a = \frac{2\Delta}{a} = \frac{2s}{a}$, the altitude of triangle $ABC$ on the side $BC$. Simplifying this, we obtain $\frac{Y_3Z_2}{a} = \frac{s-a}{s}$. From this, the inradius of the triangle $AY_3Z_2$ is given by $r_a = \frac{s-a}{s} \cdot r$. Similarly, the inradii of the triangles $BZ_1X_3$ and $CX_2BY_1$ are $r_b = \frac{s-b}{s} \cdot r$ and $r_c = \frac{s-c}{s} \cdot r$ respectively. From this, we have

$$r_a + r_b + r_c = r.$$

We summarize this in the following proposition.

**Proposition**

If tangents to the incircles of a triangle are drawn parallel to the sides, cutting out three triangles each similar to the given one, the sum of the inradii of the three triangles is equal to the inradius of the given triangle.
4.5 Mixtilinear excircles

The mixtilinear excircles are analogously defined. The mixtilinear exradius in the angle $A$ is given by

$$\rho_A = r_a \sec^2 \frac{\alpha}{2},$$

where $r_a = \frac{\triangle}{s-a}$ is the corresponding exradius.
Chapter 5

The shoemaker’s knife

5.1 The shoemaker’s knife

Let $P$ be a point on a segment $AB$. The region bounded by the three semicircles (on the same side of $AB$) with diameters $AB$, $AP$ and $PB$ is called a shoemaker’s knife. Suppose the smaller semicircles have radii $a$ and $b$ respectively. Let $Q$ be the intersection of the largest semicircle with the perpendicular through $P$ to $AB$. This perpendicular is an internal common tangent of the smaller semicircles.

Exercise

1. Show that the area of the shoemaker’s knife is $\pi ab$.

2. Let $UV$ be the external common tangent of the smaller semicircles. Show that $UPQV$ is a rectangle.

3. Show that the circle through $U$, $P$, $Q$, $V$ has the same area as the shoemaker’s knife.
5.1.1 Archimedes’ Theorem

The two circles each tangent to $CP$, the largest semicircle $AB$ and one of the smaller semicircles have equal radii $t$, given by

$$t = \frac{ab}{a + b}.$$

Proof. Consider the circle tangent to the semicircles $O(a + b)$, $O_1(a)$, and the line $PQ$. Denote by $t$ the radius of this circle. Calculating in two ways the height of the center of this circle above the line $AB$, we have

$$(a + b - t)^2 - (a - b - t)^2 = (a + t)^2 - (a - t)^2.$$

From this,

$$t = \frac{ab}{a + b}.$$

The symmetry of this expression in $a$ and $b$ means that the circle tangent to $O(a + b)$, $O_2(b)$, and $PQ$ has the same radius $t$. This proves the theorem.

5.1.2 Construction of the Archimedean circles

Let $Q_1$ and $Q_2$ be points on the semicircles $O_1(a)$ and $O_2(b)$ respectively such that $O_1Q_1$ and $O_2Q_2$ are perpendicular to $AB$. The lines $O_1Q_2$ and $O_2Q_1$ intersect at a point $C_3$ on $PQ$, and

$$C_3P = \frac{ab}{a + b}.$$

Note that $C_3P = t$, the radius of the Archimedean circles. Let $M_1$ and $M_2$ be points on $AB$ such that $PM_1 = PM_2 = C_3P$. The center $C_1$ of the
Archimedean circle $C_1(t)$ is the intersection of the circle $O_1(M_2)$ and the perpendicular through $M_1$ to $AB$. Likewise, $C_2$ is the intersection of the circle $O_2(M_1)$ and the perpendicular through $M_2$ to $AB$.

5.1.3 Incircle of the shoemaker’s knife

The circle tangent to each of the three semicircles has radius given by

$$\rho = \frac{ab(a + b)}{a^2 + ab + b^2}.$$

**Proof.** Let $\angle COO_2 = \theta$. By the cosine formula, we have

\[
\begin{align*}
(a + \rho)^2 &= (a + b - \rho)^2 + b^2 + 2b(a + b - \rho) \cos \theta, \\
(b + \rho)^2 &= (a + b - \rho)^2 + a^2 - 2a(a + b - \rho) \cos \theta.
\end{align*}
\]

Eliminating $\rho$, we have

$$a(a + \rho)^2 + b(b + \rho)^2 = (a + b)(a + b - \rho)^2 + ab^2 + ba^2.$$

The coefficients of $\rho^2$ on both sides are clearly the same. This is a linear equation in $\rho$:

$$a^3 + b^3 + 2(a^2 + b^2)\rho = (a + b)^3 + ab(a + b) - 2(a + b)^2\rho,$$
from which
\[ 4(a^2 + ab + b^2)\rho = (a + b)^3 + ab(a + b) - (a^3 + b^3) = 4ab(a + b), \]
and \( \rho \) is as above.

5.1.4 Bankoff’s Theorem

If the incircle \( C(\rho) \) of the shoemaker’s knife touches the smaller semicircles at \( X \) and \( Y \), then the circle through the points \( P, X, Y \) has the same radius as the Archimedean circles.

Proof. The circle through \( P, X, Y \) is clearly the incircle of the triangle \( CO_1O_2 \), since
\[ CX = CY = \rho, \quad O_1X = O_1P = a, \quad O_2Y = O_2P = b. \]
The semiperimeter of the triangle \( CO_1O_2 \) is
\[ a + b + \rho = (a + b) + \frac{ab(a + b)}{a^2 + ab + b^2} = \frac{(a + b)^3}{a^2 + ab + b^2}. \]
The inradius of the triangle is given by
\[ \sqrt{\frac{ab\rho}{a + b + \rho}} = \sqrt{\frac{ab \cdot ab(a + b)}{(a + b)^3}} = \frac{ab}{a + b}. \]
This is the same as \( t \), the radius of the Archimedean circles.
5.1.5 Construction of incircle of shoemaker’s knife

Locate the point $C_3$ as in §???. Construct circle $C_3(P)$ to intersect $O_1(a)$ and $O_2(b)$ at $X$ and $Y$ respectively. Let the lines $O_1X$ and $O_2Y$ intersect at $C$. Then $C(X)$ is the incircle of the shoemaker’s knife.

Note that $C_3(P)$ is the Bankoff circle, which has the same radius as the Archimedean circles.

Exercise

1. Show that the area of triangle $CO_1O_2$ is

\[ \frac{ab(a + b)^2}{a^2 + ab + b^2} \]

2. Show that the center $C$ of the incircle of the shoemaker’s knife is at a distance $2\rho$ from the line $AB$.

3. Show that the area of the shoemaker’s knife to that of the heart (bounded by semicircles $O_1(a)$, $O_2(b)$ and the lower semicircle $O(a+b)$) is as $\rho$ to $a + b$. 
4. Show that the points of contact of the incircle $C(p)$ with the semicircles can be located as follows: $Y$, $Z$ are the intersections with $Q_1(A)$, and $X$, $Z$ are the intersections with $Q_2(B)$.

5. Show that $PZ$ bisects angle $AZB$.

### 5.2 Archimedean circles in the shoemaker’s knife

Let $t = \frac{ab}{a+b}$ as before.

#### 5.2.1

Let $UV$ be the external common tangent of the semicircles $O_1(a)$ and $O_2(b)$, which extends to a chord $HK$ of the semicircle $O(a+b)$.

Let $C_4$ be the intersection of $O_1V$ and $O_2U$. Since

$$O_1U = a, \quad O_2V = b, \quad \text{and} \quad O_1P : PO_2 = a : b,$$

$C_4P = \frac{ab}{a+b} = t$. This means that the circle $C_4(t)$ passes through $P$ and touches the common tangent $HK$ of the semicircles at $N$. 
Let $M$ be the midpoint of the chord $HK$. Since $O$ and $P$ are symmetric (isotomic conjugates) with respect to $O_1O_2$,

$$OM + PN = O_1U + O_2V = a + b.$$ 

it follows that $(a + b) - QM = PN = 2t$. From this, the circle tangent to $HK$ and the minor arc $HK$ of $O(a+b)$ has radius $t$. This circle touches the minor arc at the point $Q$.

5.2.2

Let $OI'$, $O_1Q_1$, and $O_2Q_2$ be radii of the respective semicircles perpendicular to $AB$. Let the perpendiculars to $AB$ through $O$ and $P$ intersect $Q_1Q_2$ at $I$ and $J$ respectively. Then $PJ = 2t$, and since $O$ and $P$ are isotomic conjugates with respect to $O_1O_2$,

$$OI = (a + b) - 2t.$$ 

It follows that $II' = 2t$. Note that $OQ_1 = OQ_2$. Since $I$ and $J$ are isotomic conjugates with respect to $Q_1Q_2$, we have $JJ' = II' = 2t$. 
It follows that each of the circles through $I$ and $J$ tangent to the minor arc of $O(a + b)$ has the same radius $t$.  

5.2.3

The circles $C_1(t)$ and $M_2(t)$ have two internal common tangents, one of which is the line $PQ$. The second internal common tangent passes through the point $B$.  

---

1These circles are discovered by Thomas Schoch of Essen, Germany.  
2Dodge, in in *Eves’ Circle*. 
5.2.4
The external common tangent of \( P(t) \) and \( O_1(a) \) passes through \( O_2 \).

5.3 The Schoch line

5.3.1
The incircle of the curvilinear triangle bounded by the semicircle \( O(a + b) \) and the circles \( A(2a) \) and \( B(2b) \) has radius \( t = \frac{ab}{a+b} \).

\[ (2a + x)^2 + (2b + x)^2 = 2[(a + b)^2 + (a + b - x)^2]. \]

From this,
\[ x = \frac{ab}{a+b} = t. \]

5.3.2 Theorem (Schoch)

If a circle of radius \( t = \frac{ab}{a+b} \) is tangent externally to each of the semicircles \( O_1(a) \) and \( O_2(b) \), its center lies on the perpendicular to \( AB \) through \( S \).

5.3.3 Theorem (Woo)

For \( k > 0 \), consider the circular arcs through \( P \), centers on the line \( AB \) (and on opposite sides of \( P \)), radii \( kr_1 \), \( kr_2 \) respectively. If a circle of radius \( t = \frac{ab}{a+b} \) is tangent externally to both of them, then its center lies on the Schoch line, the perpendicular to \( AB \) through \( S \).
Proof. Let $A_k(ka)$ and $B_k(ka)$ be two circles tangent externally at $P$, and $S_k(t)$ the circle tangent externally to each of these. The distance from the center $S_k$ to the “vertical” line through $P$ is, by the cosine formula

$$(ka + t) \cos \angle S_k A_k P - 2a$$

$$= \frac{(ka + t)^2 + k^2(a + b)^2 - (kb + t)^2}{2k(a + b)} - ka$$

$$= \frac{2k(a - b)t + k^2(a^2 - b^2) + k^2(a + b)^2 - 2k^2a(a + b)}{2k(a + b)}$$

$$= \frac{a - b}{a + b}t.$$  

Remark

For $k = 2$, this is the circle in the preceding proposition. It happens to be tangent to $O(a + b)$ as well, internally.

5.3.4 Proposition

The circle $S_k(t)$ tangent externally to the semicircle $O(a + b)$ touches the latter at $Q$.

Proof. If the ray $OQ$ is extended to meet the Schoch line at a point $W$, then

$$\frac{QW}{OQ} = \frac{PK}{OP}.$$
and
\[ QW = \frac{OQ}{OP} \cdot PK = \frac{a+b}{a-b} \cdot \frac{a-b}{a+b} t = t. \]

**Exercise**

1. The height of the center \( S_k \) above \( AB \) is
\[ \frac{2ab}{(a+b)^2} \sqrt{k(a+b)^2 + ab}. \]

2. Find the value of \( k \) for the circle in Proposition ??.

**5.3.5**

Consider the semicircle \( M(\frac{a+b}{2}) \) with \( O_1O_2 \) as a diameter. Let \( \frac{S'}{S''} \) be the intersection of Schoch line with the semicircle \( O(a+b), (M) \), and \( T \) the intersection of \( (M) \) with the radius \( OS' \).

**Exercise**

1. Show that \( PT \) is perpendicular to \( AB \).

2. Show that \( S'' \) is \( \sqrt{(a+x)(b-x)} \) above \( AB \), and \( PS'' = 2t \).

---

\(^3\text{Answer: } k = \frac{a^2 + 4ab + b^2}{ab}.\)
3. $S''$ is the point $S_k$ for $k = \frac{3}{4}$.

4. $S'$ is $\sqrt{(2a + x)(2b - x)}$ above $AB$, and

$$MS' = \frac{a^2 + 6ab + b^2}{2(a + b)}.$$

From this, deduce that $TS' = 2t$.

5. $S'$ is the point $S_k$ for $k = \frac{2a^2 + 11ab + 2b^2}{4ab}$.

6. $PTS'S''$ is a parallelogram.
Chapter 6

The Use of Complex Numbers

6.1 Review on complex numbers

A complex number $z = x + yi$ has a real part $x$ and an imaginary part $y$. The conjugate of $z$ is the complex number $\bar{z} = x - yi$. The norm is the nonnegative number $|z|$ given by $|z|^2 = x^2 + y^2$. Note that

$$|z|^2 = z\bar{z} = \bar{z}z.$$

$z$ is called a unit complex number if $|z| = 1$. Note that $|z| = 1$ if and only if $\bar{z} = \frac{1}{z}$.

Identifying the complex number $z = x + yi$ with the point $(x, y)$ in the plane, we note that $|z_1 - z_2|$ measures the distance between $z_1$ and $z_2$. In particular, $|z|$ is the distance between $|z|$ and the origin 0. Note also that $\bar{z}$ is the mirror image of $z$ in the horizontal axis.

6.1.1 Multiplicative property of norm

For any complex numbers $z_1$ and $z_2$, $|z_1z_2| = |z_1||z_2|$.

6.1.2 Polar form

Each complex number $z$ can be expressed in the form $z = |z|(\cos \theta + i \sin \theta)$, where $\theta$ is unique up to a multiple of $2\pi$, and is called the argument of $z$. 
6.1.3 De Moivre Theorem

\[(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) = \cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)\].

In particular,

\[(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta\].

6.2 Coordinatization

Given \(\triangle ABC\), we set up a coordinate system such that the circumcenter \(O\) corresponds to the complex number 0, and the vertices \(A, B, C\) correspond to unit complex numbers \(z_1, z_2, z_3\) respectively. In this way, the circumradius \(R\) is equal to 1.

![Diagram of \(\triangle ABC\) with complex number coordinates]

Exercise

1. The centroid \(G\) has coordinates \(\frac{1}{3}(z_1 + z_2 + z_3)\).

2. The orthocenter \(H\) has coordinates \(z_1 + z_2 + z_3\).

3. The nine-point center \(F\) has coordinates \(\frac{1}{2}(z_1 + z_2 + z_3)\).

4. Let \(X, Y, Z\) be the midpoints of the minor arcs \(BC, CA, AB\) of the circumcircle of \(\triangle ABC\) respectively. Prove that \(AX\) is perpendicular to \(YZ\). [Hint: Consider the tangents at \(Y\) and \(Z\). Show that these are parallel to \(AC\) and \(AB\) respectively.] Deduce that the orthocenter of \(\triangle XYZ\) is the incenter \(I\) of \(\triangle ABC\).
6.2.1 The incenter

Now, we try to identify the coordinate of the incenter $I$. This, according to the preceding exercise, is the orthocenter of $\triangle XYZ$.

It is possible to choose *unit* complex numbers $t_1$, $t_2$, $t_3$ such that

$$z_1 = t_1^2, \quad z_2 = t_2^2, \quad z_3 = t_3^2,$$

and $X$, $Y$, $Z$ are respectively the points $-t_2t_3$, $-t_3t_1$ and $-t_1t_2$. From these, the incenter $I$, being the orthocenter of $\triangle XYZ$, is the point $-(t_2t_3 + t_3t_1 + t_1t_2) = -t_1t_2t_3(t_1 + t_2 + t_3)$.

**Exercise**

1. Show that the excenters are the points

$$I_A = t_1t_2t_3(-t_1 + t_2 + t_3),$$
$$I_B = t_1t_2t_3(t_1 - t_2 + t_3),$$
$$I_C = t_1t_2t_3(t_1 + t_2 - t_3).$$

6.3 The Feuerbach Theorem

The nine-point circle of a triangle is tangent internally to the incircle, and externally to each of the excircles.

**Proof.** Note that the distance between the incenter $I$ and the nine-point center $F$ is

$$IF = \left| \frac{1}{2}(t_1^2 + t_2^2 + t_3^2) + (t_1t_2 + t_2t_3 + t_3t_1) \right|$$

$$= \left| \frac{1}{2}(t_1 + t_2 + t_3)^2 \right|$$

$$= \frac{1}{2}|t_1 + t_2 + t_3|^2.$$

Since the circumradius $R = 1$, the radius of the nine-point circle is $\frac{1}{2}$. We apply Theorem 3.5.1 to calculate the inradius $r$: 
Feuerbach Theorem.

\[ r = \frac{1}{2} (1 - OI^2) \]
\[ = \frac{1}{2} \left( 1 - |t_1 t_2 t_3(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3})|^2 \right) \]
\[ = \frac{1}{2} \left( 1 - |t_1 + t_2 + t_3|^2 \right) \]
\[ = \frac{1}{2} - IF. \]

This means that \( IF \) is equal to the difference between the radii of the nine-point circle and the incircle. These two circles are therefore tangent internally.
Exercise

Complete the proof of the Feuerbach theorem.

1. \( I_A F = \frac{1}{2} | - t_1 + t_2 + t_3 |^2 \).
2. If \( d_A \) is the distance from \( O \) to \( I_A \), then \( d_A = | - t_1 + t_2 + t_3 | \).
3. The exradius \( r_A = I_A F - \frac{1}{2} \).

6.3.1 The Feuerbach point

Indeed, the three lines each joining the point of contact of the nine-point with an excircle to the opposite vertex of the triangle are concurrent.

Exercise

1. Let \( D \) be the midpoint of the side \( BC \) of triangle \( ABC \). Show that one of the common tangents of the circles \( I(N) \) and \( D(N) \) is parallel to \( BC \).

2. The nine-point circle is tangent to the circumcircle if and only if the triangle is right.

3. More generally, the nine-point circle intersects the circumcircle only if one of \( \alpha, \beta, \gamma \geq \frac{\pi}{2} \). In that case, they intersect at an angle \( \arccos(1 + 2 \cos \alpha \cos \beta \cos \gamma) \).
6.4

The shape and orientation of a triangle with vertices $z_1$, $z_2$, $z_3$ is determined by the ratio

$$\frac{z_3 - z_1}{z_2 - z_1}.$$

6.4.1

Two triangles with vertices $(z_1, z_2, z_3)$ and $(w_1, w_2, w_3)$ are similar with the same orientation if and only if

$$\frac{z_3 - z_1}{z_2 - z_1} = \frac{w_3 - w_1}{w_2 - w_1}.$$

Equivalently,

$$\det \begin{pmatrix} z_1 & w_1 & 1 \\ z_2 & w_2 & 1 \\ z_3 & w_3 & 1 \end{pmatrix} = 0.$$

Exercise

1. Three distinct points $z_1$, $z_2$, $z_3$ are collinear if and only if $\frac{z_3 - z_1}{z_2 - z_1}$ is a real number.

2. Three distinct points $z_1$, $z_2$, $z_3$ are collinear if and only if

$$\det \begin{pmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{pmatrix} = 0.$$

3. The equation of the line joining two distinct points $z_1$ and $z_2$ is

$$\overline{z} = Az + B,$$

where

$$A = \frac{\overline{z_1} - \overline{z_2}}{z_1 - z_2}, \quad B = \frac{z_1\overline{z_2} - z_2\overline{z_1}}{z_1 - z_2}.$$

4. Show that if $\overline{z} = Az + B$ represents a line of slope $\lambda$, then $A$ is a unit complex number, and $\lambda = -\frac{1}{1+\overline{A}}$.

5. The mirror image of a point $z$ in the line $\overline{z} = Az + B$ is the point $\overline{A\overline{z} + B}$. 
6.4.2
Let $\omega$ denote a complex cube root of unity:
\[
\omega = \frac{1}{2}(-1 + \sqrt{3}i).
\]
This is a root of the quadratic equation $x^2 + x = 1 = 0$, the other root being
\[
\omega^2 = \overline{\omega} = \frac{1}{2}(-1 - \sqrt{3}i).
\]
Note that $1, \omega, \omega^2$ are the vertices of an equilateral triangle (with counter - clockwise orientation).

6.4.3
$z_1, z_2, z_3$ are the vertices of an equilateral triangle (with counter clockwise orientation) if and only if
\[
z_1 + \omega z_2 + \omega^2 z_3 = 0.
\]

Exercise

1. If $u$ and $v$ are two vertices of an equilateral triangle, find the third vertex.  

2. If $z_1, z_2, z_3$ and $w_1, w_2, w_3$ are the vertices of equilateral triangles (with counter clockwise orientation), then so are the midpoints of the segments $z_1w_1, z_2w_2, and z_3w_3$.

3. If $z_1, z_2$ are two adjacent vertices of a square, find the coordinates of the remaining two vertices, and of the center of the square.

4. On the three sides of triangle $ABC$, construct outward squares. Let $A', B', C'$ be the centers of the squares on $BC, CA, AB$ respectively, show that $AA'$ is perpendicular to, and has the same length as $B'C'$.

5. $OAB, OCD, DAX, and BCY$ are equilateral triangles with the same orientation. Show that the latter two have the same center. 

\[^1\text{If } uvw \text{ is an equilateral triangle with counterclockwise orientation, } w = -\omega w - \omega^2 v = -\omega u + (1 + \omega)v. \text{ If it has clockwise orientation, } w = (1 + \omega)u - \omega v.\]

\[^2\text{More generally, if } OAB \text{ (counterclockwise) and } OCD \text{ (clockwise) are similar triangles. The triangles } CAX \text{ (counterclockwise) and } DYB \text{ (clockwise), both similar to the first triangle, have the same circumcenter. (J.Dou, AMME 2866, 2974).}\]
6.4.4 Napoleon’s Theorem

If on each side of a given triangle, equilateral triangles are drawn, either all outside or all inside the triangle, the centers of these equilateral triangles form an equilateral triangle.

Proof. Let \(\omega\) be a complex cube root of unity, so that the third vertex of an equilateral triangle on \(z_1z_2\) is \(z'_3 := -(\omega z_1 + \omega^2 z_2)\). The center of this
equilateral triangle is
\[ w_3 = \frac{1}{3}(1 - \omega)z_1 + (1 - \omega^2)z_2 = \frac{1 - \omega}{3}[z_1 + (1 + \omega)z_2] = \frac{1 - \omega}{3}[z_1 - \omega^2 z_2]. \]

Likewise, the centers of the other two similarly oriented equilateral triangles are
\[ w_1 = \frac{1 - \omega}{3}[z_3 - \omega^2 z_1], \quad w_2 = \frac{1 - \omega}{3}[z_2 - \omega^2 z_3]. \]

These form an equilateral triangle since
\[
\begin{align*}
& w_1 + \omega w_2 + \omega^2 w_3 \\
& = \frac{1 - \omega}{3}[z_3 + \omega z_2 + \omega^2 z_1 - \omega^2(z_1 + \omega z_3 + \omega^2 z_2)] \\
& = 0.
\end{align*}
\]

**Exercise**

1. (Fukuta’s generalization of Napoleon’s Theorem) Given triangle ABC,
   \[ X_1 \]
   let \( Y_1 \) be points dividing the sides \( CA \) in the same ratio \( 1 - k : k \). Denote by \( Y_2 \) their isotomic conjugate on the respective sides. Complete the following equilateral triangles, all with the same orientation,
   \[ X_1X_2X_3, \; Y_1Y_2Y_3, \; Z_1Z_2Z_3, \; Y_2Z_1X_3', \; Z_2X_1Y_3', \; X_2Y_1Z_3'. \]

   (a) Show that the segments \( X_3X_3', \; Y_3Y_3' \) and \( Z_3Z_3' \) have equal lengths, \( 60^\circ \) angles with each other, and are concurrent.

   (b) Consider the hexagon \( X_3Z_3'Y_3X_3'Z_3Y_3' \). Show that the centroids of the 6 triangles formed by three consecutive vertices of this hexagon are themselves the vertices of a regular hexagon, whose center is the centroid of triangle \( ABC \).

---

\(^3\text{Mathematics Magazine, Problem 1493.}\)
6.5 Concyclic points

Four non-collinear points $z_1, z_2, z_3, z_4$ are concyclic if and only if the cross ratio
\[
(z_1, z_2; z_3, z_4) := \frac{z_4 - z_1}{z_3 - z_1} \frac{z_4 - z_2}{z_3 - z_2} = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}
\]
is a real number.
Proof. Suppose \( z_1 \) and \( z_2 \) are on the same side of \( z_3z_4 \). The four points are concyclic if the counter clockwise angles of rotation from \( z_1z_3 \) to \( z_1z_4 \) and from \( z_2z_3 \) to \( z_2z_4 \) are equal. In this case, the ratio

\[
\frac{z_4 - z_1}{z_3 - z_1} / \frac{z_4 - z_2}{z_3 - z_2}
\]

of the complex numbers is real, (and indeed positive).

On the other hand, if \( z_1, z_2 \) are on opposite sides of \( z_3z_4 \), the two angles differ by \( \pi \), and the cross ratio is a negative real number.

### 6.6 Construction of the regular 17-gon

#### 6.6.1 Gauss’ analysis

Suppose a regular 17-gon has center 0 \( \in \mathbb{C} \) and one vertex represented by the complex number 1. Then the remaining 16 vertices are the roots of the equation

\[
\frac{x^{17} - 1}{x - 1} = x^{16} + x^{15} + \cdots + x + 1 = 0.
\]

If \( \omega \) is one of these 16 roots, then these 16 roots are precisely \( \omega, \omega^2, \ldots, \omega^{15}, \omega^{16} \).

(Note that \( \omega^{17} = 1 \).) Geometrically, if \( A_0, A_1 \) are two distinct vertices of a regular 17-gon, then successively marking vertices \( A_2, A_3, \ldots, A_{16} \) with

\[
A_0A_1 = A_1A_2 = \ldots = A_{14}A_{15} = A_{15}A_{16},
\]

we obtain all 17 vertices. If we write \( \omega = \cos \theta + i \sin \theta \), then \( \omega + \omega^{16} = 2 \cos \theta \). It follows that the regular 17-gon can be constructed if one can construct the number \( \omega + \omega^{16} \). Gauss observed that the 16 complex numbers \( \omega^k, k = 1, 2, \ldots, 16 \), can be separated into two “groups” of eight, each with a sum constructible using only ruler and compass. This is decisively the hardest step. But once this is done, two more applications of the same idea eventually isolate \( \omega + \omega^{16} \) as a constructible number, thereby completing the task of construction. The key idea involves the very simple fact that if the coefficients \( a \) and \( b \) of a quadratic equation \( x^2 - ax + b = 0 \) are constructible, then so are its roots \( x_1 \) and \( x_2 \). Note that \( x_1 + x_2 = a \) and \( x_1x_2 = b \).

Gauss observed that, modulo 17, the first 16 powers of 3 form a permutation of the numbers 1, 2, \ldots, 16:

\[
\begin{align*}
k & \quad 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\
3^k & \quad 1 & 3 & 9 & 10 & 13 & 5 & 15 & 11 & 16 & 14 & 8 & 7 & 4 & 12 & 2 & 6
\end{align*}
\]
Let

\begin{align*}
y_1 &= \omega + \omega^9 + \omega^{13} + \omega^{15} + \omega^{16} + \omega^8 + \omega^4 + \omega^2, \\
y_2 &= \omega^3 + \omega^{10} + \omega^5 + \omega^{11} + \omega^{14} + \omega^7 + \omega^{12} + \omega^6.
\end{align*}

Note that

\[ y_1 + y_2 = \omega + \omega^2 + \cdots + \omega^{16} = -1. \]

Most crucial, however, is the fact that the product \( y_1y_2 \) does not depend on the choice of \( \omega \). We multiply these directly, but adopt a convenient bookkeeping below. Below each power \( \omega^k \), we enter a number \( j \) (from 1 to 8 meaning that \( \omega^k \) can be obtained by multiplying the \( j \)th term of \( y_1 \) by an appropriate term of \( y_2 \) (unspecified in the table but easy to determine):

\begin{center}
\begin{tabular}{cccccccccccccccc}
\( \omega \) & \( \omega^2 \) & \( \omega^3 \) & \( \omega^4 \) & \( \omega^5 \) & \( \omega^6 \) & \( \omega^7 \) & \( \omega^8 \) & \( \omega^9 \) & \( \omega^{10} \) & \( \omega^{11} \) & \( \omega^{12} \) & \( \omega^{13} \) & \( \omega^{14} \) & \( \omega^{15} \) & \( \omega^{16} \) \\
3 & 2 & 2 & 1 & 4 & 1 & 1 & 1 & 4 & 3 & 1 & 1 & 1 & 2 & 1 & 2 \\
4 & 3 & 3 & 2 & 5 & 2 & 3 & 3 & 5 & 4 & 5 & 2 & 5 & 6 & 2 & 3 \\
6 & 5 & 4 & 4 & 6 & 3 & 7 & 4 & 7 & 5 & 6 & 4 & 6 & 7 & 6 & 7 \\
7 & 6 & 6 & 5 & 8 & 5 & 8 & 8 & 8 & 7 & 7 & 8 & 8 & 8 & 7 & 8
\end{tabular}
\end{center}

From this it is clear that

\[ y_1y_2 = 4(\omega + \omega^2 + \cdots + \omega^{16}) = -4. \]

It follows that \( y_1 \) and \( y_2 \) are the roots of the quadratic equation

\[ y^2 + y - 4 = 0, \]

and are constructible. We may take

\[ y_1 = \frac{-1 + \sqrt{17}}{2}, \quad y_2 = \frac{-1 - \sqrt{17}}{2}. \]

Now separate the terms of \( y_1 \) into two “groups” of four, namely,

\[ z_1 = \omega + \omega^{13} + \omega^{16} + \omega^4, \quad z_2 = \omega^9 + \omega^{15} + \omega^8 + \omega^2. \]

Clearly, \( z_1 + z_2 = y_1 \). Also,

\[ z_1z_2 = (\omega + \omega^{13} + \omega^{16} + \omega^4)(\omega^9 + \omega^{15} + \omega^8 + \omega^2) = \omega + \omega^2 + \cdots + \omega^{16} = -1. \]

It follows that \( z_1 \) and \( z_2 \) are the roots of the quadratic equation

\[ z^2 - y_1z - 1 = 0, \]
and are constructible, since \( y_1 \) is constructible. Similarly, if we write
\[
\begin{align*}
z_3 &= \omega^3 + \omega^5 + \omega^{14} + \omega^{12}, \\
z_4 &= \omega^{10} + \omega^{11} + \omega^7 + \omega^6,
\end{align*}
\]
we find that \( z_3 + z_4 = y_2 \), and \( z_3z_4 = \omega + \omega^2 + \cdots + \omega^{16} = -1 \), so that \( z_3 \) and \( z_4 \) are the roots of the quadratic equation
\[
z^2 - y_2z - 1 = 0
\]
and are also constructible.

Finally, further separating the terms of \( z_1 \) into two pairs, by putting
\[
\begin{align*}
t_1 &= \omega + \omega^{16}, \\
t_2 &= \omega^{13} + \omega^4,
\end{align*}
\]
we obtain
\[
\begin{align*}
t_1 + t_2 &= z_1, \\
t_1t_2 &= (\omega + \omega^{16})(\omega^{13} + \omega^4) = \omega^{14} + \omega^5 + \omega^{12} + \omega^3 = z_3.
\end{align*}
\]
It follows that \( t_1 \) and \( t_2 \) are the roots of the quadratic equation
\[
t^2 - z_1t + z_3 = 0,
\]
and are constructible, since \( z_1 \) and \( z_3 \) are constructible.

### 6.6.2 Explicit construction of a regular 17-gon

To construct two vertices of the regular 17-gon inscribed in a given circle \( O(A) \).

1. On the radius \( OB \) perpendicular to \( OA \), mark a point \( J \) such that \( OJ = \frac{1}{4}OA \).

2. Mark a point \( E \) on the segment \( OA \) such that \( \angle OJE = \frac{1}{5}\angle OJA \).

3. Mark a point \( F \) on the diameter through \( A \) such that \( O \) is between \( E \) and \( F \) and \( \angle EJF = 45^\circ \).

4. With \( AF \) as diameter, construct a circle intersecting the radius \( OB \) at \( K \).

---

5. Mark the intersections of the circle $E(K)$ with the diameter of $O(A)$ through $A$. Label the one between $O$ and $A$ points $P_4$, and the other and $P_6$.

6. Construct the perpendicular through $P_4$ and $P_6$ to intersect the circle $O(A)$ at $A_4$ and $A_6$.  

Then $A_4, A_6$ are two vertices of a regular 17-gon inscribed in $O(A)$. The polygon can be completed by successively laying off arcs equal to $A_4A_6$, leading to $A_8, A_{10}, \ldots A_{16}, A_1 = A, A_3, A_5, \ldots, A_{15}, A_{17}, A_2$.

\[^5\text{Note that $P_4$ is not the midpoint of $AF$.}\]
Chapter 7

The Menelaus and Ceva Theorems

7.1

7.1.1 Sign convention

Let $A$ and $B$ be two distinct points. A point $P$ on the line $AB$ is said to divide the segment $AB$ in the ratio $AP : PB$, positive if $P$ is between $A$ and $B$, and negative if $P$ is outside the segment $AB$.

7.1.2 Harmonic conjugates

Two points $P$ and $Q$ on a line $AB$ are said to divide the segment $AB$ harmonically if they divide the segment in the same ratio, one externally and the other internally:

\[ \frac{AP}{PB} = -\frac{AQ}{QB} \]

We shall also say that $P$ and $Q$ are harmonic conjugates with respect to the segment $AB$. 
7.1.3
Let $P$ and $Q$ be harmonic conjugates with respect to $AB$. If $AB = d$, $AP = p$, and $AQ = q$, then $d$ is the harmonic mean of $p$ and $q$, namely,

$$\frac{1}{p} + \frac{1}{q} = \frac{2}{d}.$$

Proof. This follows from

$$\frac{p}{d - p} = -\frac{q}{d - q}.$$

7.1.4
We shall use the abbreviation $(A, B; P, Q)$ to stand for the statement $P, Q$ divide the segment $AB$ harmonically.

Proposition
If $(A, B; P, Q)$, then $(A, B; Q, P)$, $(B, A; P, Q)$, and $(P, Q; A, B)$.

Therefore, we can speak of two collinear (undirected) segments dividing each other harmonically.

Exercise
1. Justify the following construction of harmonic conjugate.

Given $AB$, construct a right triangle $ABC$ with a right angle at $B$ and $BC = AB$. Let $M$ be the midpoint of $BC$. 
For every point $P$ (except the midpoint of $AB$), let $P'$ be the point on $AC$ such that $PP' \perp AB$.

The intersection $Q$ of the lines $P'M$ and $AB$ is the harmonic conjugate of $P$ with respect to $AB$.

### 7.2 Apollonius Circle

#### 7.2.1 Angle bisector Theorem

If the internal (respectively external) bisector of angle $BAC$ intersect the line $BC$ at $X$ (respectively $X'$), then

$$BX : XC = c : b; \quad BX' : X'C = c : -b.$$  

#### 7.2.2 Example

The points $X$ and $X'$ are harmonic conjugates with respect to $BC$, since

$$BX : XC = c : b, \quad \text{and} \quad BX' : X'C = c : -b.$$  

#### 7.2.3

$A$ and $B$ are two fixed points. For a given positive number $k \neq 1$, \footnote{If $k = 1$, the locus is clearly the perpendicular bisector of the segment $AB$.} the locus of points $P$ satisfying $AP : PB = k : 1$ is the circle with diameter $XY$, where $X$ and $Y$ are points on the line $AB$ such that $AX : XB = k : 1$ and $AY : YB = k : -1$. 

---

\[\text{Diagram} \quad BX : XC = c : b. \quad BX : XC = c : -b.\]
Proof. Since \( k \neq 1 \), points \( X \) and \( Y \) can be found on the line \( AB \) satisfying the above conditions.

Consider a point \( P \) not on the line \( AB \) with \( AP : PB = k : 1 \). Note that \( PX \) and \( PY \) are respectively the internal and external bisectors of angle \( APB \). This means that angle \( XPY \) is a right angle.

Exercise

1. The bisectors of the angles intersect the sides \( BC, CA, AB \) respectively at \( P, Q, \) and \( R \). \( P', Q', \) and \( R' \) on the sides \( CA, AB, \) and \( BC \) respectively such that \( PP'//BC, QQ'//CA, \) and \( RR'//AB \). Show that

\[
\frac{1}{PP'} + \frac{1}{QQ'} + \frac{1}{RR'} = 2\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).
\]

2. Suppose \( ABC \) is a triangle with \( AB \neq AC \), and let \( D, E, F, G \) be points on the line \( BC \) defined as follows: \( D \) is the midpoint of \( BC \), \( AE \) is the bisector of \( \angle BAC \), \( F \) is the foot of the perpendicular from
A to $BC$, and $AG$ is perpendicular to $AE$ (i.e. $AG$ bisects one of the exterior angles at $A$). Prove that $AB \cdot AC = DF \cdot EG$.

3. If $AB = d$, and $k \neq 1$, the radius of the Apollonius circle is $\frac{k}{k^2 - 1}d$.

4. Given two disjoint circles $(A)$ and $(B)$, find the locus of the point $P$ such that the angle between the pair of tangents from $P$ to $(A)$ and that between the pair of tangents from $P$ to $(B)$ are equal. \(^2\)

### 7.3 The Menelaus Theorem

Let $X, Y, Z$ be points on the lines $BC, CA, AB$ respectively. The points $X, Y, Z$ are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = -1.$$

\(^2\)Let $a$ and $b$ be the radii of the circles. Suppose each of these angles is $2\theta$. Then $\frac{a}{AP} = \sin \theta = \frac{b}{BP}$, and $AP : BP = a : b$. From this, it is clear that the locus of $P$ is the circle with the segment joining the centers of similitude of $(A)$ and $(B)$ as diameter.
Proof. \((\Rightarrow)\) Let \(W\) be the point on \(AB\) such that \(CW//XY\). Then,
\[
\frac{BX}{XC} = \frac{BZ}{ZW}, \quad \text{and} \quad \frac{CY}{YA} = \frac{WZ}{ZA}.
\]
It follows that
\[
\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{BZ}{ZW} \cdot \frac{WZ}{ZA} = (−1)(−1)(−1) = −1.
\]
\((\Leftarrow)\) Suppose the line joining \(X\) and \(Z\) intersects \(AC\) at \(Y'\). From above,
\[
\frac{BX}{XC} \cdot \frac{CY'}{Y'A} \cdot \frac{AZ}{ZB} = −1 = \frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB}
\]
It follows that
\[
\frac{CY'}{Y'A} = \frac{CY}{YA}.
\]
The points \(Y'\) and \(Y\) divide the segment \(CA\) in the same ratio. These must be the same point, and \(X, Y, Z\) are collinear.

Exercise

1. \(M\) is a point on the median \(AD\) of \(\triangle ABC\) such that \(AM : MD = p : q\).
   The line \(CM\) intersects the side \(AB\) at \(N\). Find the ratio \(AN : NB\). \(^3\)

2. The incircle of \(\triangle ABC\) touches the sides \(BC, CA, AB\) at \(D, E, F\) respectively. Suppose \(AB \neq AC\). The line joining \(E\) and \(F\) meets \(BC\) at \(P\). Show that \(P\) and \(D\) divide \(BC\) harmonically.

\(^3\)Answer: \(AN : NB = p : 2q\).
3. The incircle of $\triangle ABC$ touches the sides $BC$, $CA$, $AB$ at $D$, $E$, $F$ respectively. $X$ is a point inside $\triangle ABC$ such that the incircle of $\triangle XBC$ touches $BC$ at $D$ also, and touches $CX$ and $XB$ at $Y$ and $Z$ respectively. Show that $E$, $F$, $Z$, $Y$ are concyclic.  

4. Given a triangle $ABC$, let the incircle and the ex-circle on $BC$ touch the side $BC$ at $X$ and $X'$ respectively, and the line $AC$ at $Y$ and $Y'$ respectively. Then the lines $XY$ and $X'Y'$ intersect on the bisector of angle $A$, at the projection of $B$ on this bisector.

7.4 The Ceva Theorem

Let $X$, $Y$, $Z$ be points on the lines $BC$, $CA$, $AB$ respectively. The lines $AX$, $BY$, $CZ$ are concurrent if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = +1.$$  

Proof. ($\implies$) Suppose the lines $AX$, $BY$, $CZ$ intersect at a point $P$. Consider the line $BPY$ cutting the sides of $\triangle CAX$. By Menelaus’ theorem,

$$\frac{CY}{YA} \cdot \frac{AP}{PX} \cdot \frac{XB}{BC} = -1, \quad \text{or} \quad \frac{CY}{YA} \cdot \frac{PA}{XP} \cdot \frac{BX}{BC} = +1.$$  

\footnote{IMO 1996.}
Also, consider the line $CPZ$ cutting the sides of $\triangle ABX$. By Menelaus’ theorem again,

\[
\frac{AZ}{ZB} \cdot \frac{BC}{CX} \cdot \frac{XP}{PA} = -1, \quad \text{or} \quad \frac{AZ}{ZB} \cdot \frac{BC}{XC} \cdot \frac{XP}{PA} = +1.
\]

Multiplying the two equations together, we have

\[
\frac{CY}{YA} \cdot \frac{AZ}{ZB} \cdot \frac{BX}{XC} = +1.
\]

($\iff$) Exercise.

### 7.5 Examples

#### 7.5.1 The centroid

If $D$, $E$, $F$ are the midpoints of the sides $BC$, $CA$, $AB$ of $\triangle ABC$, then clearly

\[
\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.
\]

The medians $AD$, $BE$, $CF$ are therefore concurrent (at the centroid $G$ of the triangle).

Consider the line $BGE$ intersecting the sides of $\triangle ADC$. By the Menelaus theorem,

\[
-1 = \frac{AG}{GD} \cdot \frac{DB}{BC} \cdot \frac{CE}{EA} = \frac{AG}{GD} \cdot \frac{-1}{2} \cdot \frac{1}{1}.
\]

It follows that $AG : GD = 2 : 1$. The centroid of a triangle divides each median in the ratio 2:1.
7.5.2 The incenter

Let $X$, $Y$, $Z$ be points on $BC$, $CA$, $AB$ such that

\[
AX \quad \angle BAC \\
BY \quad \text{bisects} \quad \angle CBA, \\
CZ \quad \angle ACB
\]

then

\[
\frac{AZ}{ZB} = \frac{b}{a}, \quad \frac{BX}{XC} = \frac{c}{b}, \quad \frac{CY}{YA} = \frac{a}{c}.
\]

It follows that

\[
\frac{AZ \cdot BX \cdot CY}{ZB \cdot XC \cdot YA} = \frac{b \cdot c \cdot a}{a \cdot b \cdot c} = +1,
\]

and $AX$, $BY$, $CZ$ are concurrent, at the incenter $I$ of the triangle.

Exercise

1. Use the Ceva theorem to justify the existence of the excenters of a triangle.

2. Let $AX$, $BY$, $CZ$ be cevians of $\triangle ABC$ intersecting at a point $P$.
   (i) Show that if $AX$ bisects angle $A$ and $BX \cdot CY = XC \cdot BZ$, then $\triangle ABC$ is isosceles.
   (ii) Show if $AX$, $BY$, $CZ$ are bisectors and $BP \cdot ZP = BZ \cdot AP$, then $\triangle ABC$ is a right triangle.

3. Suppose three cevians, each through a vertex of a triangle, trisect each other. Show that these are the medians of the triangle.

4. $ABC$ is a right triangle. Show that the lines $AP$, $BQ$, and $CR$ are concurrent.
5. If three equal cevians divide the sides of a triangle in the same ratio and the same sense, the triangle must be equilateral.

6. Suppose the bisector of angle $A$, the median on the side $b$, and the altitude on the side $c$ are concurrent. Show that

$$\cos \alpha = \frac{c}{b + c}.$$

7. Given triangle $ABC$, construct points $A'$, $B'$, $C'$ such that $ABC'$, $BCA'$ and $CAB'$ are isosceles triangles satisfying

$$\angle BCA' = \angle CBA' = \alpha, \quad \angle CAB' = \angle ACB' = \beta, \quad \angle ABC' = \angle BAC' = \gamma.$$

Show that $AA'$, $BB'$, and $CC'$ are concurrent.

### 7.6 Trigonometric version of the Ceva Theorem

#### 7.6.1

Let $X$ be a point on the side $BC$ of triangle $ABC$ such that the directed angles $\angle BAX = \alpha_1$ and $\angle XAC = \alpha_2$. Then

$$\frac{BX}{XC} = \frac{BX/AX}{XC/AX} = \frac{\sin \alpha_1/\sin \beta}{\sin \alpha_2/\sin \gamma} = \frac{\sin \gamma \cdot \sin \alpha_1}{\sin \beta \cdot \sin \alpha_2} = \frac{c \cdot \sin \alpha_1}{b \cdot \sin \alpha_2}.$$

**Proof.** By the sine formula,

$$\frac{BX}{XC} = \frac{BX/AX}{XC/AX} = \frac{\sin \alpha_1/\sin \beta}{\sin \alpha_2/\sin \gamma} = \frac{\sin \gamma \cdot \sin \alpha_1}{\sin \beta \cdot \sin \alpha_2} = \frac{c \cdot \sin \alpha_1}{b \cdot \sin \alpha_2}.$$
7.6.2

Let \( X, Y, Z \) be points on the lines \( BC, CA, AB \) respectively. The lines \( AX, BY, CZ \) are concurrent if and only if

\[
\frac{\sin \alpha_1}{\sin \alpha_2} \cdot \frac{\sin \beta_1}{\sin \beta_2} \cdot \frac{\sin \gamma_1}{\sin \gamma_2} = +1.
\]

**Proof.** Analogous to

\[
\frac{BX}{XC} = \frac{c}{b} \cdot \frac{\sin \alpha_1}{\sin \alpha_2}
\]

are

\[
\frac{CY}{YA} = \frac{a}{c} \cdot \frac{\sin \beta_1}{\sin \beta_2}, \quad \frac{AZ}{ZB} = \frac{b}{a} \cdot \frac{\sin \gamma_1}{\sin \gamma_2}.
\]

Multiplying the three equations together,

\[
\frac{AZ}{ZB} \cdot \frac{BX}{XC} \cdot \frac{CY}{YA} = \frac{\sin \alpha_1}{\sin \alpha_2} \cdot \frac{\sin \beta_1}{\sin \beta_2} \cdot \frac{\sin \gamma_1}{\sin \gamma_2}.
\]

**Exercise**

1. Show that the three altitudes of a triangle are concurrent (at the orthocenter \( H \) of the triangle).

2. Let \( A', B', C' \) be points outside \( \triangle ABC \) such that \( A'BC, B'CA \) and \( C'AB \) are similar isosceles triangles. Show that \( AA', BB', CC' \) are concurrent. 

\[8\] Solution. Let \( X \) be the intersection of \( AA' \) and \( BC \). Then

\[
\frac{BX}{XC} = \frac{\sin(\beta + \omega)}{\sin(\gamma + \omega)} \cdot \frac{\sin \gamma}{\sin \beta}.
\]
3. Show that the perpendiculars from $I_A$ to $BC$, from $I_B$ to $CA$, and from $I_C$ to $AB$ are concurrent.

### 7.7 Mixtilinear incircles

Suppose the mixtilinear incircles in angles $A$, $B$, $C$ of triangle $ABC$ touch the circumcircle respectively at the points $A'$, $B'$, $C'$. The segments $AA'$, $BB'$, and $CC'$ are concurrent.

**Proof.** We examine how the mixtilinear incircle divides the minor arc $BC$ of the circumcircle. Let $A'$ be the point of contact. Denote $\alpha_1 := \angle A'AB$ and $\alpha_2 := \angle A'AC$. Note that the circumcenter $O$, and the points $K$, $A'$ are collinear. In triangle $KOC$, we have

$$OK = R - \rho_1, \quad OC = R, \quad \angle KOC = 2\alpha_2,$$

where $R$ is the circumradius of triangle $ABC$. Note that $CX_2 = \frac{b(s-c)}{s}$, and $KC^2 = \rho_1^2 + CX_2^2$. Applying the cosine formula to triangle $KOC$, we have

$$2R(R - \rho_1) \cos 2\alpha_2 = (R - \rho_1)^2 + R^2 - \rho_1^2 - \left(\frac{b(s-c)}{s}\right)^2.$$

Since $\cos 2\alpha_2 = 1 - 2\sin^2 \alpha_2$, we obtain, after rearrangement of the terms,

$$\sin \alpha_2 = \frac{b(s-c)}{s} \cdot \frac{1}{\sqrt{2R(R - \rho_1)}}.$$

---

9Consider these as cevians of triangle $I_AI_BI_C$. 

---
Similarly, we obtain
\[ \sin \alpha_1 = \frac{c(s - b)}{s} \cdot \frac{1}{\sqrt{2R(R - \rho)}}. \]

It follows that
\[ \frac{\sin \alpha_1}{\sin \alpha_2} = \frac{c(s - b)}{b(s - c)}. \]

If we denote by \( B' \) and \( C' \) the points of contact of the circumcircle with the mixtilinear incircles in angles \( B \) and \( C \) respectively, each of these divides the respective minor arcs into the ratios
\[ \frac{\sin \beta_1}{\sin \beta_2} = \frac{a(s - c)}{c(s - a)}, \quad \frac{\sin \gamma_1}{\sin \gamma_2} = \frac{b(s - a)}{a(s - b)}. \]

From these,
\[ \frac{\sin \alpha_1}{\sin \alpha_2} \cdot \frac{\sin \beta_1}{\sin \beta_2} \cdot \frac{\sin \gamma_1}{\sin \gamma_2} = \frac{a(s - c)}{c(s - a)} \cdot \frac{b(s - a)}{a(s - b)} \cdot \frac{c(s - b)}{b(s - c)} = +1. \]

By the Ceva theorem, the segments \( AA' \), \( BB' \) and \( CC' \) are concurrent.

**Exercise**

1. The mixtilinear incircle in angle \( A \) of triangle \( ABC \) touches its circumcircle at \( A' \). Show that \( AA' \) is a common tangent of the mixtilinear incircles of angle \( A \) in triangle \( AA'B \) and of angle \( A \) in triangle \( AA'C \).
7.8 Duality

Given a triangle $ABC$, let

$$X, \quad X'$$  
$$Y, \quad Y'$$  
$$Z, \quad Z'$$

be harmonic conjugates with respect to the side $BC$. $CA$. $AB$.

The points $X', Y', Z'$ are collinear if and only if the cevians $AX, BY, CZ$ are concurrent.

**Proof.** By assumption,

$$\frac{BX'}{XC} = -\frac{BX}{XC}, \quad \frac{CY'}{YA} = -\frac{CY}{YA}, \quad \frac{AZ'}{Z'B} = -\frac{AZ}{ZB}.$$

It follows that

$$\frac{BX'}{XC} \cdot \frac{CY'}{YA} \cdot \frac{AZ'}{Z'B} = -1 \quad \text{if and only if} \quad \frac{BX}{XC'} \cdot \frac{CY}{YA} \cdot \frac{AZ}{Z'B} = +1.$$

The result now follows from the Menelaus and Ceva theorems.

7.8.1 Ruler construction of harmonic conjugate

Given two points $A$ and $B$, the harmonic conjugate of a point $P$ can be constructed as follows. Choose a point $C$ outside the line $AB$. Draw the
lines $CA$, $CB$, and $CP$. Through $P$ draw a line intersecting $CA$ at $Y$ and $CB$ at $X$. Let $Z$ be the intersection of the lines $AX$ and $BY$. Finally, let $Q$ be the intersection of $CZ$ with $AB$. $Q$ is the harmonic conjugate of $P$ with respect to $A$ and $B$.

### 7.8.2 Harmonic mean

Let $O$, $A$, $B$ be three collinear points such that $OA = a$ and $OB = b$. If $H$ is the point on the same ray $OA$ such that $h = OH$ is the harmonic mean of $a$ and $b$, then $(O, H; A, B)$. Since this also means that $(A, B; O, H)$, the point $H$ is the harmonic conjugate of $O$ with respect to the segment $AB$.

### 7.9 Triangles in perspective

#### 7.9.1 Desargues Theorem

Given two triangles $ABC$ and $A'B'C'$, the lines $AA'$, $BB'$, $CC'$ are concurrent if and only if the intersections of the pairs of lines $BC, B'C'$ are $CA, C'A'$ collinear.

*Proof.* Suppose $AA'$, $BB'$, $CC'$ intersect at a point $X$. Applying Menelaus’
theorem to the triangle $XBC$ and transversal $B'C'P$, we have

\[
\frac{XA'}{AA'} \cdot \frac{AR}{BB'} \cdot \frac{BB'}{B'B} = -1, \quad \frac{XB'}{BB'} \cdot \frac{BP}{PC} \cdot \frac{CC'}{C'C} = -1, \quad \frac{XC'}{C'C} \cdot \frac{CQ}{QA} \cdot \frac{AA'}{AA} = -1.
\]

Multiplying these three equation together, we obtain

\[
\frac{AR}{RB} \cdot \frac{BP}{PC} \cdot \frac{CQ}{QA} = -1.
\]

By Menelaus’ theorem again, the points $P, Q, R$ are concurrent.

7.9.2

Two triangles satisfying the conditions of the preceding theorem are said to be \textit{perspective}. $X$ is the center of perspectivity, and the line $PQR$ the axis of perspectivity.

7.9.3

Given two triangles $ABC$ and $A'B'C'$, if the lines $AA', BB', CC'$ are parallel, then the intersections of the pairs of lines $BC \parallel B'C'$ are collinear. $CA \parallel C'A'$
Proof.

\[ \frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = \left( -\frac{BB'}{CC'} \right) \left( -\frac{CC'}{AA'} \right) \left( -\frac{AA'}{BB'} \right) = -1. \]

7.9.4

If the corresponding sides of two triangles are pairwise parallel, then the lines joining the corresponding vertices are concurrent.

Proof. Let \( X \) be the intersection of \( BB' \) and \( CC' \). Then

\[ \frac{CX}{XC'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}. \]

The intersection of \( AA' \) and \( CC' \) therefore coincides with \( X \).

7.9.5

Two triangles whose sides are parallel in pairs are said to be homothetic. The intersection of the lines joining the corresponding vertices is the homothetic center. Distances of corresponding points to the homothetic center are in the same ratio as the lengths of corresponding sides of the triangles.
Chapter 8

Homogeneous coordinates

8.1 Coordinates of points on a line

8.1.1

Let $B$ and $C$ be two distinct points. Each point $X$ on the line $BC$ is uniquely determined by the ratio $BX : XC$. If $BX : XC = \lambda : \lambda'$, then we say that $X$ has homogeneous coordinates $\lambda : \lambda'$ with respect to the segment $BC$. Note that $\lambda + \lambda' \neq 0$ unless $X$ is the point at infinity on the line $BC$. In this case, we shall normalize the homogeneous coordinates to obtain the barycentric coordinate of $X : \frac{\lambda}{\lambda + \lambda'} B + \frac{\lambda'}{\lambda + \lambda'} C$.

Exercise

1. Given two distinct points $B$, $C$, and real numbers $y$, $z$, satisfying $y + z = 1$, $yB + zC$ is the point on the line $BC$ such that $BX : XC = z : y$.

2. If $\lambda \neq \frac{1}{2}$, the harmonic conjugate of the point $P = (1 - \lambda)A + \lambda B$ is the point

$$P' = \frac{1 - \lambda}{1 - 2\lambda} A - \frac{\lambda}{1 - 2\lambda} B.$$

8.2 Coordinates with respect to a triangle

Given a triangle $ABC$ (with positive orientation), every point $P$ on the plane has barycentric coordinates of the form $P : xA + yB + zC$, $x + y + z = 1$. 
This means that the areas of the oriented triangles $PBC$, $PCA$ and $PAB$ are respectively
\[ \triangle PBC = x\triangle, \quad \triangle PCA = y\triangle, \quad \text{and} \quad \triangle PAB = z\triangle. \]

We shall often identify a point with its barycentric coordinates, and write $P = xA + yB + zC$. In this case, we also say that $P$ has homogeneous coordinates $x : y : z$ with respect to triangle $ABC$.

Exercises

If $P$ has homogeneous coordinates of the form $0 : y : z$, then $P$ lies on the line $BC$.

8.2.1

Let $X$ be the intersection of the lines $AP$ and $BC$. Show that $X$ has homogeneous coordinates $0 : y : z$, and hence barycentric coordinates
\[ X = \frac{y}{y+z}B + \frac{z}{y+z}C. \]
This is the point at infinity if and only if $y+z = 0$. Likewise, if $Y$ and $Z$ are respectively the intersections of $BP$ with $CA$, and of $CP$ with $AB$, then
\[ Y = \frac{x}{z+x}A + \frac{z}{z+x}C, \quad Z = \frac{x}{x+y}A + \frac{y}{x+y}B. \]

8.2.2 Ceva Theorem

If $X$, $Y$, and $Z$ are points on the lines $BC$, $CA$, and $AB$ respectively such that
\[ BX : XC = \mu : \nu, \quad AY : YC = \lambda : \nu, \quad AZ : ZB = \lambda : \mu, \]
and if \( \frac{1}{\lambda} + \frac{1}{\mu} + \frac{1}{\nu} \neq 0 \), then the lines \( AX, BY, CZ \) intersect at the point \( P \) with homogeneous coordinates

\[
\frac{1}{\lambda} : \frac{1}{\mu} : \frac{1}{\nu} = \mu\nu : \lambda\nu : \lambda\mu
\]

with respect to the triangle \( ABC \). In barycentric coordinates,

\[
P = \frac{1}{\mu\nu + \lambda\nu + \lambda\mu}(\mu\nu \cdot A + \lambda\nu \cdot B + \lambda\mu \cdot C).
\]

### 8.2.3 Examples

**Centroid**

The midpoints \( D, E, F \) of the sides of triangle \( ABC \) divide the sides in the ratios

\[
BD : DC = 1 : 1, \\
AE : EC = 1 : 1, \\
AF : FB = 1 : 1.
\]

The medians intersect at the centroid \( G \), which has homogeneous coordinates \( 1 : 1 : 1 \), or

\[
G = \frac{1}{3}(A + B + C).
\]

**Incenter**

The (internal) bisectors of the sides of triangle \( ABC \) intersect the sides at \( X, Y, Z \) respectively with

\[
BX : XC = c : b = ac : ab, \\
AY : YC = a = bc : ab, \\
AZ : ZB = b : a = bc : ac.
\]

These bisectors intersect at the incenter \( I \) with homogeneous coordinates

\[
\frac{1}{bc} : \frac{1}{ca} : \frac{1}{ab} = a : b : c.
\]
8.2.4 Menelaus Theorem

If $X$, $Y$, and $Z$ are points on the lines $BC$, $CA$, and $AB$ respectively such that

\begin{align*}
BX &: \ XC = \mu : -\nu, \\
AY &: \ YC = -\lambda : \nu, \\
AZ &: \ ZB = \lambda : -\mu,
\end{align*}

then the points $X$, $Y$, $Z$ are collinear.

8.2.5 Example

Consider the tangent at $A$ to the circumcircle of triangle $ABC$. Suppose $AB \neq AC$. This intersects the line $BC$ at a point $X$. To determine the coordinates of $X$ with respect to $BC$, note that $BX \cdot CX = AX^2$. From this,

\[
\frac{BX}{CX} = \frac{BX \cdot CX}{CX^2} = \frac{AX^2}{CX^2} = \left(\frac{AX}{CX}\right)^2 = \left(\frac{AB}{CA}\right)^2 = \frac{c^2}{b^2},
\]

where we have made use of the similarity of the triangles $ABX$ and $CAX$. Therefore,

\[
BX : XC = c^2 : -b^2.
\]

Similarly, if the tangents at $B$ and $C$ intersect respectively the lines $CA$ and $AB$ at $Y$ and $Z$, we have

\begin{align*}
AY &: \ YC = -c^2 : a^2 = -\frac{1}{a^2} : -\frac{1}{c^2}, \\
AZ &: \ ZB = b^2 : -a^2 = \frac{1}{b^2} : -\frac{1}{a^2}.
\end{align*}

From this, it follows that the points $X$, $Y$, $Z$ are collinear.
8.3 The centers of similitude of two circles

8.3.1 External center of similitude

Consider two circles, centers $A$, $B$, and radii $r_1$ and $r_2$ respectively.

Suppose $r_1 \neq r_2$. Let $AP$ and $BQ$ be (directly) parallel radii of the circles. The line $PQ$ always passes a fixed point $K$ on the line $AB$. This is the external center of similitude of the two circles, and divides the segment $AB$ externally in the ratio of the radii:

$$AK : KB = r_1 : -r_2.$$

The point $K$ has homogeneous coordinates $r_2 : -r_1$ with respect to the segment $AB$.

8.3.2 Internal center of similitude

If $AP$ and $BQ'$ are oppositely parallel radii of the circles, then the line $PQ'$ always passes a fixed point $H$ on the line $AB$. This is the internal center of similitude of the two circles, and divides the segment $AB$ internally in the ratio of the radii:

$$AH : HB = r_1 : r_2.$$

The point $H$ has homogeneous coordinates $r_2 : r_1$ with respect to the segment $AB$.

Note that $H$ and $K$ divide the segment $AB$ harmonically.
Example

Consider three circles $O_i(r_i)$, $i = 1, 2, 3$, whose centers are not collinear and whose radii are all distinct. Denote by $C_k$, $k = 1, 2, 3$, the internal center of similitude of the circles $(O_i)$ and $(O_j)$, $i, j \neq k$. Since

\[
\begin{align*}
O_2C_1 & : C_1O_3 = r_2 : r_3, \\
O_1C_2 & : C_2O_3 = r_1 : r_3, \\
O_1C_3 & : C_3O_2 = r_1 : r_2,
\end{align*}
\]

the lines $O_1A_1$, $O_2A_2$, $O_3A_3$ are concurrent, their intersection being the point

\[
\frac{1}{r_1} : \frac{1}{r_2} : \frac{1}{r_3}
\]

with respect to the triangle $O_1O_2O_3$.

Exercise

1. Let $P_2$ be the external center of similitude of the circles $(O_2), (O_3)$. Find the homogeneous coordinates of the points $P_1$, $P_2$, $P_3$ with respect to the triangle $O_1O_2O_3$, and show that they are collinear.
2. Given triangle $ABC$, the perpendiculars from the excenters $I_B$ to $AB$, $I_C$ to $AC$, and $I_A$ to $BC$ and $I_A$ to $CA$ intersect at $A'$. Show that the lines $AA'$, $BB'$, and $CC'$ are concurrent.\footnote{CMJ408.894.408.S905.}

8.4

Consider a circle with center $K$, radius $\rho$, tangent to the sides $AB$ and $AC$, and the circumcircle of triangle $ABC$. Let $\epsilon = 1$ or $-1$ according as the tangency with the circumcircle is external or internal. Since $AK : AI = \rho : r$, $AK : KI = \rho : -(\rho - r)$,

$$K = \frac{1}{r}[\rho I - (\rho - r)A].$$

Also, let $P$ be the point of contact with the circumcircle. Since $OP : KP = R : \epsilon \rho$, we have $OP : PK = R : -\epsilon \rho$, and

$$P = \frac{1}{R + \epsilon \rho}(R \cdot K - \epsilon \rho \cdot O) = \frac{-\epsilon \rho}{R - \epsilon \rho} \cdot O + \frac{R}{R - \epsilon \rho} \cdot K.$$

Now, every point on the line $AP$ is of the form

$$\lambda P + (1 - \lambda)A = \frac{\lambda \rho}{r(R - \epsilon \rho)}(-\epsilon r \cdot O + R \cdot I) + f(\lambda)A,$$
for some real number $\lambda$. Assuming $A$ not on the line $OI$, it is clear that $AP$ intersects $OI$ at a point with homogeneous $-er : R$ with respect to the segment $OI$. In other words,

$$OX : XI = R : -er.$$ 

This is the external center of similitude of the circumcircle $(O)$ and the incircle $(I)$ according as $\epsilon = \frac{-1}{1}$, i.e., the circle $(K)$ touching the circumcircle of $ABC$ internally externally.

In barycentric coordinates, this is the point

$$X = \frac{1}{R - er}(-er \cdot O + R \cdot I).$$

This applies to the mixtilinear incircles (excircles) at the other two vertices too.

8.4.1 Theorem

Let $ABC$ be a given triangle. The three segments joining the each vertex of the triangle to the point of contact of the corresponding mixtilinear incircles excircles are concurrent at external internal center of similitude of the circumcircle and the incircle.
8.5 Isotomic conjugates

Let $X$ be a point on the line $BC$. The unique point $X'$ on the line satisfying $BX = -CX'$ is called the isotomic conjugate of $X$ with respect to the segment $BC$. Note that

$$\frac{BX'}{X'C} = \left(\frac{BX}{XC}\right)^{-1}.$$ 

8.5.1

Let $P$ be a point with homogeneous coordinates $x : y : z$ with respect to a triangle $ABC$. Denote by $X, Y, Z$ the intersections of the lines $AP, BP, CP$ with the sides $BC, CA, AB$. Clearly,

$$BX : XC = z : y, \quad CY : YA = x : z, \quad AZ : ZB = y : x.$$ 

If $X', Y', Z'$ are the isotomic conjugates of $X, Y, Z$ on the respective sides, then

$$BX' : X'C = y : z, \quad AY' : Y'C = x : z, \quad AZ' : Z'B = x : y.$$ 

It follows that $AX', BY', CZ'$ are concurrent. The intersection $P'$ is called the isotomic conjugate of $P$ (with respect to the triangle $ABC$). It has homogeneous coordinates

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$
Exercise

1. If \( X = yB + zC \), then the isotomic conjugate is \( X' = zB + yC \).

2. \( X', Y', Z' \) are collinear if and only if \( X, Y, Z \) are collinear.

8.5.2 Gergonne and Nagel points

Suppose the incircle \( I(r) \) of triangle \( ABC \) touches the sides \( BC, CA, \) and \( AB \) at the points \( X, Y, \) and \( Z \) respectively.

\[
BX : XC = s - b : s - c, \\
AY : YC = s - a : s - c, \\
AZ : ZB = s - a : s - b.
\]

This means the cevians \( AX, BY, CZ \) are concurrent. The intersection is called the Gergonne point of the triangle, sometimes also known as the Gergonne point.

Let \( X', Y', Z' \) be the isotomic conjugates of \( X, Y, Z \) on the respective sides. The point \( X' \) is indeed the point of contact of the excircle \( I_A(r_1) \) with the side \( BC \); similarly for \( Y' \) and \( Z' \). The cevians \( AX', BY', CZ' \) are
concurrent. The intersection is the Nagel point of the triangle. This is the isotomic conjugate of the Gergonne point \( L \).

**Exercise**

1. Which point is the isotomic conjugate of itself with respect to a given triangle. \(^2\)

2. Suppose the excircle on the side \( BC \) touches this side at \( X' \). Show that \( AN : NX' = a : s \). \(^3\)

3. Suppose the incircle of \( \triangle ABC \) touches its sides \( BC, CA, AB \) at \( X, Y, Z \) respectively. Let \( A', B', C' \) be the points on the incircle diametrically opposite to \( X, Y, Z \) respectively. Show that \( AA', BB' \) and \( CC' \) are concurrent. \(^4\)

### 8.6 Isogonal conjugates

**8.6.1**

Given a triangle, two cevians through a vertex are said to be *isogonal* if they are symmetric with respect to the internal bisector of the angle at the vertex.

\(^2\)The centroid.

\(^3\)Let the excircle on the side \( CA \) touch this side at \( Y' \). Apply the Menelaus theorem to \( \triangle AX'C \) and the line \( BNY' \) to obtain \( \frac{AN}{NX'} = \frac{a}{s-a} \). From this the result follows.

\(^4\)The line \( AX' \) intersects the side \( BC \) at the point of contact \( X' \) of the excircle on this side. Similarly for \( BY' \) and \( CZ' \). It follows that these three lines intersect at the Nagel point of the triangle.
Exercise

1. Show that
\[ \frac{BX^*}{X^*C} = \frac{c^2}{y^2} \cdot \frac{XC}{BX} \]

2. Given a triangle \( ABC \), let \( D \) and \( E \) be points on \( BC \) such that \( \angle BAD = \angle CAE \). Suppose the incircles of the triangles \( ABD \) and \( ACE \) touch the side \( BC \) at \( M \) and \( N \) respectively. Show that
\[ \frac{1}{BM} + \frac{1}{MD} = \frac{1}{CN} + \frac{1}{NE}. \]

8.6.2

Given a point \( P \), let \( l_a, l_b, l_c \) be the respective cevians through \( P \) the vertices \( A, B, C \) of \( \triangle ABC \). Denote by \( l_a^*, l_b^*, l_c^* \) their isogonal cevians. Using the trigonometric version of the Ceva theorem, it is easy to see that the cevians \( l_a^*, l_b^*, l_c^* \) are concurrent if and only if \( l_a, l_b, l_c \) are concurrent. Their intersection \( P^* \) is called the isogonal conjugate of \( P \) with respect to \( \triangle ABC \).

8.6.3

Suppose \( P \) has homogeneous coordinates \( x : y : z \) with respect to triangle \( \triangle ABC \). If the cevian \( BP \) and its isogonal cevian respectively meet the side \( CP \) at \( X \) and \( X^* \), then since \( AB \) at \( Y \) and \( Y^* \), then since \( AC \) at \( Z \) and \( Z^* \),

\[ BX : XC = z : y, \quad AY : YC = z : x, \quad AZ : ZB = y : x, \]
we have

\[
\begin{align*}
BX^* : X^*C &= c^2 y : b^2 z = \frac{y}{b^2} : \frac{z}{c^2}, \\
AY^* : Y^*C &= b^2 y : a^2 z = \frac{z}{a^2} : \frac{y}{b^2}, \\
AZ^* : Z^*B &= a^2 y : c^2 z = \frac{x}{c^2} : \frac{z}{a^2}.
\end{align*}
\]

From this it follows that the isogonal conjugate \( P^* \) has homogeneous coordinates

\[
\frac{a^2}{x} : \frac{b^2}{y} : \frac{c^2}{z}.
\]

8.6.4 Circumcenter and orthocenter as isogonal conjugates

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{circumcenter_orthocenter_isogonal_conjugates.png}
\caption{Circumcenter and orthocenter as isogonal conjugates}
\end{figure}

The homogeneous coordinates of the circumcenter are

\[
a \cos \alpha : b \cos \beta : c \cos \gamma = a^2(b^2 + c^2 - a^2) : b^2(c^2 + a^2 - b^2) : c^2(a^2 + b^2 - c^2).
\]

Exercise

1. Show that a triangle is isosceles if its circumcenter, orthocenter, and an excenter are collinear.  

8.6.5 The symmedian point

The symmedian point \( K \) is the isogonal conjugate of the centroid \( G \). It has homogeneous coordinates,

\[
K = a^2 : b^2 : c^2.
\]

\footnotesize{Solution (Leon Bankoff) This is clear when \( \alpha = 90^\circ \). If \( \alpha \neq 90^\circ \), the lines \( AO \) and \( AH \) are isogonal with respect to the bisector \( AI_A \); if \( O, H, I_A \) are collinear, then \( \angle OAI_A = \angle HAI_A = 0 \) or \( 180^\circ \), and the altitude \( AH \) falls along the line \( AI_A \). Hence, the triangle is isosceles.
Exercise

1. Show that the lines joining each vertex to a common corner of the squares meet at the symmedian point of triangle $ABC$.

8.6.6 The symmedians

If $D^*$ is the point on the side $BC$ of triangle $ABC$ such that $AD^*$ is the isogonal cevian of the median $AD$, $AD^*$ is called the symmedian on the side $BC$. The length of the symmedian is given by

$$t_a = \frac{2bc}{b^2 + c^2}, \quad m_a = \frac{bc\sqrt{2(b^2 + c^2) - a^2}}{b^2 + c^2}.$$

Exercise

1. $t_a = t_b$ if and only if $a = b$.

2. If an altitude of a triangle is also a symmedian, then either it is isosceles or it contains a right angle.  

8.6.7 The exsymmedian points

Given a triangle $ABC$, complete it to a parallelogram $BACA'$. Consider the isogonal cevian $BP$ of the side $BA'$. Since each of the pairs $BP$, $BA'$, and $BA$, $BC$ is symmetric with respect to the bisector of angle $B$, $\angle PBA = \angle A'BC = \angle BCA$. It follows that $BP$ is tangent to the circle $ABC$ at $B$. Similarly, the isogonal cevian of $CA'$ is the tangent at $C$ to the circumcircle of triangle $ABC$. The intersection of these two tangents at $B$ and $C$ to the circumcircle is therefore the isogonal conjugate of $A'$ with respect to $\angle BCA$.

---

$^6$Crux 960.
the triangle. This is the *exsymmedian point* $K_A$ of the triangle. Since $A'$ has homogeneous coordinates $-1:1:1$ with respect to triangle $ABC$, the exsymmedian point $K_A$ has homogeneous coordinates $-a^2:b^2:c^2$. The other two exsymmedian points $K_B$ and $K_C$ are similarly defined. These exsymmedian points are the vertices of the *tangential triangle* bounded by the tangents to the circumcircle at the vertices.

$$
K_A = -a^2:b^2:c^2,
K_B = a^2:-b^2:c^2,
K_C = a^2:b^2:-c^2.
$$

**Exercise**

1. What is the isogonal conjugate of the incenter $I$?

2. Given $\lambda, \mu, \nu$, there is a (unique) point $P$ such that

$$
PP_1:PP_2:PP_3 = \lambda:\mu:\nu
$$

if and only if each “nontrivial” sum of $a\lambda$, $b\mu$ and $c\nu$ is nonzero. This is the point

$$
\frac{a\lambda}{a\lambda + b\mu + c\nu} A + \frac{b\mu}{a\lambda + b\mu + c\nu} B + \frac{c\nu}{a\lambda + b\mu + c\nu} C.
$$

3. Given a triangle $ABC$, show that its tangential triangle is finite unless $ABC$ contains a right angle.
(a) The angles of the tangential triangle are $180^\circ - 2\alpha$, $180^\circ - 2\beta$, and $180^\circ - 2\gamma$, (or $2\alpha$, $2\beta$ and $2\gamma - 180^\circ$ if the angle at $C$ is obtuse).

(b) The sides of the tangential triangle are in the ratio
\[
\sin 2\alpha : \sin 2\beta : \sin 2\gamma = a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2).
\]

4. Justify the following table for the homogeneous coordinates of points associated with a triangle.

<table>
<thead>
<tr>
<th>Point</th>
<th>Symbol</th>
<th>Homogeneous coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>Centroid</td>
<td>$G$</td>
<td>1:1:1</td>
</tr>
<tr>
<td>Incenter</td>
<td>$I$</td>
<td>$a:b:c$</td>
</tr>
<tr>
<td>Excenters</td>
<td>$I_A$</td>
<td>$-a:b:c$</td>
</tr>
<tr>
<td></td>
<td>$I_B$</td>
<td>$a:-b:c$</td>
</tr>
<tr>
<td></td>
<td>$I_C$</td>
<td>$a:b:-c$</td>
</tr>
<tr>
<td>Gergonne point</td>
<td>$L$</td>
<td>$(s-b)(s-c) : (s-c)(s-a) : (s-a)(s-b)$</td>
</tr>
<tr>
<td>Nagel point</td>
<td>$N$</td>
<td>$s-a : s-b : s-c$</td>
</tr>
<tr>
<td>Symmedian point</td>
<td>$K$</td>
<td>$a^2 : b^2 : c^2$</td>
</tr>
<tr>
<td>Exsymmedian points</td>
<td>$K_A$</td>
<td>$-a^2 : b^2 : c^2$</td>
</tr>
<tr>
<td></td>
<td>$K_B$</td>
<td>$a^2 : -b^2 : c^2$</td>
</tr>
<tr>
<td></td>
<td>$K_C$</td>
<td>$a^2 : b^2 : -c^2$</td>
</tr>
<tr>
<td>Circumcenter</td>
<td>$O$</td>
<td>$a^2(b^2+c^2-a^2) : b^2(c^2+a^2-b^2) : c^2(a^2+b^2-c^2)$</td>
</tr>
<tr>
<td>Orthocenter</td>
<td>$H$</td>
<td>$(a^2+b^2-c^2)(c^2+a^2-b^2) : (b^2+c^2-a^2)(a^2+b^2-c^2) : (c^2+a^2-b^2)(b^2+c^2-a^2)$</td>
</tr>
</tbody>
</table>

5. Show that the incenter $I$, the centroid $G$, and the Nagel point $N$ are collinear. Furthermore, $IG : GN = 1 : 2$.

6. Find the barycentric coordinates of the incenter of $\triangle O_1O_2O_3$. \(^7\)

\(^7\)Solution. \(\frac{1}{2}[(b+c)A + (c+a)B + (a+b)C] = \frac{3}{2}M - \frac{1}{2}I\).
7. The Gergonne point of the triangle $K_A K_B K_C$ is the symmedian point $K$ of $\triangle ABC$.

8. Characterize the triangles of which the midpoints of the altitudes are collinear.  

9. Show that the mirror image of the orthocenter $H$ in a side of a triangle lies on the circumcircle.

10. Let $P$ be a point in the plane of $\triangle ABC$, $G_A$, $G_B$, $G_C$ respectively the centroids of $\triangle PBC$, $\triangle PCA$ and $\triangle PAB$. Show that $AG_A$, $BG_B$, and $CG_C$ are concurrent.

11. If the sides of a triangle are in arithmetic progression, then the line joining the centroid to the incenter is parallel to a side of the triangle.

12. If the squares of a triangle are in arithmetic progression, then the line joining the centroid and the symmedian point is parallel to a side of the triangle.

8.6.8

In §7 we have established, using the trigonometric version of Ceva theorem, the concurrency of the lines joining each vertex of a triangle to the point of contact of the circumcircle with the mixtilinear incircle in that angle. Suppose the line $AA'$, $BB'$, $CC'$ intersects the sides $BC$, $CA$, $AB$ at points $X$, $Y$, $Z$ respectively. We have

$$\frac{BX}{XC} = \frac{c \sin \alpha_1}{b \sin \alpha_2} = \frac{(s-b)/b^2}{(s-c)/c^2}.$$ 

$$\frac{BX}{XC} = \frac{s-b}{c^2} : \frac{s-c}{b^2},$$

$$\frac{AY}{YC} = \frac{s-c/a^2}{s-b/b^2} : \frac{s-c/c^2}{s-b/b^2},$$

$$\frac{AZ}{ZB} = \frac{s-c/a^2}{s-b/b^2} : \frac{s-c/c^2}{s-b/b^2}.$$ 

---

8More generally, if $P$ is a point with nonzero homogeneous coordinates with respect to $\triangle ABC$, and $AP$, $BP$, $CP$ cut the opposite sides at $X$, $Y$ and $Z$ respectively, then the midpoints of $AX$, $BY$, $CZ$ are never collinear. It follows that the orthocenter must be a vertex of the triangle, and the triangle must be right. See MG1197.844.S854.

9At the centroid of $A$, $B$, $C$, $P$; see MGQ781.914.
These cevians therefore intersect at the point with homogeneous coordinates
\[
\frac{a^2}{s-a} : \frac{b^2}{s-b} : \frac{c^2}{s-c}.
\]
This is the isogonal conjugate of the point with homogeneous coordinates
\(s - a : s - b : s - c\), the Nagel point.

8.6.9
The isogonal conjugate of the Nagel point is the external center of similitude of the circumcircle and the incircle.

Exercise
1. Show that the isogonal conjugate of the Gergonne point is the internal center of similitude of the circumcircle and the incircle.

8.7 Point with equal parallel intercepts
Given a triangle \(ABC\), we locate the point \(P\) through which the parallels to the sides of \(ABC\) make equal intercepts by the lines containing the sides of \(ABC\).\(^{10}\) It is easy to see that these intercepts have lengths \((1 - x)a\), \((1 - y)b\), and \((1 - z)c\) respectively. For the equal - parallel - intercept point \(P\),
\[
1 - x : 1 - y : 1 - z = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}.
\]
Note that
\[
(1 - x)A + (1 - y)B + (1 - z)C = 3G - 2P.
\]
This means that \(3G - P = 2I'\), the isotomic conjugate of the incenter \(I\). From this, the points \(I', G, P\) are collinear and
\[
I'G : GP = 1 : 2.
\]

\(^{10}\)AMM E396. D.L. MacKay - C.C. Oursler.
Exercise

1. Show that the triangles $OII'$ and $HNP$ are homothetic at the centroid $G$. 11

2. Let $P$ be a point with homogeneous coordinates $x : y : z$. Suppose the parallel through $P$ to $BC$ intersects $AC$ at $Y$ and $AB$ at $Z$. Find the homogeneous coordinates of the points $Y$ and $Z$, and the length of the segment $YZ$. 12

3. Make use of this to determine the homogeneous coordinates of the equal - parallel - intercept point 13 of triangle $ABC$ and show that the equal parallel intercepts have a common length

$$\frac{2abc}{ab + bc + ca}.$$  

4. Let $K$ be a point with homogeneous coordinates $p : q : r$ with respect to triangle $ABC$, $X, Y, Z$ the traces of $K$ on the sides of the triangle.

---

11The centroid $G$ divides each of the segments $OH$, $IN$, and $I'P$ in the ratio $1 : 2$.

12$Y$ and $Z$ are respectively the points $x : 0 : y + z$ and $x : y + z : 0$. The segment $YZ$ has length $\frac{2(y+z)}{2+y+z}$.

13$x : y : z = -\frac{1}{a} + \frac{1}{b} + \frac{1}{c} : \frac{1}{a} - \frac{1}{b} + \frac{1}{c} : \frac{1}{a} + \frac{1}{b} - \frac{1}{c}$. 
If the triangle $ABC$ is completed into parallelograms $ABA'C$, $BCB'A$, and $CAC'B$, then the lines $A'X$, $B'Y$, and $C'Z$ are concurrent at the point $Q$ with homogeneous coordinates

$$
\frac{1}{p} + \frac{1}{q} + \frac{1}{r} : \frac{1}{p} - \frac{1}{q} + \frac{1}{r} : \frac{1}{p} + \frac{1}{q} - \frac{1}{r}.
$$

\[14\]

The trace of $K$ on the line $BC$ is the point $X$ with homogeneous coordinates $0 : q : r$. If the triangle $ABC$ is completed into a parallelogram $ABA'C$, the fourth vertex $A'$ is the point $-1 : 1 : 1$. The line $A'X$ has equation $(q - r)x - ry + qz = 0$; similarly for the lines $B'Y$ and $C'Z$. From this it is straightforward to verify that these three lines are concurrent at the given point.
8.8 Area formula

If \( P, Q \) and \( R \) are respectively the points
\[ P = x_1A + y_1B + z_1C, \quad Q = x_2A + y_2B + z_2C, \quad R = x_3A + y_3B + z_3C, \]
then the area of triangle \( PQR \) is given by
\[ \Delta PQR = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \Delta. \]

Exercise

1. Let \( X, Y, \) and \( Z \) be points on \( BC, CA, \) and \( AB \) respectively such that
\[ BX : XC = \lambda : \lambda', \quad CY : YA = \mu : \mu', \quad AZ : ZB = \nu : \nu'. \]
The area of triangle \( XYZ \) is given by
\[ \Delta XYZ = \frac{\lambda \mu \nu + \lambda' \mu' \nu'}{(\lambda + \lambda')(\mu + \mu')(\nu + \nu')}. \]

2. Deduce that the points \( X, Y, Z \) are collinear if and only if \( \lambda \mu \nu = -\lambda' \mu' \nu' \).

3. If \( X', Y', Z' \) are isotomic conjugates of \( X, Y, Z \) on their respective sides, show that the areas of the triangles \( XYZ \) and \( X'Y'Z' \) are equal.

\[ \text{Proof. These have barycentric coordinates} \]
\[ X = \frac{\lambda'}{\lambda + \lambda'}B + \frac{\lambda}{\lambda + \lambda'}C, \quad Y = \frac{\mu'}{\mu + \mu'}C + \frac{\mu}{\mu + \mu'}A, \quad Z = \frac{\nu'}{\nu + \nu'}A + \frac{\nu}{\nu + \nu'}B. \]

By the preceding exercise,
\[ \Delta XYZ = \frac{1}{(\lambda + \lambda')(\mu + \mu')(\nu + \nu')} \begin{vmatrix} 0 & \lambda' & \lambda \\ \mu & 0 & \mu' \\ \nu' & \nu & 0 \end{vmatrix} \]
\[ = \frac{\lambda \mu \nu + \lambda' \mu' \nu'}{(\lambda + \lambda')(\mu + \mu')(\nu + \nu')}. \]
8.9 Routh’s Theorem

8.9.1 Intersection of two cevians

Let $Y$ and $Z$ be points on the lines $CA$ and $AB$ respectively such that $CY : YA = \mu : \mu'$ and $AZ : ZB = \nu : \nu'$. The lines $BY$ and $CZ$ intersect at the point $P$ with homogeneous coordinates $\mu \nu : \mu \nu' : \mu' \nu' : \mu_0 \nu_0$:

$$P = \frac{1}{\mu \nu + \mu' \nu' + \mu \nu'}(\mu \nu' A + \mu \nu B + \mu' \nu' C).$$

8.9.2 Theorem

Let $X$, $Y$ and $Z$ be points on the lines $BC$, $CA$ and $AB$ respectively such that $BX : XC = \lambda : \lambda'$, $CY : YA = \mu : \mu'$, $AZ : ZB = \nu : \nu'$.

The lines $AX$, $BY$ and $CZ$ bound a triangle of area

$$\frac{(\lambda \mu \nu - \lambda' \mu' \nu')^2}{(\lambda \mu + \lambda' \mu' + \lambda \mu')(\mu \nu + \mu' \nu' + \mu \nu')(\nu \lambda + \nu' \lambda' + \nu \lambda')} \triangle.$$

Exercise

1. In each of the following cases, $BX : XC = \lambda : 1$, $CY : YA = \mu : 1$, and $AZ : ZB = \nu : 1$. Find $\frac{\triangle'}{\triangle}$.

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\mu$</th>
<th>$\nu$</th>
<th>$\lambda \mu \nu - 1$</th>
<th>$\lambda \mu + \lambda + 1$</th>
<th>$\mu \nu + \mu + 1$</th>
<th>$\nu \lambda + \nu + 1$</th>
<th>$\frac{\triangle'}{\triangle}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>4</td>
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<tr>
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<td>3</td>
<td>6</td>
<td>7</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

2. The cevians $AX$, $BY$, $CZ$ are such that $BX : XC = CY : YA = AZ : ZB = \lambda : 1$. Find $\lambda$ such that the area of the triangle intercepted by the three cevians $AX$, $BY$, $CZ$ is $\frac{1}{7}$ of $\triangle ABC$. 


3. The cevians $AD$, $BE$, $CF$ intersect at $P$. Show that
\[
\frac{[DEF]}{[ABC]} = 2 \frac{PD}{PA} \cdot \frac{PE}{PB} \cdot \frac{PF}{PC}.
\]

4. The cevians $AD$, $BE$, and $CF$ of triangle $ABC$ intersect at $P$. If the areas of the triangles $BDP$, $CEP$, and $AFP$ are equal, show that $P$ is the centroid of triangle $ABC$.

8.10 Distance formula in barycentric coordinates

8.10.1 Theorem
The distance between two points $P = xA + yB + zC$ and $Q = uA + vB + wC$ is given by
\[
PQ^2 = \frac{1}{2}[(x-u)^2(b^2+c^2-a^2) + (y-v)^2(c^2+a^2-b^2) + (z-w)^2(a^2+b^2-c^2)]
\]

Proof. It is enough to assume $Q = C$. The distances from $P$ to the sides $BC$ and $CA$ are respectively $PP_1 = 2\triangle \cdot \frac{x}{a}$ and $PP_2 = 2\triangle \cdot \frac{y}{b}$. By the cosine formula,
\[
P_1P_2^2 = PP_1^2 + PP_2^2 + 2 \cdot PP_1 \cdot PP_2 \cdot \cos \gamma
\]
\[
= 4\triangle^2 \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 + \frac{xy}{a^2b^2} \left( a^2 + b^2 - c^2 \right)
\]
\[
= 4\triangle^2 \left( \frac{x}{a} \right)^2 + \left( \frac{y}{b} \right)^2 + \frac{1}{2} [(1-z)^2 - x^2 - y^2] \cdot (a^2 + b^2 - c^2)
\]
\[
= \frac{2\triangle^2}{a^2b^2} \left[ x^2(b^2+c^2-a^2) + y^2(c^2+a^2-b^2) + (z-1)^2(a^2+b^2-c^2) \right]
\]

It follows that $CP = \frac{PP_1 \cdot PP_2}{\sin \gamma} = \frac{ab \cdot PP_1 \cdot PP_2}{2\triangle}$ is given by
\[
CP^2 = \frac{1}{2} \left[ x^2(b^2+c^2-a^2) + y^2(c^2+a^2-b^2) + (z-1)^2(a^2+b^2-c^2) \right]
\]

The general formula follows by replacing $x$, $y$, $z-1$ by $x-u$, $y-v$, $z-w$ respectively.

\textsuperscript{16}Crux 2161.
Chapter 9

Circles inscribed in a triangle

9.1

Given a triangle $ABC$, to locate a point $P$ on the side $BC$ so that the incircles of triangles $ABP$ and $ACP$ have equal radii.

9.1.1 Analysis

Suppose $BP : PC = k : 1 - k$, and denote the length of $AP$ by $x$. By Stewart’s Theorem,

$$x^2 = kb^2 + (1 - k)c^2 - k(1 - k)a^2.$$ 

Equating the inradii of the triangles $ABP$ and $ACP$, we have

$$\frac{2k\triangle}{c + x + ka} = \frac{2(1 - k)\triangle}{b + x + (1 - k)a}.$$ 

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This latter equation can be rewritten as

\[
\frac{c + x + ka}{k} = \frac{b + x + (1 - k)a}{1 - k},
\]

(9.1)
or

\[
\frac{c + x}{k} = \frac{b + x}{1 - k},
\]

(9.2)
from which

\[
k = \frac{x + c}{2x + b + c}.
\]

Now substitution into (1) gives

\[
x^2(2x + b + c)^2 = (2x + b + c)[(x + c)b^2 + (x + b)c^2] - (x + b)(x + c)a^2.
\]

Rearranging, we have

\[
(x + b)(x + c)a^2 = (2x + b + c)[(x + c)b^2 + (x + b)c^2 - x^2[(x + b) + (x + c)]]
\]

\[
= (2x + b + c)[(x + b)(c^2 - x^2) + (x + c)(b^2 - x^2)]
\]

\[
= (2x + b + c)(x + b)(x + c) [(c - x) + (b - x)]
\]

\[
= (2x + b + c)(x + b)(x + c)(b + c - 2x)
\]

\[
= (x + b)(x + c)[(b + c)^2 - 4x^2].
\]

From this,

\[
x^2 = \frac{1}{4}((b + c)^2 - a^2) = \frac{1}{4}(b + c + a)(b + c - a) = s(s - a).
\]

9.1.2

Lau\textsuperscript{1} has proved an interesting formula which leads to a simple construction of the point $P$. If the angle between the median $AD$ and the angle bisector $AX$ is $\theta$, then

\[
m_a \cdot w_a \cdot \cos \theta = s(s - a).
\]

\textsuperscript{1}Solution to Crux 1097.
This means if the perpendicular from $X$ to $AD$ is extended to intersect the circle with diameter $AD$ at a point $Y$, then $AY = \sqrt{s(s-a)}$. Now, the circle $A(Y)$ intersects the side $BC$ at two points, one of which is the required point $P$.

9.1.3 An alternative construction of the point $P$

Let $X$ and $Y$ be the projections of the incenter $I$ and the excenter $I_A$ on the side $AB$. Construct the circle with $XY$ as diameter, and then the tangents from $A$ to this circle. $P$ is the point on $BC$ such that $AP$ has the same length as these tangents.
Exercise

1. Show that

\[ r' = \frac{s - \sqrt{s(s-a)}}{a} \cdot r. \]

2. Show that the circle with \( XY \) as diameter intersects \( BC \) at \( P \) if and only if \( \triangle ABC \) is isosceles. \(^2\)

**9.1.4 Proof of Lau’s formula**

Let \( \theta \) be the angle between the median and the bisector of angle \( A \).

Complete the triangle \( ABC \) into a parallelogram \( ABA'C \). In triangle \( AA'C \), we have

\[ AA' = 2m_a, \quad AC = b, \quad A'C = c; \]
\[ \angle ACA' = 180^\circ - \alpha, \quad \angle AAC = \frac{\alpha}{2} + \theta, \quad \angle A'AC = \frac{\alpha}{2} - \theta. \]

By the sine formula,

\[ \frac{b + c}{2m_a} = \frac{\sin\left(\frac{\alpha}{2} + \theta\right) + \sin\left(\frac{\alpha}{2} - \theta\right)}{\sin(180^\circ - \alpha)} = \frac{2 \sin \frac{\alpha}{2} \cos \theta}{\sin \alpha} = \frac{\cos \theta}{\cos \frac{\alpha}{2}}. \]

From this it follows that

\[ m_a \cdot \cos \theta = \frac{b + c}{2} \cdot \cos \frac{\alpha}{2}. \]

\(^2\)Hint: \( AP \) is tangent to the circle \( XYP \).
Now, since \( w_a = \frac{2bc}{b+c} \cos \frac{a}{2} \), we have
\[
m_a \cdot w_a \cdot \cos \theta = bc \cos^2 \frac{\alpha}{2} = s(s - a).
\]
This proves Lau’s formula.

9.1.5

Here, we make an interesting observation which leads to a simpler construction of \( P \), bypassing the calculations, and leading to a stronger result: (3) remains valid if instead of inradii, we equate the exradii of the same two subtriangles on the sides \( BP \) and \( CP \). Thus, the two subtriangles have equal inradii if and only if they have equal exradii on the sides \( BP \) and \( CP \).

Let \( \theta = \angle APB \) so that \( \angle APC = 180^\circ - \theta \). If we denote the inradii by \( r' \) and the exradii by \( \rho \), then
\[
\frac{r'}{\rho} = \tan \frac{\beta}{2} \tan \frac{\theta}{2} = \tan(90^\circ - \frac{\theta}{2}) \tan \frac{\gamma}{2}.
\]
Since \( \tan \frac{\theta}{2} \tan(90^\circ - \frac{\theta}{2}) = 1 \), we also have
\[
\left( \frac{r'}{\rho} \right)^2 = \tan \frac{\beta}{2} \tan \frac{\gamma}{2}.
\]
This in turn leads to
\[
\tan \frac{\theta}{2} = \sqrt{\left( \frac{\tan \frac{\beta}{2}}{\tan \frac{\gamma}{2}} \right)}.
\]
In terms of the sides of triangle $ABC$, we have

$$\tan \frac{\theta}{2} = \sqrt{\frac{s-b}{s-c}} = \frac{\sqrt{(s-b)(s-c)}}{s-c} = \frac{\sqrt{BX \cdot XC}}{XC}. $$

This leads to the following construction of the point $P$.

Let the incircle of $\triangle ABC$ touch the side $BC$ at $X$.

Construct a semicircle with $BC$ as diameter to intersect the perpendicular to $BC$ through $X$ at $Y$.

Mark a point $Q$ on the line $BC$ such that $AQ \parallel YC$.

The intersection of the perpendicular bisector of $AQ$ with the side $BC$ is the point $P$ required.

Exercise

1. Let $ABC$ be an isosceles triangle with $AB = BC$. $F$ is the midpoint of $AB$, and the side $BA$ is extended to a point $K$ with $AK = \frac{1}{2}AC$.

The perpendicular through $A$ to $AB$ intersects the circle $F(K)$ at a point $Q$. $P$ is the point on $BC$ (the one closer to $B$ if there are two) such that $AP = AQ$. Show that the inradii of triangles $ABP$ and $ACP$ are equal.
2. Given triangle $ABC$, let $P_0, P_1, P_2, \ldots, P_n$ be points on $BC$ such that $P_0 = B, P_n = C$ and the inradii of the subtriangles $AP_{k-1}P_k$, $k = 1, \ldots, n$, are all equal. For $k = 1, 2, \ldots, n$, denote $\angle AP_kP_{k-1} = \theta_k$. Show that $\tan \frac{\theta_k}{2}, k = 1, \ldots, n - 1$ are $n - 1$ geometric means between $\cot \frac{\beta}{2}$ and $\tan \frac{\gamma}{2}$, i.e.,

$$\frac{1}{\tan \frac{\beta}{2}}, \tan \frac{\theta_1}{2}, \tan \frac{\theta_2}{2}, \ldots \tan \frac{\theta_{n-1}}{2}, \tan \frac{\gamma}{2}$$

form a geometric progression.

3. Let $P$ be a point on the side $BC$ of triangle $ABC$ such that the excircle of triangle $ABP$ on the side $BP$ and the incircle of triangle $ACP$ have the same radius. Show that \(^3\)

$$BP : PC = -a + b + 3c : a + 3b + c,$$

and

$$AP = \frac{(b + c)^2 - a(s - c)}{2(b + c)}.$$

\(^3\)If $BP : PC = k : 1 - k$, and $AP = x$, then

$$\frac{k}{c + x - ka} = \frac{(1 - k)}{b + x + (1 - k)a}.$$

Also, by Stewart’s Theorem $x^2 = kb^2 + (1 - k)c^2 - k(1 - k)a^2$. 
4. Let $\triangle ABC$ be an isosceles triangle, $D$ the midpoint of the base $BC$. On the minor arc $BC$ of the circle $A(B)$, mark a point $X$ such that $CX = CD$. Let $Y$ be the projection of $X$ on the side $AC$. Let $P$ be a point on $BC$ such that $AP = AY$. Show that the inradius of triangle $\triangle ABP$ is equal to the exradius of triangle $\triangle ACP$ on the side $CP$. 
9.2

Given a triangle, to construct three circles through a common point, each tangent to two sides of the triangle, such that the 6 points of contact are concyclic.

Let $G$ be the common point of the circles, and $X_2$, $X_3$ on the side $BC$, $Y_1$, $Y_3$ on $CA$, and $Z_1$, $Z_2$ on $AB$, the points of contact.

9.2.1 Analysis

Consider the circle through the 6 points of contact. The line joining the center to each vertex is the bisector of the angle at that vertex. This center is indeed the incenter $I$ of the triangle. It follows that the segments $X_2X_3$, $Y_3Y_1$, and $Z_1Z_2$ are all equal in length. Denote by $X$, $Y$, $Z$ the projections of $I$ on the sides. Then $XX_2 = XX_3$. Also,

$$AZ_2 = AZ_1 + Z_1Z_2 = AY_1 + Y_1Y_3 = AY_3.$$ 

This means that $X$ and $A$ are both on the radical axis of the circles $(K_2)$ and $(K_3)$. The line $AX$ is the radical axis. Similarly, the line $BY$ is the radical axis of the pair of circles $(K_1)$ and $(K_2)$. The common point $G$ of the circles, being the intersection of $AX$, $BY$, and $CZ$, is the Gergonne point of the triangle.

\[\text{Thébault - Eves, AMM E457.}\]
The center $K_1$ is the intersection of the segment $AI$ and the parallel through $G$ to the radius $XI$ of the incircle.

The other two centers $K_2$ and $K_3$ can be similarly located.

9.3

Given a triangle, to construct three congruent circles through a common point, each tangent to two sides of the triangle.

9.3.1 Analysis

Let $I_1$, $I_2$, $I_3$ be the centers of the circles lying on the bisectors $IA$, $IB$, $IC$ respectively. Note that the lines $I_2I_3$ and $BC$ are parallel; so are the pairs $I_3I_1$, $CA$, and $I_1I_2$, $AB$. It follows that triangles $I_1I_2I_3$ and $ABC$ are per-
spective from their common incenter \( I \). The line joining their circumcenters passes through \( I \). Note that \( T \) is the circumcenter of triangle \( I_1I_2I_3 \), the circumradius being the common radius \( t \) of the three circles. This means that \( T, O \) and \( I \) are collinear. Since

\[
\frac{I_3I_1}{CA} = \frac{I_1I_2}{AB} = \frac{I_2I_3}{BC} = \frac{r - t}{r},
\]

we have \( t = \frac{r-t}{r} \cdot R \), or

\[
\frac{t}{R} = \frac{r}{R + r}.
\]

This means \( I \) divides the segment \( OT \) in the ratio

\[TI : IO = -r : R + r.\]

Equivalently, \( OT : TI = R : r \), and \( T \) is the internal center of similitude of the circumcircle and the incircle.

### 9.3.2 Construction

Let \( O \) and \( I \) be the circumcenter and the incenter of triangle \( ABC \).

1. Construct the perpendicular from \( I \) to \( BC \), intersecting the latter at \( X \).
2. Construct the perpendicular from \( O \) to \( BC \), intersecting the circumcircle at \( M \) (so that \( IX \) and \( OM \) are directly parallel).
3. Join \( OX \) and \( IM \). Through their intersection \( P \) draw a line parallel to \( IX \), intersecting \( OI \) at \( T \), the internal center of similitude of the circumcircle and incircle.
4. Construct the circle \( T(P) \) to intersect the segments \( IA, IB, IC \) at \( I_1, I_2, I_3 \) respectively.
5. The circles \( I_j(T), j = 1,2,3 \) are three equal circles through \( T \) each tangent to two sides of the triangle.
9.4

9.4.1 Proposition

Let $I$ be the incenter of $\triangle ABC$, and $I_1, I_2, I_3$ the incenters of the triangles $IBC$, $ICA$, and $IAB$ respectively. Extend $II_1$ beyond $I_1$ to intersect $BC$ at $A'$, and similarly $II_2$ beyond $I_2$ to intersect $CA$ at $B'$, $II_3$ beyond $I_3$ to intersect $AB$ at $C'$. Then, the lines $AA'$, $BB'$, $CC'$ are concurrent at a point with homogeneous barycentric coordinates

$$a \sec \frac{\alpha}{2} : b \sec \frac{\beta}{2} : c \sec \frac{\gamma}{2}.$$

Proof. The angles of triangle $IBC$ are

$$\pi - \frac{1}{2}(\beta + \gamma), \quad \frac{\beta}{2}, \quad \frac{\gamma}{2}.$$  

The homogeneous coordinates of $I_1$ with respect to $IBC$ are

$$\cos \frac{\alpha}{2} : \sin \frac{\beta}{2} : \sin \frac{\gamma}{2}.$$  

---

5This point apparently does not appear in Kimberling’s list.
Since \( I = \frac{1}{2}(a \cdot A + b \cdot B + c \cdot C) \), the homogeneous coordinates of \( I_1 \) with respect to \( ABC \) are

\[
a \cos \frac{\alpha}{2} : b \cos \frac{\alpha}{2} + 2s \sin \frac{\beta}{2} : c \cos \frac{\alpha}{2} + 2s \sin \frac{\gamma}{2}
\]

\[
= a : b(1 + 2 \cos \frac{\gamma}{2}) : c(1 + 2 \cos \frac{\beta}{2}).
\]

Here, we have made use of the sine formula:

\[
\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma} = \frac{2s}{\sin \alpha + \sin \beta + \sin \gamma} = \frac{2s}{4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}}.
\]

Since \( I \) has homogeneous coordinates \( a : b : c \), it is easy to see that the line \( II_1 \) intersects \( BC \) at the point \( A' \) with homogeneous coordinates

\[
0 : b \cos \frac{\gamma}{2} : c \cos \frac{\beta}{2} = 0 : b \sec \frac{\beta}{2} : c \sec \frac{\gamma}{2}.
\]

Similarly, \( B' \) and \( C' \) have coordinates

\[
A' : 0 : b \sec \frac{\beta}{2} : c \sec \frac{\gamma}{2}
\]

\[
B' : a \sec \frac{\alpha}{2} : 0 : c \sec \frac{\gamma}{2}
\]

\[
C' : a \sec \frac{\alpha}{2} : b \sec \frac{\beta}{2} : 0.
\]

From these, it is clear that \( AA', BB', CC' \) intersect at a point with homogeneous coordinates

\[
a \sec \frac{\alpha}{2} : b \sec \frac{\beta}{2} : c \sec \frac{\gamma}{2}.
\]

**Exercise**

1. Let \( O_1, O_2, O_3 \) be the circumcenters of triangles \( I_1BC, I_2CA, I_3AB \) respectively. Are the lines \( O_1I_1, O_2I_2, O_3I_3 \) concurrent?

## 9.5 Malfatti circles

### 9.5.1 Construction Problem

Given a triangle, to construct three circles mutually tangent to each other, each touching two sides of the triangle.
Construction

Let $I$ be the incenter of triangle $ABC$.

(1) Construct the incircles of the subtriangles $IBC$, $ICA$, and $IAB$.

(2) Construct the external common tangents of each pair of these incircles. (The incircles of $ICA$ and $IAB$ have $IA$ as a common tangent. Label the other common tangent $Y_1Z_1$ with $Y_1$ on $CA$ and $Z_1$ on $AB$ respectively. Likewise the common tangent of the incircles of $IAB$ and $IBC$ is $Z_2X_2$ with $Z_2$ on $AB$ and $X_2$ on $BC$, and that of the incircles of $IBC$ and $ICA$ is $X_3Y_3$ with $X_3$ on $BC$ and $Y_3$ on $CA$.) These common tangents intersect at a point $P$.

(3) The incircles of triangles $AY_1Z_1$, $BZ_2X_2$, and $CX_3Y_3$ are the required Malfatti circles.

Exercise

1. Three circles of radii $r_1$, $r_2$, $r_3$ are mutually tangent to each other. Find the lengths of the sides of the triangle bounded by their external
9.6

9.6.1

Given a circle $K(a)$ tangent to $O(R)$ at $A$, and a point $B$, to construct a circle $K'(b)$ tangent externally to $K(a)$ and internally to $(O)$ at $B$.

Construction

Extend $OB$ to $P$ such that $BP = a$. Construct the perpendicular bisector of $KP$ to intersect $OB$ at $K'$, the center of the required circle.

9.6.2

Two circles $H(a)$ and $K(b)$ are tangent externally to each other, and internally to a third, larger circle $O(R)$, at $A$ and $B$ respectively.

$$AB = 2R \sqrt{\frac{a}{R-a} \cdot \frac{b}{R-b}}.$$  

\[ a = \frac{r}{r-r_2}(\sqrt{r_2r_3} - \sqrt{r_3r_1} + \sqrt{r_1r_2}) + \frac{r}{r-r_3}(\sqrt{r_2r_3} + \sqrt{r_3r_1} - \sqrt{r_1r_2}) \]

where

$$r = \frac{\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} + \sqrt{r_1 + r_2 + r_3}}{\sqrt{r_2r_3} + \sqrt{r_3r_1} + \sqrt{r_1r_2}} \cdot \sqrt{r_1r_2r_3}.$$
Proof. Let $\angle AOB = \theta$. Applying the cosine formula to triangle $AOB$,

$$AB^2 = R^2 + R^2 - 2R^2 \cos \theta,$$

where

$$\cos \theta = \frac{(R - a)^2 + (R - b)^2 - (a + b)^2}{2(R - a)(R - b)},$$

by applying the cosine formula again, to triangle $OHK$.

Exercise

1. Given a circle $K(A)$ tangent externally to $O(A)$, and a point $B$ on $O(A)$, construct a circle tangent to $O(A)$ at $B$ and to $K(A)$ externally (respectively internally).

2. Two circles $H(a)$ and $K(b)$ are tangent externally to each other, and also externally to a third, larger circle $O(R)$, at $A$ and $B$ respectively. Show that

$$AB = 2R \sqrt{\frac{a}{R + a} \cdot \frac{b}{R + b}}.$$  

9.6.3

Let $H(a)$ and $K(b)$ be two circles tangent internally to $O(R)$ at $A$ and $B$ respectively. If $(P)$ is a circle tangent internally to $(O)$ at $C$, and externally to each of $(H)$ and $(K)$, then

$$AC : BC = \sqrt{\frac{a}{R - a}} : \sqrt{\frac{b}{R - b}}.$$  

Proof. The lengths of $AC$ and $BC$ are given by

$$AC = 2R \sqrt{\frac{ac}{(R - a)(R - c)}}, \quad BC = 2R \sqrt{\frac{bc}{(R - b)(R - c)}}.$$  

Construction of the point $C$

(1) On the segment $AB$ mark a point $X$ such that the cevians $AK$, $BH$, and $OX$ intersect. By Ceva theorem,

$$AX : XB = \frac{a}{R - a} : \frac{b}{R - b}.$$
(2) Construct a circle with $AB$ as diameter. Let the perpendicular through $X$ to $AB$ intersect this circle at $Q$ and $Q'$. Let the bisectors angle $AQB$ intersect the line $AB$ at $Y$.

Note that $AQ^2 = AX \cdot AB$ and $BQ^2 = XB \cdot AB$. Also, $AY : YB = AQ : QB$. It follows that

$$AY : YB = \sqrt{\frac{a}{R-a}} : \sqrt{\frac{b}{R-b}}.$$ 

(3) Construct the circle through $Q$, $Y$, $Q'$ to intersect $(O)$ at $C$ and $C'$. Then $C$ and $C'$ are the points of contact of the circles with $(O)$, $(H)$, and $(K)$. Their centers can be located by the method above.

9.6.4

Given three points $A$, $B$, $C$ on a circle $(O)$, to locate a point $D$ such that there is a chain of 4 circles tangent to $(O)$ internally at the points $A$, $B$, $C$, $D$.

Bisect angle $ABC$ to intersect $AC$ at $E$ and the circle $(O)$ at $X$. Let $Y$ be the point diametrically opposite to $X$. The required point $D$ is the intersection of the line $YE$ and the circle $(O)$. 
Beginning with any circle $K(A)$ tangent internally to $O(A)$, a chain of four circles can be completed to touch $(O)$ at each of the four points $A, B, C, D$.

**Exercise**

1. Let $A, B, C, D, E, F$ be six consecutive points on a circle. Show that the chords $AD, BE, CF$ are concurrent if and only if $AB \cdot CD \cdot EF = BC \cdot DE \cdot FA$.

2. Let $A_1A_2\ldots A_{12}$ be a regular 12–gon. Show that the diagonals $A_1A_5, A_3A_6$ and $A_4A_8$ are concurrent.

3. Inside a given circle $C$ is a chain of six circles $C_i, i = 1, 2, 3, 4, 5, 6,$
such that each $C_i$ touches $C_{i-1}$ and $C_{i+1}$ externally. (Remark: $C_7 = C_1$).
Suppose each $C_i$ also touches $C$ internally at $A_i$, $i = 1, 2, 3, 4, 5, 6$.
Show that $A_1A_4$, $A_2A_5$ and $A_3A_6$ are concurrent. 

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Rabinowitz, The seven circle theorem, Pi Mu Epsilon Journal, vol 8, no. 7 (1987) pp.441 – 449. The statement is still valid if each of the circles $C_i$, $i = 1, 2, 3, 4, 5, 6$, is outside the circle $C$. 

Chapter 10

Quadrilaterals

10.1 Area formula

Consider a quadrilateral $ABCD$ with sides

$AB = a, \quad BC = b, \quad CD = c, \quad DA = d,$

angles

$\angle DAB = \alpha, \quad \angle ABC = \beta, \quad \angle BCD = \gamma, \quad \angle CDA = \delta,$

and diagonals

$AC = x, \quad BD = y.$

Applying the cosine formula to triangles $ABC$ and $ADC$, we have

$$x^2 = a^2 + b^2 - 2ab \cos \beta,$$
\[ x^2 = c^2 + d^2 - 2cd \cos \delta. \]

Eliminating \( x \), we have
\[ a^2 + b^2 - c^2 - d^2 = 2ab \cos \beta - 2cd \cos \delta, \]

Denote by \( S \) the area of the quadrilateral. Clearly,
\[ S = \frac{1}{2}ab \sin \beta + \frac{1}{2}cd \sin \delta. \]

Combining these two equations, we have
\[
16S^2 + (a^2 + b^2 - c^2 - d^2)^2
= 4(ab \sin \beta + cd \sin \delta)^2 + 4(ab \cos \beta - cd \cos \delta)^2
= 4(a^2b^2 + c^2d^2 - 8abcd(\cos \beta \cos \delta - \sin \beta \sin \delta)
= 4(a^2b^2 + c^2d^2 - 8abcd)(\beta + \delta)
= 4(a^2b^2 + c^2d^2 - 8abcd[2 \cos^2 \frac{\beta + \delta}{2} - 1])
= 4(ab + cd)^2 - 16abcd \cos^2 \frac{\beta + \delta}{2}.
\]

Consequently,
\[
16S^2
= 4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2 - 16abcd \cos^2 \frac{\beta + \delta}{2}
= [2(ab + cd) + (a^2 + b^2 - c^2 - d^2)][2(ab + cd) - (a^2 + b^2 - c^2 - d^2)]
- 16abcd \cos^2 \frac{\beta + \delta}{2}
= [(a + b)^2 - (c - d)^2][(c + d)^2 - (a - b)^2] - 16abcd \cos^2 \frac{\beta + \delta}{2}
= (a + b + c - d)(a + b - c + d)(c + d + a - b)(c + d - a - b)
- 16abcd \cos^2 \frac{\beta + \delta}{2}.
\]

Writing
\[ 2s := a + b + c + d, \]
we reorganize this as
\[ S^2 = (s - a)(s - b)(s - c)(s - d) - abed \cos^2 \frac{\beta + \delta}{2}. \]
10.1.1 Cyclic quadrilateral
If the quadrilateral is cyclic, then $\beta + \delta = 180^\circ$, and $\cos \frac{\beta + \delta}{2} = 0$. The area formula becomes

$$S = \sqrt{(s-a)(s-b)(s-c)(s-d)},$$

where $s = \frac{1}{2}(a + b + c + d)$.

**Exercise**

1. If the lengths of the sides of a quadrilateral are fixed, its area is greatest when the quadrilateral is cyclic.

2. Show that the Heron formula for the area of a triangle is a special case of this formula.

10.2 Ptolemy’s Theorem

Suppose the quadrilateral $ABCD$ is cyclic. Then, $\beta + \delta = 180^\circ$, and $\cos \beta = -\cos \delta$. It follows that

$$\frac{a^2 + b^2 - x^2}{2ab} + \frac{c^2 + d^2 - x^2}{2cd} = 0,$$

and

$$x^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}.$$

Similarly, the other diagonal $y$ is given by

$$y^2 = \frac{(ab + cd)(ac + bd)}{(ad + bc)}.$$

From these, we obtain

$$xy = ac + bd.$$

This is Ptolemy’s Theorem. We give a synthetic proof of the theorem and its converse.
10.2.1 Ptolemy’s Theorem

A convex quadrilateral $ABCD$ is cyclic if and only if

$$AB \cdot CD + AD \cdot BC = AC \cdot BD.$$  

Proof. (Necessity) Assume, without loss of generality, that $\angle BAD > \angle ABD$. Choose a point $P$ on the diagonal $BD$ such that $\angle BAP = \angle CAD$. Triangles $BAP$ and $CAD$ are similar, since $\angle ABP = \angle ACD$. It follows that $AB : AC = BP : CD$, and

$$AB \cdot CD = AC \cdot BP.$$  

Now, triangles $ABC$ and $APD$ are also similar, since $\angle BAC = \angle BAP + \angle PAC = \angle DAC + \angle PAC = \angle PAD$, and $\angle ACB = \angle ADP$. It follows that $AC : BC = AD : PD$, and

$$BC \cdot AD = AC \cdot PD.$$  

Combining the two equations, we have

$$AB \cdot CD + BC \cdot AD = AC(BP + PD) = AC \cdot BD.$$  

(Sufficiency). Let $ABCD$ be a quadrilateral satisfying (**). Locate a point $P'$ such that $\angle BAP' = \angle CAD$ and $\angle ABP' = \angle ACD$. Then the triangles $ABP$ and $ACD$ are similar. It follows that

$$AB : AP' : BP' = AC : AD : CD.$$  

From this we conclude that

(i) $AB \cdot CD = AC' \cdot BP'$, and
(ii) triangles $ABC$ and $AP'D$ are similar since $\angle BAC = \angle P'AD$ and $AB : AC = AP' : AD$.

Consequently, $AC : BC = AD : P'D$, and

$$AD \cdot BC = AC \cdot P'D.$$ 

Combining the two equations,

$$AC(BP' + P'D) = AB \cot CD + AD \cdot BC = AC \cdot BD.$$ 

It follows that $BP' + P'D = BC$, and the point $P'$ lies on diagonal $BD$. From this, $\angle ABD = \angle ABP' = \angle ACD$, and the points $A, B, C, D$ are concyclic.

**Exercise**

1. Let $P$ be a point on the minor arc $BC$ of the circumcircle of an equilateral triangle $ABC$. Show that $AP = BP + CP$.

![Diagram](image)

2. $P$ is a point on the incircle of an equilateral triangle $ABC$. Show that $AP^2 + BP^2 + CP^2$ is constant. \(^1\)

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\(^1\)If each side of the equilateral triangle has length $2a$, then $AP^2 + BP^2 + CP^2 = 5a^2$. 
3. Each diagonal of a convex quadrilateral bisects one angle and trisects the opposite angle. Determine the angles of the quadrilateral. 

4. If three consecutive sides of a convex, cyclic quadrilateral have lengths $a$, $b$, $c$, and the fourth side $d$ is a diameter of the circumcircle, show that $d$ is the real root of the cubic equation

$$x^3 - (a^2 + b^2 + c^2)x - 2abc = 0.$$ 

5. One side of a cyclic quadrilateral is a diameter, and the other three sides have lengths 3, 4, 5. Find the diameter of the circumcircle.

6. The radius $R$ of the circle containing the quadrilateral is given by

$$R = \frac{(ab + cd)(ac + bd)(ad + bc)}{4S}.$$ 

10.2.2

If $ABCD$ is cyclic, then

$$\tan \frac{\alpha}{2} = \sqrt{\frac{(s - a)(s - d)}{(s - b)(s - c)}}.$$ 

**Proof.** In triangle $ABD$, we have $AB = a$, $AD = d$, and $BD = y$, where

$$y^2 = \frac{(ab + cd)(ac + bd)}{ad + bc}.$$ 

\[2\] Answer: Either $A = D = 72^\circ$, $B = C = 108^\circ$, or $A = D = \frac{720^\circ}{7}$, $B = C = \frac{540^\circ}{7}$. 

\[2\]
By the cosine formula,
\[
\cos \alpha = \frac{a^2 + d^2 - y^2}{2ad} = \frac{a^2 - b^2 - c^2 + d^2}{2(ad + bc)}.
\]

In an alternative form, this can be written as
\[
\tan^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{1 + \cos \alpha} = \frac{(b + c)^2 - (a - d)^2}{(a + d)^2 - (b - c)^2} = \frac{(-a + b + c + d)(a + b + c - d)}{(a-b + c + d)(a + b - c + d)} = \frac{(s-a)(s-d)}{(s-b)(s-c)}.
\]

**Exercise**

1. Let \(Q\) denote an arbitrary convex quadrilateral inscribed in a fixed circle, and let \(F(Q)\) be the set of inscribed convex quadrilaterals whose sides are parallel to those of \(Q\). Prove that the quadrilaterals in \(F(Q)\) of maximum area is the one whose diagonals are perpendicular to one another.  

2. Let \(a, b, c, d\) be positive real numbers.

   (a) Prove that \(a + b > |c - d|\) and \(c + d > |a - b|\) are necessary and sufficient conditions for there to exist a convex quadrilateral that admits a circumcircle and whose side lengths, in cyclic order, are \(a, b, c, d\).

   (b) Find the radius of the circumcircle.

3. Determine the maximum area of the quadrilateral with consecutive vertices \(A, B, C,\) and \(D\) if \(\angle A = \alpha, BC = b,\) and \(CD = c\) are given.

10.2.3 **Construction of cyclic quadrilateral of given sides**

10.2.4 **The anticenter of a cyclic quadrilateral**

Consider a cyclic quadrilateral \(ABCD\), with circumcenter \(O\). Let \(X, Y, Z, W\) be the midpoints of the sides \(AB, BC, CD, DA\) respectively. The midpoint of \(XZ\) is the centroid \(G\) of the quadrilateral. Consider the perpendicular \(X\) to the opposite side \(CD\). Denote by \(O'\) the intersection of this

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3MG1472.952. (E.Greil)
4CMJ545.951.S961. (J.Fukuta)
5CMJ538.945.S955. (M.S.Klamkin)
perpendicular with the line $OG$. Since $O'X'//ZO$ and $G$ is the midpoint of $XA$, it is clear that $O'G = GO$.

It follows that the perpendiculars from the midpoints of the sides to the opposite sides of a cyclic quadrilateral are concurrent at the point $O'$, which is the symmetric of the circumcenter in the centroid. This is called the *anticenter* of the cyclic quadrilateral.

**10.2.5**

Let $P$ be the midpoint of the diagonal $AC$. Since $AXPW$ is a parallelogram, $\angle XPW = \angle XAW$. Let $X'$ and $W'$ be the projections of the midpoints $X$ and $W$ on their respective opposite sides. The lines $XX'$ and $WW'$ intersect at $O'$. Clearly, $O'$, $W'$, $C$, $X'$ are concyclic. From this, we have

$$\angle XO'W = \angle X'O'W' = 180^\circ - \angle X'C'W' = \angle XAW = \angle XPW.$$
It follows that the four points $P$, $X$, $W$, and $O'$ are concyclic. Since $P$, $X$, $W$ are the midpoints of the sides of triangle $ABD$, the circle through them is the nine-point circle of triangle $ABD$. From this, we have

**Proposition**

The nine-point circles of the four triangles determined by the four vertices of a cyclic quadrilateral pass through the anticenter of the quadrilateral.

**10.2.6 Theorem**

The incenters of the four triangles determined by the vertices of a cyclic quadrilateral form a rectangle.

*Proof.* The lines $AS$ and $DP$ intersect at the midpoint $H$ of the arc $BC$ on the other side of the circle $ABCD$. Note that $P$ and $S$ are both on the circle $H(B) = H(C)$. If $K$ is the midpoint of the arc $AD$, then $HK$, being the bisector of angle $AHD$, is the perpendicular bisector of $PS$. For the same reason, it is also the perpendicular bisector of $QR$. It follows that $PQRS$ is an isosceles trapezium.

The same reasoning also shows that the chord joining the midpoints of the arcs $AB$ and $CD$ is the common perpendicular bisector of $PQ$ and $RS$. From this, we conclude that $PQRS$ is indeed a rectangle.

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6Court, p.133.
10.2.7 Corollary

The inradii of these triangles satisfy the relation

\[ r_a + r_c = r_b + r_d. \]

**Proof.** If \( AB \) and \( CD \) are parallel, then each is parallel to \( HK \). In this case, \( r_a = r_b \) and \( r_c = r_d \). More generally,

\[ r_a - r_b = PQ \sin \frac{1}{2}(\angle BDC - \angle AHD) \]

and

\[ r_d - r_c = SR \sin \frac{1}{2}(\angle BAC - \angle AHD). \]

Since \( PQ = SR \) and \( \angle BDC = \angle BAC \), it follows that \( r_a - r_b = r_d - r_c \), and

\[ r_a + r_c = r_b + r_d. \]

**Exercise**

1. Suppose the incircles of triangles \( ABC \) and \( ABD \) touch the diagonal \( AC \) at \( X \) and \( BD \) at \( Y \) respectively.

Show that

\[ XY = ZW = \frac{1}{2}|a - b + c - d|. \]

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10.3 Circumscribable quadrilaterals

A quadrilateral is said to be *circumscribable* if it has an incircle.

10.3.1 Theorem

A quadrilateral is circumscribable if and only if the two pairs of opposite sides have equal total lengths.

*Proof.* (Necessity) Clear.

(Sufficiency) Suppose \( AB + CD = BC + DA \), and \( AB < AD \). Then \( BC < CD \), and there are points \( X \) on \( AD \) and \( Y \) on \( CD \) such that \( AX = AB \) and \( CY = CD \). Then \( DX = DY \). Let \( K \) be the circumcircle of triangle \( BXY \). \( AK \) bisects angle \( A \) since the triangles \( AKX \) and \( AKB \) are congruent. Similarly, \( CK \) and \( DK \) are bisectors of angles \( B \) and \( C \) respectively. It follows that \( K \) is equidistant from the sides of the quadrilateral. The quadrilateral admits of an incircle with center \( K \).

10.3.2 Let \( ABCD \) be a circumscribable quadrilateral, \( X, Y, Z, W \) the points of contact of the incircle with the sides. The diagonals of the quadrilaterals \( ABCD \) and \( XYZW \) intersect at the same point.

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8See Crux 199. This problem has a long history, and usually proved using projective geometry. Charles Trigg remarks that the Nov.-Dec. issue of Math. Magazine, 1962, contains nine proofs of this theorem. The proof here was given by Joseph Konhauser.
Furthermore, \( YZW \) is orthodiagonal if and only if \( ABCD \) is orthodiagonal.

**Proof.** We compare the areas of triangles \( APX \) and \( CPZ \). This is clearly

\[
\frac{\triangle APX}{\triangle CPZ} = \frac{AP \cdot PX}{CP \cdot PZ}.
\]

On the other hand, the angles \( PCZ \) and \( PAX \) are supplementary, since \( YZ \) and \( XW \) are tangents to the circle at the ends of the chord \( CA \). It follows that

\[
\frac{\triangle APX}{\triangle CPZ} = \frac{AP \cdot AX}{CP \cdot CZ}.
\]

From these, we have

\[
\frac{PX}{PZ} = \frac{AX}{CZ}.
\]

This means that the point \( P \) divides the diagonal \( XZ \) in the ratio \( AX : CZ \).

Now, let \( Q \) be the intersection of the diagonal \( XZ \) and the chord \( BD \). The same reasoning shows that \( Q \) divides \( XZ \) in the ratio \( BX : DZ \). Since \( BX = AX \) and \( DZ = CZ \), we conclude that \( Q \) is indeed the same as \( P \).

The diagonal \( XZ \) passes through the intersection of \( AC \) and \( BD \). Likewise, so does the diagonal \( YW \).

**Exercise**

1. The area of the circumscribable quadrilateral is given by

\[
S = \sqrt{abcd} \cdot \sin \frac{\alpha + \gamma}{2}.
\]
In particular, if the quadrilateral is also cyclic, then

\[ S = \sqrt{abcd}. \]

2. If a cyclic quadrilateral with sides \( a, b, c, d \) (in order) has area \( S = \sqrt{abcd} \), is it necessarily circumscribable?

3. If the consecutive sides of a convex, cyclic and circumscribable quadrilateral have lengths \( a, b, c, d \), and \( d \) is a diameter of the circumcircle, show that

\[ (a + c)b^2 - 2(a^2 + 4ac + c^2)b + ac(a + c) = 0. \]

4. Find the radius \( r' \) of the circle with center \( I \) so that there is a quadrilateral whose vertices are on the circumcircle \( O(R) \) and whose sides are tangent to \( I(r') \).

5. Prove that the line joining the midpoints of the diagonals of a circumscribable quadrilateral passes through the incenter of the quadrilateral.

10.4 Orthodiagonal quadrilateral

10.4.1

A quadrilateral is orthodiagonal if its diagonals are perpendicular to each other.

10.4.2

A quadrilateral is orthodiagonal if and only if the sum of squares on two opposite sides is equal to the sum of squares on the remaining two opposite sides.

\(^9\)No, when the quadrilateral is a rectangle with unequal sides. Consider the following three statements for a quadrilateral.
(a) The quadrilateral is cyclic.
(b) The quadrilateral is circumscribable.
(c) The area of the quadrilateral is \( S = \sqrt{abcd} \).
Apart from the exception noted above, any two of these together implies the third.
(Crux 777).
\(^{10}\)Is it possible to find integers \( a \) and \( c \) so that \( b \) is also an integer?
\(^{11}\)PME417.78S.S79S.(C.W.Dodge)
Proof. Let $K$ be the intersection of the diagonals, and $\angle AKB = \theta$. By the cosine formula,

\begin{align*}
AB^2 &= AK^2 + BK^2 - 2AK \cdot BK \cdot \cos \theta, \\
CD^2 &= CK^2 + DK^2 - 2CK \cdot DK \cdot \cos \theta; \\
BC^2 &= BK^2 + CK^2 + 2BK \cdot CK \cdot \cos \theta, \\
DA^2 &= DK^2 + AK^2 + 2DK \cdot AK \cdot \cos \theta.
\end{align*}

Now,

\[BC^2 + DA^2 - AB^2 - CD^2 = 2 \cos \theta (BK \cdot CK + DK \cdot AK + AK \cdot BK + CK \cdot DK)\]

It is clear that this is zero if and only if $\theta = 90^\circ$.

Exercise

1. Let $ABCD$ be a cyclic quadrilateral with circumcenter $O$. The quadrilateral is orthodiagonal if and only if the distance from $O$ to each side of the $ABCD$ is half the length of the opposite side.  

2. Let $ABCD$ be a cyclic, orthodiagonal quadrilateral, whose diagonals intersect at $P$. Show that the projections of $P$ on the sides of $ABCD$ form the vertices of a bicentric quadrilateral, and that the circumcircle also passes through the midpoints of the sides of $ABCD$.  

10.5 Bicentric quadrilateral

A quadrilateral is bicentric if it has a circumcircle and an incircle.

10.5.1 Theorem

The circumradius $R$, the inradius $r$, and the the distance $d$ between the circumcenter and the incenter of a bicentric quadrilateral satisfies the relation

\[\frac{1}{r^2} = \frac{1}{(R + d)^2} + \frac{1}{(R - d)^2}.\]

The proof of this theorem is via the solution of a locus problem.

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12Klamkin, Crux 1062. Court called this Brahmagupta’s Theorem.
13Crux 2209; also Crux 1866.
10.5.2 Fuss problem

Given a point $P$ inside a circle $I(r)$, $IP = c$, to find the locus of the intersection of the tangents to the circle at $X, Y$ with $\angle XY = 90^\circ$.

![Diagram showing the Fuss problem](image)

Solution

Let $Q$ be the intersection of the tangents at $X$ and $Y$, $IQ = x$, $\angle PIQ = \theta$. We first find a relation between $x$ and $\theta$.

Let $M$ be the midpoint of $XY$. Since $IXQ$ is a right triangle and $XM \perp IQ$, we have $IM \cdot IQ = IX^2$, and

$$IM = \frac{r^2}{x}.$$ 

Note that $MK = c \sin \theta$, and $PK = IM - c \cos \theta = \frac{r^2}{x} - c \cos \theta$.

Since $PK$ is perpendicular to the hypotenuse $XY$ of the right triangle $PXY$,

$$PK^2 = XK \cdot YK = r^2 - IK^2 = r^2 - IM^2 - MK^2.$$ 

From this, we obtain

$$\left(\frac{r^2}{x} - c \cos \theta\right)^2 = r^2 - \frac{r^4}{x^2} - c^2 \sin^2 \theta.$$ 

\(^{14}\text{See } \S 39 \text{ of Heinrich Dörrie, } 100 \text{ Great Problems of Elementary Mathematics, Dover, 1965.}\)
and, after rearrangement,
\[ x^2 + 2x \cdot \frac{cr^2}{r^2 - c^2} \cdot \cos \theta = \frac{2r^4}{r^2 - c^2}. \]

Now, for any point \( Z \) on the left hand side with \( IZ = d \), we have
\[ ZQ^2 = d^2 + x^2 + 2xd \cos \theta. \]

Fuss observed that this becomes constant by choosing
\[ d = \frac{cr^2}{r^2 - c^2}. \]

More precisely, if \( Z \) is the point \( O \) such that \( OI \) is given by this expression, then \( OQ \) depends only on \( c \) and \( r \):
\[ OQ^2 = \frac{c^2r^4}{(r^2 - c^2)^2} + \frac{2r^4(r^2 - c^2)}{(r^2 - c^2)^2} = \frac{r^4(2r^2 - c^2)}{(r^2 - c^2)^2} \]

This means that \( Q \) always lies on the circle, center \( O \), radius \( R \) given by
\[ R^2 = \frac{r^4(2r^2 - c^2)}{(r^2 - c^2)^2}. \]

**Proof of Theorem**

By eliminating \( c \), we obtain a relation connecting \( R, r \) and \( d \). It is easy to see that
\[ R^2 = \frac{2r^4(r^2 - c^2) + c^2r^4}{(r^2 - c^2)^2} = \frac{2r^4}{r^2 - c^2} + d^2, \]
from which
\[ R^2 - d^2 = \frac{2r^4}{r^2 - c^2}. \]

On the other hand,
\[ R^2 + d^2 = \frac{r^4(2r^2 - c^2)}{(r^2 - c^2)^2} + \frac{c^2r^4}{(r^2 - c^2)^2} = \frac{2r^6}{(r^2 - c^2)^2}. \]

From these, we eliminate \( c \) and obtain
\[ \frac{1}{r^2} = \frac{2(R^2 + d^2)}{(R^2 - d^2)^2} = \frac{1}{(R + d)^2} + \frac{1}{(R - d)^2}, \]
relating the circumradius, the inradius, and the distance between the two centers of a bicentric quadrilateral.
10.5.3 Construction problem

Given a point $I$ inside a circle $O(R)$, to construct a circle $I(r)$ and a bicentric quadrilateral with circumcircle $(O)$ and incircle $(I)$.

Construction

If $I$ and $O$ coincide, the bicentric quadrilaterals are all squares, $r = \frac{R}{\sqrt{2}}$. We shall assume $I$ and $O$ distinct.

1. Let $HK$ be the diameter through $I$, $IK < IH$. Choose a point $M$ such that $IM$ is perpendicular to $IK$, and $IK = IM$.
2. Join $H$, $M$ and construct the projection $P$ of $I$ on $HM$.

The circle $I(P)$ is the required incircle.

10.5.4 Lemma

Let $Q$ be a cyclic quadrilateral. The quadrilateral bounded by the tangents to circumcircle at the vertices is cyclic if and only if $Q$ is orthodiagonal.

Proof. Given a cyclic quadrilateral quadrilateral $XYZW$, let $ABCD$ be the quadrilateral bounded by the tangents to the circumcircle at $X, Y, Z, W$. Since $(\alpha + \gamma) + 2(\theta + \phi) = 360^\circ$, it is clear that $ABCD$ is cyclic if and only if the diagonals $XZ$ and $YW$ are perpendicular.
10.5.5 Proposition

(a) Let $ABCD$ be a cyclic, orthodiagonal quadrilateral. The quadrilateral $XYZW$ bounded by the tangents to the circumcircle at the vertices is bicentric.

(b) Let $ABCD$ be a bicentric quadrilateral. The quadrilateral $XYZW$ formed by the points of contact with the incircle is orthodiagonal (and circumscribable). Furthermore, the diagonals of $XYZW$ intersect at a point on the line joining the circumcenter and the incenter of $ABCD$.

Exercise

1. The diagonals of a cyclic quadrilateral are perpendicular and intersect at $P$. The projections of $P$ on the sides form a bicentric quadrilateral,
the circumcircle of which passes through the midpoints of the sides.

2. Characterize quadrilaterals which are simultaneously cyclic, circumscribable, and orthodiagonal.

3. The diagonals of a bicentric quadrilateral intersect at $P$. Let $HK$ be the diameter of the circumcircle perpendicular to the diagonal $AC$ (so that $B$ and $H$ are on the same side of $AC$). If $HK$ intersects $AC$ at $M$, show that $BP : PD = HM : MK$.

4. Given triangle $ABC$, construct a point $D$ so that the convex quadrilateral $ABCD$ is bicentric.

5. For a bicentric quadrilateral with diagonals $p$, $q$, circumradius $R$ and inradius $r$,

$$\frac{pq}{4r^2} - \frac{4R^2}{pq} = 1.$$
10.5.6

The circumcenter, the incenter, and the intersection of the diagonals of a bicentric quadrilateral are concurrent.

10.6

Consider a convex quadrilateral $ABCD$ whose diagonals $AC$ and $BD$ intersect at $K$. Let $A', B', C', D'$ be the projections of $K$ on the sides $AB, BC, CD, DA$ respectively.

10.6.1 Theorem

The quadrilateral $ABCD$ has a circumcircle if and only if $A'B'C'D'$ has an incircle.

We prove this in two separate propositions.

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20Crux 2149, Romero Márquez.
Proposition A.

Let $ABCD$ be a cyclic quadrilateral, whose diagonals intersect at $K$. The projections of $K$ on the sides of $ABCD$ form the vertices of a circumscriptible quadrilateral.

Proof. Note that the quadrilaterals $KA'AB'$, $KB'BC'$, $KC'CD'$, and $KD'DA'$ are all cyclic. Suppose $ABCD$ is cyclic. Then

$$\angle KAD' = \angle CAD = \angle CBD = \angle B'BK = \angle B'A'K.$$ 

This means $K$ lies on the bisector of angle $D'A'B'$. The same reasoning shows that $K$ also lies on the bisectors of each of the angles $B'$, $C'$, $D'$. From this, $A'B'C'D'$ has an incircle with center $K$.

Proposition B.

Let $ABCD$ be a circumscriptible quadrilateral, with incenter $O$. The perpendiculars to $OA$ at $A$, $OB$ at $B$, $OC$ at $C$, and $OD$ at $D$ bound a cyclic quadrilateral whose diagonals intersect at $O$.

Proof. The quadrilaterals $OAB'B$, $OBC'C$, $OCD'D$, and $ODA'A$ are all cyclic. Note that

$$\angle DOD' = \angle DCD' = \angle BCC'.$$
since \( OC \perp C'D' \). Similarly, \( \angle AOB' = \angle C'B'C' \). It follows that

\[
\angle DOD' + \angle AOD + \angle AOB = \angle
\]

10.6.2

Squares are erected outwardly on the sides of a quadrilateral.

The centers of these squares form a quadrilateral whose diagonals are equal and perpendicular to each other. 21

10.7 Centroids

The centroid \( G_0 \) is the center of

The edge-centroid \( G_1 \)

The face-centroid \( G_2 \):

10.8

10.8.1

A convex quadrilateral is circumscribed about a circle. Show that there exists a straight line segment with ends on opposite sides dividing both the perimeter and the area into two equal parts. Show that the straight line passes through the center of the incircle. Consider the converse. 22

10.8.2

Draw a straight line which will bisect both the area and the perimeter of a given convex quadrilateral. 23

10.9

Consider a quadrilateral \( ABCD \), and the quadrilateral formed by the various centers of the four triangles formed by three of the vertices.

21Crux 1179.
23E992.517.S52?531.(K.Tan)
10.9.1

(a) If \( Q \) is cyclic, then \( Q(O) \) is circumscribable.
(b) If \( Q \) is circumscribable, then \( Q(O) \) is cyclic. 24
(c) If \( Q \) is cyclic, then \( Q(I) \) is a rectangle.
(d) If \( Q \) is cyclic, then the nine-point circles of \( BCD, CDA, DAB, ABC \) have a point in common. 25.

Exercise

1. Prove that the four triangles of the complete quadrangle formed by the circumcenters of the four triangles of any complete quadrilateral are similar to those triangles. 26

2. Let \( P \) be a quadrilateral inscribed in a circle \( (O) \) and let \( Q \) be the quadrilateral formed by the centers of the four circles internally touching \( (O) \) and each of the two diagonals of \( P \). Then the incenters of the four triangles having for sides the sides and diagonals of \( P \) form a rectangle inscribed in \( Q \). 27

10.10

10.10.1

The diagonals of a quadrilateral \( ABCD \) intersect at \( P \). The orthocenters of the triangle \( PAB, PBC, PCD, PDA \) form a parallelogram that is similar to the figure formed by the centroids of these triangles. What is “centroids” is replaced by circumcenters? 28

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24E1055.532.S538.(V. Thébault)
25Crux 2276
26E619.444.S451. (W.B.Clarke)
27Thébault, AMM 3887.38.S837. See editorial comment on 837.p486.
28Crux 1820.
10.11 Quadrilateral formed by the projections of the intersection of diagonals

10.11.1
The diagonal of a convex quadrilateral $ABCD$ intersect at $K$. $P, Q, R, S$ are the projections of $K$ on the sides $AB, BC, CD, and DA$. Prove that $ABCD$ is cyclic if $PQRS$ is circumscriptible.  

10.11.2
The diagonals of a convex quadrilateral $ABCD$ intersect at $K$. $P, Q, R, S$ are the projections of $K$ on the sides $AB, BC, CD, and DA$. Prove that if $KP = KR$ and $KQ = KS$, then $ABCD$ is a parallelogram.

10.12 The quadrilateral $Q_{(\text{center})}$

10.12.1
If $Q(I)$ is cyclic, then $Q$ is circumscriptible.

10.12.2 The Newton line of a quadrilateral
$L$ and $M$ are the midpoints of the diagonals $AC$ and $BD$ of a quadrilateral $ABCD$. The lines $AB, CD$ intersect at $E$, and the lines $AD, BC$ intersect at $F$. Let $N$ be the midpoint of $EF$. Then the points $L, M, N$ are collinear.
Proof. Let $P, Q, R$ be the midpoints of the segments $AE, AD, DE$ respectively. Then $L, M, N$ are on the lines $PQ, QR, RP$ respectively. Apply the Menelaus theorem to the transversal $BCF$ of $\triangle EAD$.

\textsuperscript{29}Crux 2149.
\textsuperscript{30}W.Pompe, Crux 2257.
\textsuperscript{31}Seimiya, Crux 2338.
Exercise

1. Suppose $ABCD$ is a plane quadrilateral with no two sides parallel. Let $AB$ and $CD$ intersect at $E$ and $AD, BC$ intersect at $F$. If $M,N,P$ are the midpoints of $AC, BD, EF$ respectively, and $AE = a \cdot AB, AF = b \cdot AD$, where $a$ and $b$ are nonzero real numbers, prove that $MP = ab \cdot MN$.

2. The Gauss-Newton line of the complete quadrilateral formed by the four Feuerbach tangents of a triangle is the Euler line of the triangle.

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32 AMM E3299.8810.
33 AMM 4549.537.S549. (R.Obláth).