Algebraic Topology

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Chapter 1

General topology

The topology of a space determines which functions from or into the space are continuous. It consists of a family of subsets called *open sets* subject to the condition conditions:
(1) the intersection of a finite number of open sets is open;
(2) the union of an arbitrary number of open sets is open.

The complements of open sets are called *closed sets*. A function $f : X \longrightarrow Y$ between two topological spaces is *continuous* if $f^{-1}(U)$ is open in X for every open set $U \subset Y$.

A topology on X is usually defined by specifying
(i) a *neighborhood system* $\{U_x : x \in X\}$ at each point $X \in X$, a neighborhood of $x \in X$ being a set containing an open set containing x, or
(ii) a *basis*, a collection of (basic) open sets whose arbitrary unions give all the open sets, or
(iii) a *subbasis* consisting of a collection of (subbasic) open sets whose finite intersections constitute a basis.

The topology of the real line, for example, is defined as follows. A subset $U \subset \mathbb{R}$ is open if for every $x \in U$ there exists $\varepsilon > 0$ such that the open interval $(x - \varepsilon,\ x + \varepsilon)$ is contained in U. A subset $U \subset \mathbb{R}$ is open if and only if it is a *countable* union of disjoint open intervals.

More generally, a metric space $(X,\ d)$ has a natural metric topology: a subset $U \subset X$ is open if for each $x \in U$, there is a disk $\mathbb{D}_x(r) := \{y \in X : d(x, y) < r$ which is contained in U. For $n \geq 2$, in the euclidean topology of $\mathbb{R}^n$,
(i) the family of open disks, $\{\mathbb{D}_x(r) := \{y \in \mathbb{R}^n : ||x - y|| < r\}\}$, is a basis,

(ii) the family of open cuboids, $\{\prod_{j=1}^{n}[a_j,\ b_j] : a_j \leq b_j\}$ is also a basis.

1.1 Separation axioms

A topological space is
(1) T_1 if every singleton subset $\{x\}$ is closed;
(2) *Hausdorff* (T_2) if it is T_1 and for distinct $x, y \in X$ are are disjoint open sets U, V containing x, y separately;
(3) *regular* (T_3) if it is T_1 and if C is a closed set not containing x, there are disjoint open sets U and V containing x and C separately;
(4) *normal* (T_4) if it is T_1 and if C and C' are disjoint closed sets, there are disjoint open sets U and V containing C and C' separately.

$$T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1.$$

Proposition 1.1. (1) [Urysohn lemma] *Let A_0 and A_1 be disjoint closed subsets of a normal space X. There exists a continuous function $f : X \longrightarrow [0, 1]$ such that $f(A_0) = \{0\}$ and $f(A_1) = \{1\}$.*

(2) [Tietze extension theorem *Let A be a closed subspace of a normal space X. Every continuous function $A \longrightarrow [0, 1]$ has an extension over X.*

1.2 Compactness

Let X be a topological space. A subset $K \subset X$ is *compact* if every open cover has a finite subcover. In a metric space, a compact set is a *sequentially compact*: every infinite sequence contains a convergent subsequence.

Proposition 1.2. (1) *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

(2) *A continuous map from a compact space to a Hausdorff space is a closed map.*

(3)(Lebesque covering lemma) *Every open cover of a compact subset of a metric space has a positive Lebesque number,* i.e., *if $\{U_i : i \in I\}$ is an open cover of a compact subset A in a metric space X, there exists $\varepsilon > 0$ such that every subset of A of diameter $< \varepsilon$ is contained in some U_i, $i \in I$.*

1.3 Connectedness and path-connectedness

Let X be a topological space. A subset $C \subset X$ is *disconnected* if it is the union of two disjoint nonempty open set. A connected set of the real line $\mathbb{R}^1$ is an interval. X is path-connected if every two points $x,\ y \in X$ can be joined by a path, *i.e.*, there exists a continuous map $\alpha : [0,1] \longrightarrow X$ such that $\alpha(0) = x$ and $\alpha(1) = y$. A path-connected space is connected. But the converse is not true.

1.4 Compact-open topology on function spaces

Given topological spaces X and Y, denote by Y^X the set of continuous functions $X \longrightarrow Y$. The compact-open topology on Y^X is defined by taking as a subbasis the family

$$\{N(C,U) : C \subset X \text{ compact},\ U \subset Y \text{ open}\}$$

of subsets. Here,

$$N(C,U) := \{f \in Y^X : f(C) \subset U\}.$$

Thus, an open set of Y^X is a union of finite intersections of sets of the form $N(C,U)$.

Proposition 1.3. (1) *Given a map* $g : Y \longrightarrow Z$*, the maps*
(i) $g^X : X^Z \longrightarrow X^Y$ *defined by* $g^X(f) = f \circ g$*, and*
(ii) $g_X : Y^X \longrightarrow Z^X$ *defined by* $g_X(f) = g \circ f$ *are continuous.*

(2) *If* Y *is locally compact, Hausdorff, the evaluation map* $ev : Z^Y \times Y \longrightarrow Z$ *defined by*

$$ev(f,y) = f(y)$$

is continuous.

(3) *If* X *and* Y *are Hausdorff and* Y *is locally compact, the exponential map* $e : Z^{X\times Y} \longrightarrow (Z^Y)^X$ *given by*

$$e(f)(x)(y) = f(x,y)$$

is a homeomorphism.

Chapter 2

Homeomorphism

2.1 Homeomorphism

Two topological spaces X and Y are homeomorphic if there are continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that the composites $g \circ f : X \longrightarrow X$ and $f \circ g : Y \longrightarrow Y$ are both identities.

Examples

(1) Any two open intervals are homeomorphic. Here is a homeomorphism between (a, b) and (a', b'):

$$f(x) = a' + \frac{b' - a'}{b - a}(x - a).$$

Likewise, any two closed intervals are homemorphic.

(2) The intervals $[0, 1]$ and $(0, 1)$ are not homeomorphic, since one is compact, and the other is not.

(3) For fixed positive integers r and s, let $M_{r,s}$ be the set of all $r \times s$ matrices over $\mathbb{R}$. There is an obvious bijection $M_{r,s} \longrightarrow \mathbb{R}^{rs}$, which we use to endow $M_{r,s}$ with the euclidean topology. [1]

Let X be a topological space. A subset $A \subset X$ is said to inherit the subspace topology of X (or simply a subspace of X) if its open sets are of the form $U \cap A$ for U open in X.

Proposition 2.1. *If $f : X \longrightarrow Y$ is a homeomorphism, and $U \subset X$, then f induces a homeomorphism $X \setminus U \longrightarrow Y \setminus f(U)$.*

[1] This means that we shall regard $M_{r,s}$ as homeomorphic to $\mathbb{R}^{rs}$.

In this course, most of the spaces we consider are subspaces of euclidean spaces, or constructed from such in some natural ways. One of the fundamental theorems on euclidean spaces is the classification of euclidean spaces according to their dimensions.

THEOREM (Invariance of dimension). The euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic (if and) only if $m = n$.

A key idea of algebraic topology is to translate such a geometric (or topological) problem into an algebraic problem, and to show, by calculations, that the algebraic problem has no solution.

2.2 The unit cube, unit disk, and standard n-simplex

We consider three fundamental subspaces of $\mathbb{R}^n$ with their boundaries.

(1) The unit cube

$$I^n := \{(x_1, \dots, x_n) \in \mathbb{R}^n : \; 0 \le x_i \le 1, \; i = 1, 2, \dots, n\}$$

with boundary

$$\partial I^n := \{(x_1, \dots, x_n) \in I^n : \; x_i = 0 \text{ or } 1 \text{ for some } i = 1, 2, \dots, n\}.$$

(2) The standard n-simplex

$$\Delta^n := \{(x_1, \cdots, x_n) \in \mathbb{R}^n : 0 \le x_1, \dots, x_n, \; x_1 + \cdots + x_n \le 1\}$$

with boundary

$$\partial \Delta^n := \{(x_1, \dots, x_1) \in \Delta^n : \; x_1 + \cdots + x_n = 1 \text{ or } x_i = 0 \text{ for some } i\}.$$

(3) The unit disk

$$\mathbb{D}^n := \{x \in \mathbb{R}^n : ||x|| \le 1\}$$

with its boundary sphere

$$\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : ||x|| = 1\}.$$

These spaces are homeomorphic. We construct an explicit homeomorphism $I^n \longrightarrow \mathbb{D}^n$. Clearly, I^n is homeomorphic to $[-1,1]^n$, which is the union of n "double wedges"

$$K_j^n := \{x \in I^n : |x_i| \leq |x_j| \text{ for } i = 1, \ldots, n\}$$

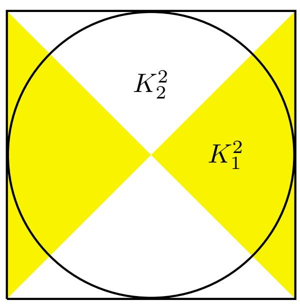
for $j = 1, 2, \ldots, n$. An obvious homeomorphism $f_j^n : K_j^n \longrightarrow K_j^n \cap \mathbb{D}^n$ is given by

$$f_j^n(x) = \begin{cases} \frac{|x_j|}{||x||} \cdot x, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Note that $f_i|_{K_i \cap K_j} = f_j^n|_{K_i \cap K_j}$. Therefore, these homeomorphisms together define a homeomorphism

$$\varphi_n : I^n \xrightarrow{\approx} [-1,1]^n = \bigcup_{j=1}^n K_j^n \longrightarrow \bigcup_{j=1}^n K_j^n \cap \mathbb{D}^n = \mathbb{D}^n.$$

This homeomorphism maps the boundary of I_n onto the sphere $\mathbb{S}^{n-1}$.



We write this map as

$$\varphi_n : (I^n, \partial I^n) \longrightarrow (\mathbb{D}^n, \mathbb{S}^{n-1}).$$

This is a very useful homeomorphism throughout this course.

2.3 The sphere $\mathbb{S}^n$

The unit vectors of $\mathbb{R}^{n+1}$ constitute the sphere $\mathbb{S}^n$:

$$\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : ||x|| = 1\}.$$

The upper and lower hemispheres of $\mathbb{S}^n$ are the subspaces

$$\mathbb{E}^n_+ := \{(x_1, \dots, x_{n+1}) \in \mathbb{S}^n :\ x_{n+1} \geq 0\},$$
$$\mathbb{E}^n_- := \{(x_1, \dots, x_{n+1}) \in \mathbb{S}^n :\ x_{n+1} \leq 0\}.$$

Clearly,

$$\mathbb{E}^n_+ \cup \mathbb{E}^n_- = \mathbb{S}^n, \qquad \mathbb{E}^n_+ \cap \mathbb{E}^n_- = \mathbb{S}^{n-1}.$$

Each of these is homeomorphic to $\mathbb{D}^n$. In each case, the projection onto "the equatorial subspace" is a homeomorphism.

$$p_\pm : (\mathbb{E}^n_\pm,\ \mathbb{S}^{n-1}) \longrightarrow (\mathbb{D}^n, \mathbb{S}^{n-1}).$$

The inverse homeomorphisms $f_\varepsilon : \mathbb{D}^n \longrightarrow \mathbb{E}^n_\pm$, $\varepsilon = \pm 1$, are given by

$$f_\varepsilon(x) = (x,\ \varepsilon\sqrt{1 - ||x||^2}).$$

2.3.1 Stereographic projection

Stereographic projection from the "north pole" $e_{n+1} \in \mathbb{S}^n$ onto the "equatorial" hyperplane gives a homeomorphism. Every point on $\mathbb{S}^n \setminus \{e_{n+1}\}$ is uniquely of the form $\cos\theta \cdot e_{n+1} + \sin\theta \cdot v$ for some unit vector $v \in e_{n+1}^\perp$. Stereographic projection from e_{n+1} sends this vector to $\cot\frac{\theta}{2} \cdot v$.

Chapter 3

Quotient topology and one-point compactification

3.1 Quotient space

Let X be a topological space with an equivalence relation R. The quotient topology on X/R is characterized by the following: a function $X/\mathsf{R} \longrightarrow Y$ is continuous if and only if the composite $X \longrightarrow X/\mathsf{R} \longrightarrow Y$ is continuous. [1]

Proposition 3.1. *Let X and Y be topological spaces with equivalence relations R and S respectively. A map $f : X \longrightarrow Y$ induces a map $\tilde{f} : X/\mathsf{R} \longrightarrow Y/\mathsf{S}$ if and only if $x\mathsf{R}x' \Rightarrow f(x)\mathsf{S}f(x')$.*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle p_X}\downarrow & & \downarrow{\scriptstyle f_Y} \\ X/\mathsf{R} & \xrightarrow[\ \tilde{f}\ ]{} & Y/\mathsf{S} \end{array}$$

3.2 The projective spaces

The real projective space $\mathbb{R}\mathrm{P}^n$ is obtained by identifying antipodal points of the sphere $\mathbb{S}^n$. The relation R on $\mathbb{S}^n$ given by $x\mathsf{R}y$ if and only if

[1]This is equivalent to: $U \subset X/\mathsf{R}$ is open (closed) if and only if $\pi^{-1}(U) \subset X$ is open (closed).

$x = \pm y$ is an equivalence relation. Each equivalence class consists of a pair of antipodal points $\{x, -x\} \subset \mathbb{S}^n$.

The real projective space $\mathbb{R}\mathrm{P}^n$ can also be viewed as the space of (punctured) lines in $\mathbb{R}^{n+1}$. This means that on $\mathbb{R}^{n+1} \setminus \{0\}$ we define a relation R' by $x\mathsf{R}'y$ if and only if $x = ky$ for some nonzero $k \in \mathbb{R}$. This is an equivalence relation. The inclusion map $\mathbb{S}^n \longrightarrow \mathbb{R}^{n+1}$ respects the relation R: clearly, $x\mathsf{R}y$ in $\mathbb{S}^n$ implies $x\mathsf{R}'y$ in $\mathbb{R}^{n+1}$. This induces a bijection $\mathbb{S}^n/\mathsf{R} \longrightarrow \mathbb{R}^{n+1}/\mathsf{R}'$, which is a homeomorphism.

3.2.1 Complex projective spaces

On the complex vector space $\mathbb{C}^{n+1}$ the equivalence relation

$$z \;\mathsf{R}\; z' \text{ if and only if } z = kz' \text{ for some nonzero } k \in \mathbb{C}$$

yields the complex projective space $\mathbb{C}\mathrm{P}^n$.

The complex projective space $\mathbb{C}\mathrm{P}^1$ is homeomorphic to $\mathbb{S}^2$.

3.3 Collapsing a closed subspace

Let $A \subset X$ be a closed subspace. By identifying points in A we obtain a space X/A with the quotient topology. [2]

Proposition 3.2. $\mathbb{E}^n_+/\mathbb{S}^{n-1}$ *is homeomorphic to* $\mathbb{S}^n$.

Proof. Every point in $\mathbb{E}^n_+$ can be written as $\cos\theta e_{n+1} + \sin\theta v$ for a unit vector $v \in \mathbb{S}^{n-1}$, and unique $\theta \in [0, \frac{\pi}{2}]$. The map which sends this to $\cos 2\theta e_{n+1} + \sin 2\theta v$ maps the equator in $-e_{n+1}$, and induces a continuous bijection $\mathbb{E}^n_+/\mathbb{S}^{n-1} \longrightarrow \mathbb{S}^n$, which is necessarily a homemorphism. $\square$

3.4 One-point compactification

Let X be a topological space, The one-point compactification of X is

$$X^\infty := X \cup \{\infty\}, \qquad \infty \notin X,$$

topologized by stipulating that $U \in X^\infty$ is open if
(i) $\infty \notin U$ and U is open in X, or
(ii) $\infty \in U$ and $X^\infty \setminus U$ is closed and compact in X.

[2] This means that we define a relation on X by $x\mathsf{R}y$ if and only if either $x, y \in A$ or $x, y \notin A$. This is an equivalence relation. The equivalence classes are the set A, and points in $X \setminus A$.

Then X^∞ is compact, and X is dense in X^∞.

Proposition 3.3. $(\mathbb{R}^n)^\infty \equiv \mathbb{S}^n$.

Proof. Consider $\mathbb{R}^n$ imbedded in $\mathbb{R}^{n+1}$ and take as ∞ the "north pole" of $\mathbb{S}^n$, namely, $N = (0, \dots, 0, 1)$. The stereographic projection $f : (\mathbb{R}^n)^\infty \longrightarrow \mathbb{S}^n$ given by

$$f(x) = \begin{cases} \frac{1}{||x||^2+1}\left(2x,\ ||x||^2 - 1\right), & x \neq N, \\ N, & x = N, \end{cases}$$

is a continuous bijection from a compact space to a Hausdorff space. Hence it is a homeomorphism. It is enough to check the continuity at the "infinite point". If U is an open neighbourhood of $N \subset \mathbb{S}^n$, its complement is a closed subset of $\mathbb{S}^n$ not containing N, and is compact. Under f, this corresponds to a compact subset of $\mathbb{R}^n$. This shows that f is continuous at N. □

Proposition 3.4. *(1) X^∞ is Hausdorff if X is locally compact and Hausdorff.*

(2) If X is compact Hausdorff and $x \in X$, then $(X \setminus \{x\})^\infty \equiv X$.

Proposition 3.5. *If X is Haudorff and regular* [3]*, $U \subset X$ open, and $\overline{U}$ compact, then*

$$X/(X \setminus U) \equiv U^\infty.$$

Proof. We construct a homeomorphism $f : U^\infty \longrightarrow X/(X\setminus U)$ by setting

$$f|U = U \xrightarrow{\subset} X \xrightarrow{p} X/(X \setminus U)$$

and $f(\infty) = [X \setminus U]$. Clearly, f is a bijection. Since it maps a compact space to a Hausdorff space, it is a homeomorphism if it is continuous. To prove this, we need only check the neighborhoods of $[X \setminus U]$. If V is an open neighborhood of $[X \setminus U]$, then $W = p^{-1}(V)$ is an open set in X, and $\overline{U} \setminus W$ is compact, being a closed set in the compact space $\overline{U}$. Now,

$$f^{-1}(V) \setminus \{\infty\} = W \cap U = U \setminus (\overline{U} \setminus W)$$

is open in U. This shows that $f^{-1}(V)$ is open in U^∞, and f is continuous. □

[3] A space is regular if each neighborhood of a point contains a closed neighborhood of the point. A compact Hausdorff space is regular.

For example, if U is a bounded open subset of $\mathbb{R}^n$, then

$$U^\infty \equiv \mathbb{R}^n/(\mathbb{R}^n \setminus U).$$

Proposition 3.6.

$$\mathbb{D}^n/\mathbb{S}^{n-1} \equiv \mathbb{S}^n.$$

Proof.

$$\mathbb{D}^n/\mathbb{S}^{n-1} \equiv I^n/\partial I^n = I^n/(I^n \setminus \mathrm{int} I^n) \equiv (\mathrm{int} I^n)^\infty \equiv (\mathbb{R}^n)^\infty \equiv \mathbb{S}^n.$$

□

Chapter 4

Homotopy

Two maps $f, g : X \longrightarrow Y$ are homotopic if there exists a continuous map $H : X \times I \longrightarrow Y$ (called a homotopy) such that

$$H(x,0) = f(x), \qquad H(x,1) = g(x)$$

for $x \in X$.

Proposition 4.1. *Homotopy is an equivalence relation on* Y^X.

$[X, Y]$ denotes the set of homotopy classes of maps from X to Y.

Examples

(1) If Y is a one-point space, there is only one mapping $f : X \longrightarrow Y$, and $[X, Y]$ is a singleton.

(2) If X is a one-point space, Y^X is homemorphic to Y. Two maps $f, g : X \longrightarrow Y$ are homotopic if and only if $f(x)$ and $g(x)$ are in the same path component of Y. $[X, Y]$ is in one-to-one correspondence with the path components of Y.

(3) Every map $f : \mathbb{R} \longrightarrow Y$ is homotopic to a constant map. Here is a null-homotopy $H : \mathbb{R} \times I \longrightarrow Y$:

$$H(x,t) = f(tx), \quad x \in \mathbb{R},\ t \in I.$$

The same reasoning shows that every map $\mathbb{R}^n \longrightarrow Y$ is null-homotopic (or homotopically trivial).

4.1 Homotopy equivalence

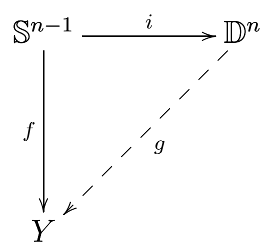
A map $f : X \longrightarrow Y$ is a *homotopy equivalence* if there exists a map $g : Y \longrightarrow X$ such that

$$g \circ f \sim \iota_X \quad \text{and} \quad f \circ g \sim \iota_Y .$$

In this case, g is said to be a *homotopy inverse* of f, and the spaces X and Y are said to be *homotopy equivalent*.

A space is *contractible* if it is homotopy equivalent to a one-point space. Show that X is contractible if and only if every map $f : X \longrightarrow Y$ is homotopic to a constant map. Every euclidean space $\mathbb{R}^n$ is contractible.

Proposition 4.2. *A map $f : \mathbb{S}^{n-1} \longrightarrow Y$ is null-homotopic if and only if it has an extension $g : \mathbb{D}^n \longrightarrow Y$.*



Proof. ($\Rightarrow$) If $H_t : \mathbb{S}^{n-1} \times I \longrightarrow Y$ is a null-homotopy such that $H_0(x) = y_0$ and $H_1(x) = f(x)$, then $g(x) = H_{||x||}(x)$ defines an extension of f to $\mathbb{D}^n$. Conversely, given such an extension, the homotopy $H_t(x) = g(tx)$ clearly satisifies $H_0(t) = g(0)$ and $H_1(x) = g(x) = f(x)$. □

Homeomorphic spaces are homotopy equivalent. The converse is not true.

Proposition 4.3. (1) *Let $p \in \mathbb{S}^n$. $\mathbb{S}^n - \{p\}$ is homeomorphic to $\mathbb{R}^n$.*

(2) *Let $q \in \mathbb{R}^{n+1}$. $\mathbb{R}^{n+1} - \{q\}$ is homotopy equivalent to $\mathbb{S}^n$.*

Proof. (1) See §2.3.1.

(2) $\mathbb{S}^n$ is a deformation retract of $\mathbb{R}^{n+1} - \{q\}$, *i.e.*, $\rho : \mathbb{R}^{n+1} - \{q\} \longrightarrow \mathbb{S}^n$ defined by $\rho(x) = \frac{x}{||x||}$ satisfies

$$\rho \circ i = \iota \text{ and } i \circ \rho \sim \iota.$$

□

4.2 Nullhomotopy of $\mathbb{S}^m \longrightarrow \mathbb{S}^n$ for $m < n$

Theorem 4.4. *If $m < n$, every map $f : \mathbb{S}^m \longrightarrow \mathbb{S}^n$ is nullhomotopic.*

Proof. Regard $\mathbb{S}^m$ as the boundary ∂K of an $(m+1)$-simplex. Subdivide ∂K by iterated barycentric subdivision until each closed $(n+1)$-simplex is mapped into an open hemisphere of $\mathbb{S}^n$. This is possible by the Lebegue covering theorem. Let ∂K be this subdivision. We construct a map $g : \partial K' \longrightarrow \mathbb{S}^n$ agreeing with f on the vertices, such that g is a homeomorphism on each face of $\partial K'$ and $g \sim f$.

First, define $h : \partial K' \longrightarrow \mathbb{R}^{n+1}$ by f on vertices and extend linearly over each simplex, *i.e.*, for $x = \sum_{i=1}^{l} \lambda_i e_i$, put $h(x) = \sum_{i=1}^{l} \lambda_i f(e_i)$. Then define $g(x) = \frac{h(x)}{||h(x)||}$. Note that $h(x)$ is never zero since the simplex $[f(e_1),\ \ldots,\ f(e_l)]$ is contained in an open hemisphere and its convex hull avoids the origin.

It is easy to construct a homotopy $f \sim g$. In fact,

$$H_t(x) = \frac{(1-t)f(x) + tg(x)}{||(1-t)f(x) + tg(x)||}$$

will do. We need only make sure that it is well defined. Since $f(x)$ and $g(x)$ lie in the same open hemisphere, the segment joining them avoids the origin.

Since g is a homeomorphism on each simplex of $\partial K'$, its image is a finite union of m-cells and cannot be $\mathbb{S}^n$ if $m < n$. Let $z \in \mathbb{S}^n$ be a point on in its image. Then g factors through a contractible space and is nullhomotopic. This shows that f is nullhomotopic. ☐

THEOREM. The spheres $\mathbb{S}^{m-1}$ and $\mathbb{S}^{n-1}$ are homotopy equivalent if and only if $m = n$.

Chapter 5

Homotopy functors and their applications

5.1 Categories and functors

Here are some examples of categories. Each consists of objects and morphisms between them.

Category	Objects	Morphisms
$\mathbf{Set}$	sets	functions
$\mathbf{Top}$	topological spaces	continuous functions
$\mathbf{Top}_0$	topological spaces with basepoints	basepoint preserving maps
$\mathbf{Grp}$	groups	homomorphisms
$\mathbf{Ab}$	abelian groups	homomorphisms
$\mathbf{Ring}$	rings	homomorphisms
	vector spaces over $\mathbb{R}$	linear maps
Vect$(\mathbb{R})$	real vector bundles	bundle maps

5.1.1 Covariant and contravariant functors

Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *covariant functor* $\mathcal{F} : \mathcal{C} \longrightarrow \mathcal{D}$ is defined by
(i) $\mathcal{F}(X) \in \mathcal{D}$ for each object $X \in \mathcal{C}$, and
(ii) $f_* : \mathcal{F}(X) \longrightarrow \mathcal{F}(Y)$ in $\mathcal{D}$ for each $f : X \longrightarrow Y$ in $\mathcal{C}$.

These are subject to
(iii) $(g \circ f)_* = g_* \circ f_*$,
(iv) $(\iota_X)_* = \iota_{\mathcal{F}(X)}$.

There are also *contravariant functors* defined analogously, with (ii)

and (iii) replaced by
(ii') $f^* : \mathcal{F}(Y) \longrightarrow \mathcal{F}(X)$ in $\mathcal{D}$ for each $f : X \longrightarrow Y$ in $\mathcal{C}$;
(iii) $(g \circ f)^* = f^* \circ g^*$.

A functor $\mathcal{F} : \mathbf{Top}_0 \longrightarrow \mathcal{D}$ is a homotopy functor if $f \sim g \Rightarrow f_* = g_*$.

Here are some examples of homotopy functors between such categories.

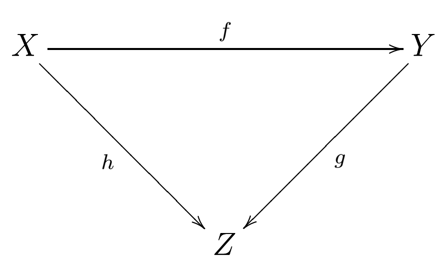
Functor	between categories	
Fundamental group	$\pi_1 : \mathbf{Top}_0 \longrightarrow \mathbf{Grp}$	
Homotopy	$\pi_n : \mathbf{Top}_0 \longrightarrow \mathbf{Ab}$	
Homology	$H_n : \mathbf{Top} \longrightarrow \mathbf{Ab}$	
cohomology	$H^* : \mathbf{Top} \longrightarrow \mathbf{Ring}$	contravariant

5.2 An example: Brouwer's fixed point theorem

The translation of a topological problem into an algebraic one is through the application of a **functor** from the category $\mathbf{Top}$ of topological spaces and continuous maps into, for example, the category $\mathbf{Ab}$ of abelian groups and homomorphisms. Let n be a positive integer. Take for granted that there is one such functor H_n, called the n-th homology functor (with integer coefficients), which associates
(i) with each topological space X an abelian group $H_n(X)$, and
(ii) with each continuous map $f : X \longrightarrow Y$ a homomorphism $H_n(f) : H_n(X) \longrightarrow H_n(Y)$, such that
(iii) for each space X, $H_n(\iota_X) = \iota_{H_n(X)}$, and
(iv) for $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$,

$$H_n(g \circ f) = H_n(g) \circ H_n(f).$$

In other words, a commutative diagram



in topology induces one in algebra:

$$\begin{array}{ccc} H_n(X) & \xrightarrow{H_n(f)} & H_n(Y) \\ & {\scriptstyle H_n(h)} \searrow \quad \swarrow {\scriptstyle H_n(g)} & \\ & H_n(Z) & \end{array}$$

It is known that

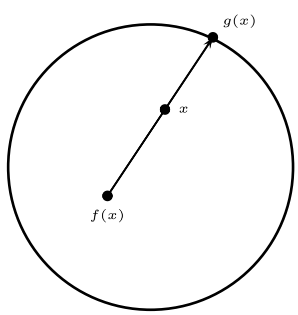
$$H_n(\mathbb{S}^m) = \begin{cases} \mathbb{Z}, & m = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

Also,

$$H_n(\mathbb{D}^m) = 0.$$

Theorem 5.1 (Brouwer's fixed point theorem). *Every map* $f : \mathbb{D}^{n+1} \longrightarrow \mathbb{D}^{n+1}$ *must have a fixed point.*

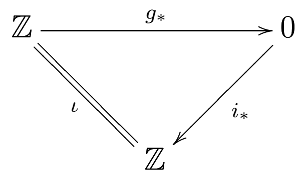
Proof. Suppose, for a contradiction, that $f : \mathbb{D}^{n+1} \longrightarrow \mathbb{D}^{n+1}$ does not have a fixed point. We can define a map $g : \mathbb{D}^{n+1} \longrightarrow \mathbb{S}^n$ by letting $g(x)$ be the intersection of the boundary sphere with the half line joining $f(x)$ to the distinct point x.



Clearly, $g(x) = x$ for $x \in \mathbb{S}^n$. This means we have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}^n & \xrightarrow{g} & \mathbb{D}^{n+1} \\ & {\scriptstyle \iota} \,\|\!\searrow \quad \swarrow {\scriptstyle i} & \\ & \mathbb{S}^n & \end{array}$$

This induces, by the functor H_n, the following commutative diagram



which is clearly absurd since the identity map $\mathbb{Z} \longrightarrow \mathbb{Z}$ cannot factor through the trivial group 0. This contradiction shows that f must have a fixed point. □

Chapter 6

Euclidean spaces and their linear maps

We shall consider the euclidean space $\mathbb{R}^n$ equipped with an inner product

$$\langle x,\ y\rangle := xy^t$$

for $x,\ y \in \mathbb{R}^n$. Two vectors $x, y \in \mathbb{R}^n$ are orthogonal if $\langle x,\ y\rangle = 0$. A basis of $\mathbb{R}^n$ consisting of mutually orthogonal unit vectors is called an orthonormal basis. A most convenient orthonormal basis of $\mathbb{R}^n$ is the canonical basis

$$e_1,\ e_2,\ \ldots,\ e_n$$

in which, for $k \leq n$, e_k is the vector in $\mathbb{R}^n$ with a single nonzero entry 1 in its k-th position. Thus,

$$\begin{aligned} e_1 =& (1, 0, \ldots, 0), \\ e_2 =& (0, 1, \ldots, 0), \\ &\vdots \\ e_n =& (0, 0, \ldots, 0). \end{aligned}$$

For $k \leq m \leq n$, we shall identify e_k in $\mathbb{R}^m$ as e_k in $\mathbb{R}^n$. In fact, there is a filtration of euclidean spaces

$$\mathbb{R}^1 \subset \mathbb{R}^2 \subset \cdots \subset \mathbb{R}^n \subset \cdots$$

for which we can speak of the (co-)limit

$$\mathbb{R}^\infty := \bigcup_{n\geq 1} \mathbb{R}^n$$

consisting of vectors each of finitely many nonzero components. The space $\mathbb{R}^\infty$ is given the colimit topology. [1]

6.1 Linear maps on euclidean spaces

6.1.1 The matrix of a linear map

Let $f : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a linear map, with

$$f(e_i) = \sum_{j=1}^{n} a_{i,j} e_j, \qquad i = 1, 2, \ldots, m.$$

It is conveniently represented by the $m \times n$ matrix

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} \\ & \cdots & & \\ a_{m,1} & a_{m,2} & \ldots & a_{n,n} \end{pmatrix}$$

If $x = \sum_{i=1}^m x_i e_i$ is abbreviated to the $1 \times m$ matrix $x = \begin{pmatrix} x_1 & x_2 & \ldots & x_m \end{pmatrix}$, then $f(x) = xA$.

6.1.2 The adjoint of a linear map

The adjoint of f is the linear map $f^\# : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by

$$\langle f(x),\ y\rangle = \langle x,\ f^\#(y)\rangle, \quad x \in \mathbb{R}^m,\ y \in \mathbb{R}^n.$$

The adjoint $f^\#$ is represented by the transpose matrix of A.

A map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is self-adjoint if

$$\langle f(x),\ y\rangle = \langle x,\ f(y)\rangle, \quad x,\ y \in \mathbb{R}^n.$$

A self-adjoint map is represented by a symmetric matrix, *i.e.*, $a_{i,j} = a_{j,i}$ for $1 \le i,\ j \le n$.

[1] This means that a map $f : \mathbb{R}^\infty \longrightarrow Y$ is continuous if and only if each $f \circ \iota_n : \mathbb{R}^n \longrightarrow \mathbb{R}^\infty \longrightarrow Y$ is continuous.

6.1.3 Determinant

The determinant of a linear map $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is the unique real number $\det f$ satisfying

$$f(v_1) \wedge \cdots \wedge f(v_n) = \det f \cdot (v_1 \wedge \cdots \wedge v_n).$$

6.2 The spectral theorem on self-adjoint linear maps

Theorem 6.1. *Let A be a real $n \times n$ symmetric matrix. There is an orthonormal basis of eigenvectors $v_1, \ldots, v_n$ with corresponding real eigenvalues.*

Proof. Consider the real valued function $F : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}$ defined by

$$F(x) = \langle xA,\ x\rangle.$$

Since the domain is compact, F assumes a maximum value λ, say, at $v \in \mathbb{S}^{n-1}$. We show that $vA = \lambda v$ (so that v is an eigenvector with real eigenvalue λ). From $f(v) = \lambda$, we have $\langle vA - \lambda v,\ v\rangle = 0$. This means $vA = \lambda v + w$ for some $w \perp v$. Since A is symmetric,

$$\langle wA,\ v\rangle = \langle w,\ vA\rangle = \langle w,\ \lambda v + w\rangle = \langle w,\ w\rangle.$$

If $w \neq 0$, then for arbitrary $t \in \mathbb{R}$, $F(v + tw) \leq \lambda$ implies

$$t^2\langle wA,\ w\rangle \leq (\lambda t^2 - 2t)\langle w,\ w\rangle$$

for arbitrary $t \in \mathbb{R}$. Consequently, $\langle w,\ w\rangle = 0$ and $w = 0$, contradicting the assumption $w \neq 0$.

This shows that $w = 0$, $vA = \lambda v$, and v is a unit eigenvector of A.

The same reasoning applied to the orthogonal complement of v shows that there is an eigenvector v_2 of A corresponding to an eigenvalue λ_2 which is the maximum of F restricted to $\mathbb{S}^{n-1} \cap v^\perp$. By induction, we obtain an orthonormal basis $v_1,\ v_2,\ \ldots,\ v_n$ of unit eigenvectors of A with corresponding eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ such that λ_1 is the maximum of F, and for $k = 2, \ldots, n$, λ_k is the maximum of F on $\mathbb{S}^{n-1} \cap$ span$(v_1, \ldots, v_{k-1})^\perp$. □

6.3 Isometries of $\mathbb{R}^n$

Let $\tau : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an isometry, *i.e.*, $||\tau(x)|| = ||x||$ for every $x \in \mathbb{R}^n$.

The only possible real eigenvalues of τ are $\varepsilon = \pm 1$. For each ε, we have an invariant subspace $V_\varepsilon := \{x \in \mathbb{R}^n : \tau(x) = \varepsilon x\}$.

$V := V_+ \bigoplus V_-$ is also an invariant subspace of τ. If $V \neq \mathbb{R}^n$, then τ has a complex (non-real) eigenvalue, corresponding to which is an invariant 2-plane with an orthonormal basis $\{u, v\}$ such that

$$\begin{aligned} \tau(u) &= \cos\theta \cdot u + \sin\theta \cdot v, \\ \tau(v) &= -\sin\theta \cdot u + \cos\theta \cdot v, \end{aligned}$$

for some $\theta \in \mathbb{R}$.

Theorem 6.2. *Let $\tau : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an isometry. There is an orthonormal basis of $\mathbb{R}^n$ relative to which τ is represented by a matrix of the form*

$$\begin{pmatrix} \cos\theta_1 & \sin\theta_1 & & & & & \\ -\sin\theta_1 & \cos\theta_1 & & & & & \\ & & \cos\theta_2 & \sin\theta_2 & & & \\ & & -\sin\theta_2 & \cos\theta_2 & & & \\ & & & & \ddots & & \\ & & & & & I_r & \\ & & & & & & -I_s \end{pmatrix}$$

Corollary 6.3 (Cartan - Dieudonné theorem). *Every isometry of $\mathbb{R}^n$ is a composite of no more than n hyperplane reflections.*

6.4 The orthogonal groups

The orthogonal group $O(n)$ is the group of isometries of $\mathbb{R}^n$. By the Cartan-Dieudonné theorem, it is generated by hyperplane reflections. Each element of $O(n)$ has determinant ± 1. Those with determinant $+1$ form the special orthogonal group $SO(n)$.

Proposition 6.4. *The special orthogonal group $SO(n)$ is path-connected.*

6.4.1 The hyperplane reflection map

Let $x \in \mathbb{R}^n$ be a unit vector. We consider the reflection in the hyperplane which is the orthogonal complement of x in $\mathbb{R}^n$. This is the isometry $\rho_x : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$\rho_x(y) = y - 2\langle x, y\rangle x.$$

This is an orientation reversing isometry and so has determinant -1. The hyperplane reflection map $\rho : \mathbb{S}^{n-1} \longrightarrow O(n)$ given by $\rho(x) = \rho_x$ is a very important map in this course.

Chapter 7

The quaternions and the octonions

7.1 The quaternions

The quaternions are generalizations of the complex numbers. Let $\mathbb{C}$ be the field of complex numbers, with the notions of conjugation and norm. The quaternion algebra is $\mathbb{H} = \mathbb{C} \oplus \mathbb{C}$ with conjugation and multiplication defined by

$$\begin{aligned} \overline{(u,\ v)} &:= (\overline{u},\ -v), \\ (u_1,\ v_1)(u_2,\ v_2) &:= (u_1 u_2 - \overline{v_2} v_1,\ v_1 u_2 + v_2 \overline{u_1}). \end{aligned} \tag{7.1}$$

If we regard $\mathbb{C}$ as a euclidean space with orthonormal basis $\mathbf{1}$, $\mathbf{i}$, and write $\mathbf{j} = (0, \mathbf{1})$, $\mathbf{k} = (0, \mathbf{i})$, then $\mathbb{H}$ has an orthonormal basis $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ with conjugation

$$\overline{\mathbf{1}} = \mathbf{1}, \quad \overline{\mathbf{i}} = -\mathbf{i}, \quad \overline{\mathbf{j}} = -\mathbf{j}, \quad \overline{\mathbf{k}} = -\mathbf{k},$$

and multiplication given by the following table and extension by linearity

$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{1}$	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$

The norm of a quaternion $\mathbf{q} \in \mathbb{H}$ is the nonnegative real number $||\mathbf{q}||$ satisfying

$$||\mathbf{q}||^2 = \mathbf{q}\overline{\mathbf{q}} = \overline{\mathbf{q}}\mathbf{q}.$$

More explicitly, if $\mathbf{q} = a_0\mathbf{1} + a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, then

$$||\mathbf{q}||^2 = a_0^2 + a_1^2 + a_2^2 + a_3^2.$$

Proposition 7.1. *The quaternion multiplication is associative but not commutative.*

The noncommutativity is quite easy to verify: $\mathbf{ij} = \mathbf{k}$ but $\mathbf{ji} = -\mathbf{k}$. For the proof of associativity, see §**??** below.

Proposition 7.2. *For* $\mathbf{q}_1,\ \mathbf{q}_2 \in \mathbb{H}$, $||\mathbf{q}_1\mathbf{q}_2|| = ||\mathbf{q}_1|| \cdot ||\mathbf{q}_2||$.

Proof. First note that $\overline{\mathbf{q}_1\mathbf{q}_2} = \overline{\mathbf{q}_2}\ \overline{\mathbf{q}_1}$. Making use of the associativity of quaternions, we have

$$\begin{aligned}
||\mathbf{q}_1\mathbf{q}_2||^2 =& (\mathbf{q}_1\mathbf{q}_2)\overline{(\mathbf{q}_1\mathbf{q}_2)} = (\mathbf{q}_1\mathbf{q}_2)(\overline{\mathbf{q}_2}\ \overline{\mathbf{q}_1}) \\
=& ((\mathbf{q}_1\mathbf{q}_2)\overline{\mathbf{q}_2})\overline{\mathbf{q}_1} = (\mathbf{q}_1(\mathbf{q}_2\overline{\mathbf{q}_2}))\overline{\mathbf{q}_1} \\
=& (\mathbf{q}_1||\mathbf{q}_2||^2)\overline{\mathbf{q}_1} = (\mathbf{q}_1\overline{\mathbf{q}_1})||\mathbf{q}_2||^2 \\
=& ||\mathbf{q}_1||^2||\mathbf{q}_2||^2 = (||\mathbf{q}_1||||\mathbf{q}_2||)^2.
\end{aligned}$$

□

This means that $\mathbb{S}^3$, consisting of the unit quaternions, is a topological group. The multiplication of quaternions gives a **normed** bilinear map $\mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$, which is clearly nonsingular.

7.2 The octonions

The octonions are constructed from the quaternions by exactly the same rule in (7.1). It is an 8-dimensional algebra over $\mathbb{R}$ with an orthonormal basis

$$\begin{array}{llll}
\mathbf{e}_0 = (\mathbf{1}, 0), & \mathbf{e}_1 = (\mathbf{i}, 0), & \mathbf{e}_2 = (\mathbf{j}, 0), & \mathbf{e}_3 = (\mathbf{k}, 0), \\
\mathbf{e}_4 = (0, \mathbf{1}), & \mathbf{e}_5 = (0, \mathbf{i}), & \mathbf{e}_6 = (0, \mathbf{j}), & \mathbf{e}_7 = (0, \mathbf{k}),
\end{array}$$

conjugation

$$\overline{\mathbf{e}_i} = \begin{cases} \mathbf{e}_i, & i = 0, \\ -\mathbf{e}_i, & 1 \le i \le 7, \end{cases}$$

and multiplication given by

	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_0$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_4$	$\mathbf{e}_5$	$\mathbf{e}_6$	$\mathbf{e}_7$
$\mathbf{e}_1$	$\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_5$	$-\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$
$\mathbf{e}_2$	$\mathbf{e}_2$	$-\mathbf{e}_3$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_6$	$\mathbf{e}_7$	$-\mathbf{e}_4$	$-\mathbf{e}_5$
$\mathbf{e}_3$	$\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$-\mathbf{e}_4$
$\mathbf{e}_4$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_6$	$-\mathbf{e}_7$	$-\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$
$\mathbf{e}_5$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_7$	$\mathbf{e}_6$	$-\mathbf{e}_1$	$-\mathbf{e}_0$	$-\mathbf{e}_3$	$\mathbf{e}_2$
$\mathbf{e}_6$	$\mathbf{e}_6$	$\mathbf{e}_7$	$\mathbf{e}_4$	$-\mathbf{e}_5$	$-\mathbf{e}_2$	$\mathbf{e}_3$	$-\mathbf{e}_0$	$-\mathbf{e}_1$
$\mathbf{e}_7$	$\mathbf{e}_7$	$-\mathbf{e}_6$	$\mathbf{e}_5$	$\mathbf{e}_4$	$-\mathbf{e}_3$	$-\mathbf{e}_2$	$\mathbf{e}_1$	$-\mathbf{e}_0$

The octonion algebra $\mathbb{K}$ is not associative. For example,

$$\mathbf{e}_1(\mathbf{e}_2\mathbf{e}_5) = \mathbf{e}_1\mathbf{e}_7 = \mathbf{e}_6,$$

but

$$(\mathbf{e}_1\mathbf{e}_2)\mathbf{e}_5 = \mathbf{e}_3\mathbf{e}_5 = -\mathbf{e}_6.$$

Nevertheless, it still has the norm property.

Proposition 7.3. *For* $\mathbf{x},\ \mathbf{y} \in \mathbb{K}$, $||\mathbf{x}\mathbf{y}|| = ||\mathbf{x}|| \cdot ||\mathbf{y}||$.

The multiplication of octonions gives a normed bilinear map $\mathbb{R}^8 \times \mathbb{R}^8 \longrightarrow \mathbb{R}^8$, which is also nonsingular.

7.3 The Cayley-Dickson algebras

The constructions of quaternions from complex numbers and octonions from quaternions can be extended to the Cayley-Dickson algebras $\mathbb{A}_n$.

$$\begin{array}{ccccccccccccccccc} \mathbb{A}_0 & \subseteq & \mathbb{A}_1 & \subseteq & \mathbb{A}_2 & \subseteq & \mathbb{A}_3 & \subseteq & \mathbb{A}_4 & \subseteq & \cdots & \subseteq & \mathbb{A}_{t-1} & \subseteq & \mathbb{A}_t & \subseteq & \cdots \\ \| & & \| & & \| & & \| & & & & & & & & & & \\ \mathbb{R} & & \mathbb{C} & & \mathbb{H} & & \mathbb{K} & & & & & & & & & & \end{array}$$

beginning with $\mathbb{A}_0 = \mathbb{R}$ (with trivial conjugation), defined recursively by $\mathbb{A}_t = \mathbb{A}_{t-1} \oplus \mathbb{A}_{t-1}$, and with conjugation and multiplication given by (7.1).

7.3.1 Commutators and associators in Cayley-Dickson algebras

An element x in a Cayley-Dickson algebra $\mathbb{A}_t$ is pure if $x + \overline{x} = 0$. The pure part of x is the element $\widetilde{x} := \frac{1}{2}(x - \overline{x})$. Let $x,\ y, z \in \mathbb{A}_t$.

(1) $[x, y] := xy - yx$ is called the *commutator* of x and y. The algebra A is commutative if all commutators are zero.

(2) $(x, y, z) := (xy)z - x(yz)$ is called the *associator* of x, y, z. The algebra A is associative if all associators are zero.

(3) $\mathbb{A}_t$ is said to be *alternative* if $(x, x, y) = (y, x, x) = 0$ for all $x,\ y \in \mathbb{A}_t$.

Proposition 7.4. *The commutators and associators of* $\mathbb{A}_t$ *are related to those of* $\mathbb{A}_{t-1}$ *as follows.*

$$[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2] - [y_1, y_2],\ 2(y_2\widetilde{x_1} - y_1\widetilde{x_2})).$$

$$((x_1, y_1), (x_2, y_2), (x_3, y_3)) = (\Phi, \Psi),$$

where

$$\begin{aligned}\Phi &= (x_1, x_2, x_3) - (x_3, y_2, y_1) + (x_2, y_3, y_1) - (y_3, y_2, x_1) + (y_3, y_1, x_2) \\ &\qquad + [x_1, \overline{y_3}y_2] - [x_2, \overline{y_3}y_1] + [x_3, \overline{y_2}y_1];\end{aligned}$$

$$\begin{aligned}\Psi &= (y_1, y_2, y_3) + (y_1, x_2, x_3) - (y_3, x_2, x_1) + (y_2, x_3, x_1) - (y_2, x_1, x_3) \\ &\qquad + y_1[x_2, x_3] - y_2[x_1, x_3] + y_3[x_1, x_2] - y_3[\overline{y_2}, y_1] - [y_3, y_1\overline{y_2}].\end{aligned}$$

Proposition 7.5. *The Cayley-Dickson algebra* $\mathbb{A}_t$ *is*
(i) *alternative if and only if* $\mathbb{A}_{t-1}$ *is associative;*
(ii) *associative if and only if* $\mathbb{A}_{t-1}$ *is associative and commutative.*

Here is the multiplication table of the Cayley-Dickson algebra $\mathbb{A}_4$.

$+e_0$	$+e_1$	$+e_2$	$+e_3$	$+e_4$	$+e_5$	$+e_6$	$+e_7$	$+e_8$	$+e_9$	$+e_{10}$	$+e_{11}$	$+e_{12}$	$+e_{13}$	$+e_{14}$	$+e_{15}$
$+e_1$	$-e_0$	$+e_3$	$-e_2$	$+e_5$	$-e_4$	$-e_7$	$+e_6$	$+e_9$	$-e_8$	$-e_{11}$	$+e_{10}$	$-e_{13}$	$+e_{12}$	$+e_{15}$	$-e_{14}$
$+e_2$	$-e_3$	$-e_0$	$+e_1$	$+e_6$	$+e_7$	$-e_4$	$-e_5$	$+e_{10}$	$+e_{11}$	$-e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	$+e_{12}$	$+e_{13}$
$+e_3$	$+e_2$	$-e_1$	$-e_0$	$+e_7$	$-e_6$	$+e_5$	$-e_4$	$+e_{11}$	$-e_{10}$	$+e_9$	$-e_8$	$-e_{15}$	$+e_{14}$	$-e_{13}$	$+e_{12}$
$+e_4$	$-e_5$	$-e_6$	$-e_7$	$-e_0$	$+e_1$	$+e_2$	$+e_3$	$+e_{12}$	$+e_{13}$	$+e_{14}$	$+e_{15}$	$-e_8$	$-e_9$	$-e_{10}$	$-e_{11}$
$+e_5$	$+e_4$	$-e_7$	$+e_6$	$-e_1$	$-e_0$	$-e_3$	$+e_2$	$+e_{13}$	$-e_{12}$	$+e_{15}$	$-e_{14}$	$+e_9$	$-e_8$	$+e_{11}$	$-e_{10}$
$+e_6$	$+e_7$	$+e_4$	$-e_5$	$-e_2$	$+e_3$	$-e_0$	$-e_1$	$+e_{14}$	$-e_{15}$	$-e_{12}$	$+e_{13}$	$+e_{10}$	$-e_{11}$	$-e_8$	$+e_9$
$+e_7$	$-e_6$	$+e_5$	$+e_4$	$-e_3$	$-e_2$	$+e_1$	$-e_0$	$+e_{15}$	$+e_{14}$	$-e_{13}$	$-e_{12}$	$+e_{11}$	$+e_{10}$	$-e_9$	$-e_8$
$+e_8$	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_0$	$+e_1$	$+e_2$	$+e_3$	$+e_4$	$+e_5$	$+e_6$	$+e_7$
$+e_9$	$+e_8$	$-e_{11}$	$+e_{10}$	$-e_{13}$	$+e_{12}$	$+e_{15}$	$-e_{14}$	$-e_1$	$-e_0$	$-e_3$	$+e_2$	$-e_5$	$+e_4$	$+e_7$	$-e_6$
$+e_{10}$	$+e_{11}$	$+e_8$	$-e_9$	$-e_{14}$	$-e_{15}$	$+e_{12}$	$+e_{13}$	$-e_2$	$+e_3$	$-e_0$	$-e_1$	$-e_6$	$-e_7$	$+e_4$	$+e_5$
$+e_{11}$	$-e_{10}$	$+e_9$	$+e_8$	$-e_{15}$	$+e_{14}$	$-e_{13}$	$+e_{12}$	$-e_3$	$-e_2$	$+e_1$	$-e_0$	$-e_7$	$+e_6$	$-e_5$	$+e_4$
$+e_{12}$	$+e_{13}$	$+e_{14}$	$+e_{15}$	$+e_8$	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_4$	$+e_5$	$+e_6$	$+e_7$	$-e_0$	$-e_1$	$-e_2$	$-e_3$
$+e_{13}$	$-e_{12}$	$+e_{15}$	$-e_{14}$	$+e_9$	$+e_8$	$+e_{11}$	$-e_{10}$	$-e_5$	$-e_4$	$+e_7$	$-e_6$	$+e_1$	$-e_0$	$+e_3$	$-e_2$
$+e_{14}$	$-e_{15}$	$-e_{12}$	$+e_{13}$	$+e_{10}$	$-e_{11}$	$+e_8$	$+e_9$	$-e_6$	$-e_7$	$-e_4$	$+e_5$	$+e_2$	$-e_3$	$-e_0$	$+e_1$
$+e_{15}$	$+e_{14}$	$-e_{13}$	$-e_{12}$	$+e_{11}$	$+e_{10}$	$-e_9$	$+e_8$	$-e_7$	$+e_6$	$-e_5$	$-e_4$	$+e_3$	$+e_2$	$-e_1$	$-e_0$

7.4 Appendix: zero divisors in $\mathbb{A}_4$

Proposition 7.6. *A doubly pure element of $\mathbb{A}_4$ is a zero divisor if and only if its two components are mutually orthogonal pure elements of the same length in $\mathbb{A}_3 = \mathbb{K}$.*

Proof. (Necessity) Let $u = (x_1, y_1)$ and $v = (x_2, y_2)$ be elements in $\mathbb{A}_4$ satisfying $uv = 0$. Since the components $x_1, y_1, x_2, y_2 \in \mathbb{K}$ are all pure,

$$x_1x_2 + y_2y_1 = 0, \tag{7.2}$$
$$y_2x_1 - y_1x_2 = 0. \tag{7.3}$$

Since $\mathbb{K}$ is a composition algebra, by considering norms, we obtain from these equations

$$\|x_1\|\|x_2\| = \|y_2\|\|y_1\|; \qquad \|y_2\|\|x_1\| = \|y_1\|\|x_2\|.$$

These can be combined to give

$$(\|x_1\| - \|y_1\|)(\|x_2\| + \|y_2\|) = 0.$$

Since $v \neq 0$, $\|x_2\| + \|y_2\|$ cannot be zero, and we conclude that $\|x_1\| = \|y_1\|$, *i.e.* the pure elements x_1 and y_1 are equal in length. The same is true for x_2 and y_2. Also, from (7.2, 7.3),

$$\|x_2\|^2\langle x_1,\ y_1\rangle = \langle x_1x_2,\ y_1x_2\rangle = -\langle y_2y_1,\ y_2x_1\rangle = -\|y_2\|^2\langle x_1,\ y_1\rangle,$$

from which

$$(\|x_2\|^2 + \|y_2\|^2)\langle x_1,\ y_1\rangle = 0.$$

Since $v \neq 0$, we conclude that $\langle x_1,\ y_1\rangle = 0$. The two components of a zero divisor in $\mathbb{A}_4$ are therefore mutually orthogonal pure vectors of $\mathbb{K}$ of the same length.

(Sufficiency) Let $x, y \in \mathbb{K}$ be mutually orthogonal unit pure elements. Extend these to a simple system of unit generators x, y, z of $\mathbb{K}$. There is an automorphism $\sigma : \mathbb{K} \longrightarrow \mathbb{K}$ such that $\sigma(e_1) = x$, $\sigma(e_2) = y$ and $\sigma(e_4) = z$. Note that $\sigma(e_7) = (xy)z$. Since $(e_1, e_2)(e_4, -e_7) = 0$, applying the automorphism σ to the components, we obtain

$$(x, y)(z, -(xy)z) = 0,$$

showing that (x, y) is a zero divisor of $\mathbb{A}_4$. □

Chapter 8

Quaternions and isometries in $\mathbb{R}^4$

Identifying the euclidean space $\mathbb{R}^4$ with $\mathbb{H}$ by associating $(x_1, x_{2;3}, x_4)$ with $x_1 + x_2\mathbf{i} + x_3\mathbf{j} + x_4\mathbf{k}$, we have
(1) $2\langle u,\ v\rangle = u\overline{v} + v\overline{u}$, equivalently,$\langle u,\ v\rangle = \mathrm{Re}(u\overline{v})$;
(2) $u \perp v$ if and only if $u\overline{v} + v\overline{u} = 0$.

The multiplication of quaternions is related to the vector product in $\mathbb{R}^3$ in the following way. Identify $\mathbb{R}^3$ as the space of *pure* quaternions. Then

$$uv = -\langle u,\ v\rangle + u \times v, \qquad u, v \in \mathbb{R}^3.$$

Proposition 8.1. (1) *In* $\mathbb{R}^4 = \mathbb{H}$*, the reflection in the hyperplane* $u^\perp$ *is given by* $\rho_u(x) = -u\overline{x}u$*.*

(2) *A linear transformation* $\tau : \mathbb{H} \longrightarrow \mathbb{H}$ *is an isometry if and only if* $\tau(x) = axb$ *or* $\tau(x) = a\overline{x}b$ *for fixed unit quaternions* a *and* b*, which are unique except for a common sign of* a *and* b*.*

In particular, $\tau_{a,\overline{a}} : \mathbb{H} \longrightarrow \mathbb{H}$ preserves 1 is induces an isometry on $\mathbb{R}^3 =$ the space of *pure* quaternions. The map $\Psi : \mathbb{S}^3 \longrightarrow SO(3)$ given by

$$\Psi(a) = \tau_{a,\overline{a}}$$

is indeed a group homomorphism with kernel $\{\pm 1\}$.

Corollary 8.2. $SO(3)$ *is isomorphic to* $\mathbb{S}^3/\{\pm 1\} = \mathbb{R}\mathrm{P}^3$.

Chapter 9

Normed bilinear maps and their Hopf constructions

9.1 Hurwitz's theorem

A normed bilinear map $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ endows the euclidean space $\mathbb{R}^n$ with a composition algebra structure. Assuming a multiplicative identity e_0, an orthogonal decomposition $\mathbb{R}^n = \mathbb{R}e_0 \oplus e_0^\perp$ leads to a conjuation in $\mathbb{R}^n$: if $x = \langle x,\ e_0\rangle + y$ for $y \perp e_0$, then its conjugate is $\overline{x} = \langle x,\ e_0\rangle - y$.

The conjugation satisfies
(1) $\overline{\overline{x}} = x$,
(2) $\overline{x+y} = \overline{x} + \overline{y}$,
(3) $\overline{xy} = \bar{y}\bar{x}$,
(4) $x\overline{x} = \overline{x}x = \|x\|^2 e_0$,
(5) $\langle xy,\ z\rangle = \langle x,\ z\overline{y}\rangle = \langle y,\ \overline{x}z\rangle$.

A composition algebra with multiplicative identity is necessarily a Cayley-Dickson algebra $\mathbb{A}_t$ for some integer t.

Lemma 9.1. *Let A be an algebra with euclidean involution. A is a composition algebra if and only if it is an alternative algebra.*

The following statement are equivalent.
(1) $\mathbb{R}^n$ is a composition algebra.
(2) Every left multiplication L_x is a similarity:

$$\langle xy,\ xy'\rangle = \|x\|^2\langle y,\ y'\rangle \text{ for all } x,\ y,\ y' \in \mathbb{R}^n. \tag{9.1}$$

(2') Every right multiplication R_y is a similarity:

$$\langle xy,\ x'y\rangle = \|y\|^2\langle x,\ x'\rangle \text{ for all } x,\ x',\ y \in \mathbb{R}^n. \tag{9.2}$$

(3) For all $x,\ y,\ x',\ y' \in \mathbb{R}^n$,

$$\langle xy,\ x'y'\rangle + \langle xy',\ x'y\rangle = 2\langle x,\ x'\rangle\langle y,\ y'\rangle. \tag{9.3}$$

Theorem 9.2 (Hurwitz). *The euclidean space $\mathbb{R}^n$ has a composition algebra structure if and only if $n = 1,\ 2,\ 4, 8$.*

9.2 Normed bilinear maps and Hopf construction

A bilinear map $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ is said to be normed if

$$||f(x,y)|| = ||x|| \cdot ||y||, \quad x \in \mathbb{R}^r,\ y \in \mathbb{R}^s.$$

A normed bilinear map is nonsingular. Such a map gives a genuine map between euclidean spheres.

Proposition 9.3. *A normed bilinear map $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ induces a homogeneous quadratic map $F : \mathbb{S}^{r+s-1} \longrightarrow \mathbb{S}^n$ via the Hopf construction*

$$F(x,y) = (||x||^2 - ||y||^2,\ 2f(x,y)), \quad x \in \mathbb{R}^r,\ y \in \mathbb{R}^s.$$

Proof. For $x \in \mathbb{R}^r$, $y \in \mathbb{R}^s$,

$$\begin{aligned} ||F(x,y)||^2 =& (||x||^2 - ||y||^2)^2 + ||2f(x,y)||^2 \\ =& (||x||^2 - ||y||^2)^2 + 4||x||^2||y||^2 \\ =& (||x||^2 + ||y||^2)^2. \end{aligned}$$

Therefore, $(x,y) \in \mathbb{S}^{r+s-1} \Rightarrow F(x,y) \in \mathbb{S}^n$. □

9.2.1 The classical Hopf maps

Applying the Hopf construction to normed bilinear maps $\mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ for $n = 1, 2, 4, 8$, we obtain the following Hopf maps.

normed bilinear map	Hopf map
$\mathbb{R}^1 \times \mathbb{R}^1 \longrightarrow \mathbb{R}^1$	$2 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$
$\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$	$\eta : \mathbb{S}^3 \longrightarrow \mathbb{S}^2$
$\mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$	$\nu : \mathbb{S}^7 \longrightarrow \mathbb{S}^4$
$\mathbb{R}^8 \times \mathbb{R}^8 \longrightarrow \mathbb{R}^8$	$\sigma : \mathbb{S}^{15} \longrightarrow \mathbb{S}^8$

The maps η, μ, σ are called the classical Hopf maps.

9.2.2 Tabulation of a bilinear map

A bilinear map $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ is most conveniently tabulated with respect to given bases $u_1, \ldots, u_r$ of $\mathbb{R}^r$ and $v_1, \ldots, v_s$ of $\mathbb{R}^s$:

	v_1	$\ldots$	v_j	$\ldots$	v_s
u_1					
$\vdots$					
u_i			$f(u_i,\ v_j)$		
$\vdots$					
u_r					

We say that f is monomial if there is a basis $w_1, \ldots, w_n$ of $\mathbb{R}^n$ such that each $f(u_i,\ v_j) = w_k$ for some $k = 1, 2 \ldots, n$.

Proposition 9.4. *In a tabulation of a normed bilinear map* $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ *with respect to* orthonormal bases $u_1, \ldots, u_r$ *of* $\mathbb{R}^r$ *and* $v_1, \ldots, v_s$ *of* $\mathbb{R}^s$*,*
(1) each entry is a unit vector in $\mathbb{R}^n$*,*
(2) each row is an orthonormal s*-frame of* $\mathbb{R}^n$*,*
(3) each column is an orthonormal r*-frame of* $\mathbb{R}^n$*,*
(4) if $f(u_i,\ v_j) = \pm f(u_{i'},\ v_{j'})$*, then* $f(u_i,\ v_{j'}) = \mp f(u_{i'},\ v_j)$.

9.2.3 A normed bilinear map $\mathbb{R}^{10} \times \mathbb{R}^{10} \longrightarrow \mathbb{R}^{16}$

The multiplication tables of quaternions and octonions give monomial bilinear maps $\mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}^4$ and $\mathbb{R}^8 \times \mathbb{R}^8 \longrightarrow \mathbb{R}^8$.The multiplication table of $\mathbb{A}_4$ does not give a normed bilinear map $\mathbb{R}^{16} \times \mathbb{R}^{16} \longrightarrow \mathbb{R}^{16}$. Nevertheless, restriction to its first 10 rows and first 10 columns yields a normed bilinear map $\mathbb{R}^{10} \times \mathbb{R}^{10} \longrightarrow \mathbb{R}^{16}$.

	$+e_0$	$+e_1$	$+e_2$	$+e_3$	$+e_4$	$+e_5$	$+e_6$	$+e_7$	$+e_8$	$+e_9$
e_0	$+e_0$	$+e_1$	$+e_2$	$+e_3$	$+e_4$	$+e_5$	$+e_6$	$+e_7$	$+e_8$	$+e_9$
e_1	$+e_1$	$-e_0$	$+e_3$	$-e_2$	$+e_5$	$-e_4$	$-e_7$	$+e_6$	$+e_9$	$-e_8$
e_2	$+e_2$	$-e_3$	$-e_0$	$+e_1$	$+e_6$	$+e_7$	$-e_4$	$-e_5$	$+e_{10}$	$+e_{11}$
e_3	$+e_3$	$+e_2$	$-e_1$	$-e_0$	$+e_7$	$-e_6$	$+e_5$	$-e_4$	$+e_{11}$	$-e_{10}$
e_4	$+e_4$	$-e_5$	$-e_6$	$-e_7$	$-e_0$	$+e_1$	$+e_2$	$+e_3$	$+e_{12}$	$+e_{13}$
e_5	$+e_5$	$+e_4$	$-e_7$	$+e_6$	$-e_1$	$-e_0$	$-e_3$	$+e_2$	$+e_{13}$	$-e_{12}$
e_6	$+e_6$	$+e_7$	$+e_4$	$-e_5$	$-e_2$	$+e_3$	$-e_0$	$-e_1$	$+e_{14}$	$-e_{15}$
e_7	$+e_7$	$-e_6$	$+e_5$	$+e_4$	$-e_3$	$-e_2$	$+e_1$	$-e_0$	$+e_{15}$	$+e_{14}$
e_8	$+e_8$	$-e_9$	$-e_{10}$	$-e_{11}$	$-e_{12}$	$-e_{13}$	$-e_{14}$	$-e_{15}$	$-e_0$	$+e_1$
e_9	$+e_9$	$+e_8$	$-e_{11}$	$+e_{10}$	$-e_{13}$	$+e_{12}$	$+e_{15}$	$-e_{14}$	$-e_1$	$-e_0$

The Hopf construction of this normed bilinear map is a quadratic map $\mathbb{S}^{19} \longrightarrow \mathbb{S}^{16}$. [1]

9.3 Appendix: Hurwitz-Radon theorem and the normed bilinear map problem

THEOREM (Hurwitz-Radon). For a given integer $n = 2^t(2c+1)$, the largest possible integer r for which there exists a normed bilinear map $\mathbb{R}^r \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is

$$\rho(n) = \begin{cases} 2t+2, & t \equiv 0 \bmod 4, \\ 2t, & t \equiv 1, 2 \bmod 4, \\ 2t+2, & t \equiv \bmod 4. \end{cases}$$

The Hopf constructions such bilinear maps give quadratic maps between euclidean spheres which are all nontrivial in homotopy.

normed bilinear map	map between spheres
$\mathbb{R}^9 \times \mathbb{R}^{16} \longrightarrow \mathbb{R}^{16}$	$\mathbb{S}^{24} \longrightarrow \mathbb{S}^{16}$
$\mathbb{R}^{10} \times \mathbb{R}^{32} \longrightarrow \mathbb{R}^{32}$	$\mathbb{S}^{41} \longrightarrow \mathbb{S}^{32}$
$\mathbb{R}^{12} \times \mathbb{R}^{64} \longrightarrow \mathbb{R}^{64}$	$\mathbb{S}^{75} \longrightarrow \mathbb{S}^{64}$
$\mathbb{R}^{16} \times \mathbb{R}^{128} \longrightarrow \mathbb{R}^{128}$	$\mathbb{S}^{143} \longrightarrow \mathbb{S}^{128}$
$\vdots$	$\vdots$

Remark. The Hurwitz-Radon number $\rho(n)$ can also be expressed as follows. If $n = 2^{4a+b}(2c+1)$ with $0 \leq b \leq 3$, then

$$\rho(n) = 8a + 2^b.$$

[1] We shall prove later that this is nontrivial in homotopy.

The normed bilinear map problem asks, for given integers r and s, the least positive integer $n := r * s$ for the existence of a normed bilinear map. Here are the values of $r * s$ for $r, s \leq 10$, all realized by taking restrictions from the normed bilinear $\mathbb{R}^{10} \times \mathbb{R}^{10} \longrightarrow \mathbb{R}^{16}$ above.

$r * s$ for $r, s \leq 10$

$r \setminus s$	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	2	4	4	6	6	8	8	10	10
3	3	4	4	4	7	8	8	8	11	12
4	4	4	4	4	8	8	8	8	12	12
5	5	6	8	8	8	8	8	8	13	14
6	6	6	8	8	8	8	8	8	14	14
7	7	8	8	8	8	8	8	8	15	16
8	8	8	8	8	8	8	8	8	16	16
9	9	10	11	12	13	14	15	16	16	16
10	10	10	12	12	14	14	16	16	16	16

Chapter 10

Polynomial maps between spheres

We consider mappings between euclidean spheres whose component functions of polynomials. A polynomial map $F : \mathbb{R}^{m+1} \to \mathbb{R}^{n+1}$ restricts to a map $f : \mathbb{S}^m \to \mathbb{S}^n$ if

$$F_1(x)^2 + F_2(x)^2 + \cdots + F_{n+1}(x)^2 = 1$$

whenever $||x||^2 = x_1^2 + x_2^2 + \cdots + x_{m+1}^2 = 1$. If the component functions are homogeneous polynomials of degree d, this condition becomes $||F(x)||^2 = ||x||^{2d}$.

10.1 Wood's theorem

Theorem 10.1 (Wood). *If $m \geq 2^t > n$ for some integer t, then every polynomial map $\mathbb{S}^m \longrightarrow \mathbb{S}^n$ must be constant.*

We shall make use of two beautiful theorems discovered in the 1960's.

THEOREM (Cassels). $x_1^2 + x_2^2 + \cdots + x_n^2$ cannot be written as a sum of fewer than n squares of rational functions in $x_1, x_2, \ldots, x_n$.

Theorem 10.2 (Pfister). *In a field of characteristic $\neq 2$, the product of two sums of 2^t squares is again a sum of 2^t squares.*

10.1.1 Proof of Wood's theorem: homogeneous case

It is enough to take $m = 2^t$ and $n = 2^t - 1$ and show that the assumption of a homogeneous map $F : \mathbb{S}^m \longrightarrow \mathbb{S}^n$ of degree d leads to a contradiction. Such a map satisfies $||F(x)||^2 = ||x||^{2d}$ for every $x \in \mathbb{S}^m$. Write $x = (y, z)$ for $y \in \mathbb{R}^m$, $z \in \mathbb{R}$, and separate F into two parts, one containing even powers of z and the other odd powers of z:

$$F(x) = P(x) + Q(x)z,$$

where the components of $P(x)$ and $Q(x)$ are polynomials containing only even powers of z. Since $||F(x)||^2 = ||x||^{2d}$, we have

$$\begin{aligned}(||y||^2 + z^2)^d =& ||P(x) + Q(x)z||^2 \\ =& ||P(x)||^2 + ||Q(x)||^2 z^2 + 2z\langle P(x),\ Q(x)\rangle.\end{aligned}$$

Matching odd and even powers of z, we conclude

$$||P(x)||^2 + ||Q(x)||^2 z^2 = (||y||^2 + z^2)^d, \tag{10.1}$$

$$\langle P(x),\ Q(x)\rangle = 0. \tag{10.2}$$

Since $P(x)$ and $Q(x)$ contain only even powers of z, it makes sense to substitute z^2 by $-||y||^2$ in (10.1) and (10.2) so that $P(x)$ and $Q(x)$ become $p(y)$ and $q(y)$ for polynomials p and q satisfying

$$||p(y)||^2 - ||q(y)||^2 ||y||^2 = 0, \tag{10.3}$$

$$\langle p(y),\ q(y)\rangle = 0. \tag{10.4}$$

From (10.3), we have

$$||y||^2 = \frac{||p(y)||^2}{||q(y)||^2}. \tag{10.5}$$

Note that this expresses a sum of $m = 2^t$ squares as a quotient (hence product) of two sums of $m = 2^t$ squares. This is in conformity with Pfister's theorem. However, Wood observed a contradiction by adding z^2 to both sides:

$$\begin{aligned}||y||^2 + z^2 =& \frac{||p(y)||^2}{||q(y)||^2} + z^2 \\ =& \frac{||p(y)||^2 + ||q(y)||^2 z^2}{||q(y)||^2} \\ =& \frac{||p(y) + q(y)z||^2}{||q(y)||^2}.\end{aligned}$$

This expresses $y_1^2 + y_2^2 + \cdots + y_m^2 + z^2$ as a quotient (hence product) of two sums of 2^t squares, which by Pfister's theorem, is a sum of $m = 2^t$ squares, contradicting Cassels' theorem.

10.1.2 Proof of Wood's theorem: general case

We reduce this to the homogeneous case. If all monomials in F have even degrees, with $2d$ the maximum of these degrees, then F can be made homogeneous, without change in value, by multiplying each monomial of degree $2e$ by $||x||^{2(d-e)}$. Such a map must be constant. Therefore, F is constant.

Suppose F contains some monomials of odd degree. The composite $F' = F \circ \alpha : \mathbb{S}^m \to \mathbb{S}^m \to \mathbb{S}^n$, where

$$\alpha(x_1, x_2, \ldots, x_{m+1}) = (x_1^2 - x_2^2 - \cdots - x_{m+1}^2,\ 2x_1x_2,\ \cdots,\ 2x_1x_{m+1})$$

has all monomials of even degrees. Such an F must be constant. Since α is surjective, F must be a constant map.

10.2 Appendix: Pfister's theorem

An identity of the form

$$(x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2) = z_1^2 + z_2^2 + \cdots + z_n^2$$

in which z_1, z_2, ..., z_n are linear in y_1, y_2, ..., y_n is equivalent to an $n \times n$ matrix M satisfying

$$M^t M = (x_1^2 + \cdots + x_n^2) I_n.$$

Note that this entails $MM^t = (x_1^2 + \cdots + x_n^2) I_n$. It is well known that such a matrix M exists for $n = 1, 2, 4, 8$, in which the entries are linear forms in x_1, x_2, ..., x_n. Assuming such matrices of order n, in which the entries are rational functions of x_1, ..., x_n, we show that a matrix of order $2n$ also exists satisfying

$$T^t T = (x_1^2 + \cdots + x_n^2 + x_{n+1}^2 + \cdots + x_{2n}^2) I_{2n}. \qquad (10.6)$$

Let M_1 and M_2 be $n \times n$ matrices satisfying

$$\begin{aligned} M_1^t M_1 = M_1 M_1^t =& (x_1^2 + \cdots + x_n^2) I_n, \\ M_2^t M_2 = M_2 M_2^t =& (x_{n+1}^2 + \cdots + x_{2n}^2) I_n. \end{aligned}$$

We try to find a matrix X such that $T = \begin{pmatrix} M_1 & M_2 \\ M_2 & X \end{pmatrix}$ satisfies (10.6).

The condition

$$\begin{aligned} T^tT &= \begin{pmatrix} M_1^t & M_2^t \\ M_2^t & X^t \end{pmatrix} \begin{pmatrix} M_1 & M_2 \\ M_2 & X \end{pmatrix} \\ &= \begin{pmatrix} M_1^tM_1 + M_2^tM_2 & M_1^tM_2 + M_2^tX \\ M_2^tM_1 + X^tM_2 & M_2^tM_2 + X^tX \end{pmatrix}. \end{aligned}$$

We require the off-diagonal blocks to be zero. These both lead to $M_2^tX = -M_1^tM_2$. Explicitly,

$$X = \left(\frac{-1}{x_{n+1}^2 + \cdots + x_{2n}^2}\right) M_2M_1^tM_2.$$

From this,

$$\begin{aligned} &(M_2^tX)^t(M_2^tX) = (-M_2^tM_1)(-M_1^tM_2) \\ \Rightarrow &X^t(M_2M_2^t)X = M_2^t(M_1M_1^t)M_2 \\ \Rightarrow &(M_2M_2^t)(X^tX) = (M_2^tM_2)(M_1M_1^t) \\ \Rightarrow &X^tX = (x_1^2 + \cdots + x_n^2)I_n, \end{aligned}$$

and (10.6) follows.

Chapter 11

Nonsingular bilinear maps

11.1 Nonsingular bilinear maps on euclidean spaces

A bilinear map $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ is said to be **nonsingular** if $f(x, y) = 0 \Rightarrow x =$ or $y = 0$. Normed bilinear maps are nonsingular.

Examples

1. There is an obvious nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^{rs}$ induced by matrix multiplication:

$$f(x, y) = xy^t.$$

2. This example can be improved by lowering the dimension of the range, by making use of polynomial multiplications. Identify $\mathbb{R}^\infty$ with the polynomial ring $\mathbb{R}[T]$. Polynomial multiplication clearly induces a nonsingular bilinear map $\mathbb{R}^\infty \times \mathbb{R}^\infty \longrightarrow \mathbb{R}^\infty$. This retricts to a nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^{r+s-1}$ since the product of two polynomials of degree $r - 1$ and $s - 1$ is one of degree $r + s - 2$.

3. By replacing the field of real numbers $\mathbb{R}$ by the field of complex numbers $\mathbb{C}$, we have nonsingular bilinear maps

$$\mathbb{R}^{2r} \times \mathbb{R}^{2s} \longrightarrow \mathbb{R}^{2r+2s-2}.$$

4. There are further improvements by replacing the complex numbers with quaternions and octonians (also known as Cayley numbers):

$$\mathbb{R}^{4r} \times \mathbb{R}^{4s} \longrightarrow \mathbb{R}^{4r+4s-4},$$
$$\mathbb{R}^{8r} \times \mathbb{R}^{8s} \longrightarrow \mathbb{R}^{8r+8s-8}.$$

For $r = s$, these give examples of symmetric nonsingular bilinear maps.

11.1.1 Imbedding of real projective spaces

Theorem 11.1. *If there is a symmetric nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^r \longrightarrow \mathbb{R}^n$, then the real projective space $\mathbb{R}\mathrm{P}^{r-1}$ imbeds in euclidean space $\mathbb{R}^{n-1}$.*

Proof. Let $f : \mathbb{R}^r \times \mathbb{R}^r \longrightarrow \mathbb{R}^n$ be a symmetric nonsingular bilinear map. Define $g : \mathbb{R}\mathrm{P}^{r-1} \longrightarrow \mathbb{R}^n \setminus \{0\}$ by

$$g([x]) = f(x,x),\ x \in \mathbb{S}^{r-1}.$$

This is clearly well-defined and nonzero. It is injective since $0 = g([x]) - g([y]) = f(x,x) - f(y,y) = f(x+y, x-y)$ implies $x+y = 0$ or $x - y = 0$, $y = \pm x$ and $[x] = [y]$. The same reasoning shows that if $g([x]) = \lambda g([y])$ for $\lambda > 0$, then x and y are linearly dependent, and $[x] = [y]$. Thus, the composite

$$\mathbb{R}\mathrm{P}^{n-1} \longrightarrow \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{S}^{n-1}$$

is injective. It clearly cannot be surjective, for otherwise, it is a homeomorphism. Thus this map is an imbedding into $\mathbb{S}^{n-1} \setminus \{\text{point}\} \equiv \mathbb{R}^{n-1}$. $\square$

11.1.2 Hopf-Stiefel theorem

One of the earliest applications of algebraic topology is the following remarkable theorem.

THEOREM (Hopf-Stiefel). If there is a nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$, then the binomial coefficients $\binom{n}{k}$ are all even for $n - s < k < r$.

The **nonsingular bilinear map problem** asks, for given integers r and s, the smallest integer $n := r\#s$ for which there exists a nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$.

11.1.3 A nonsingular bilinear map $\mathbb{R}^{16} \times \mathbb{R}^{16} \longrightarrow \mathbb{R}^{23}$

The bilinear map $f : \mathbb{K}^2 \times \mathbb{K}^2 \longrightarrow \mathbb{K}^3$ given by [1]

$$f((x_1, x_2),\ (y_1, y_2)) = ((x_1, x_2)(y_1, y_2),\ [x_2,\ y_2]).$$

is nonsingular. This gives a nonsingular bilinear $\mathbb{R}^{16} \times \mathbb{R}^{16} \longrightarrow \mathbb{R}^{23}$ with the following restrictions

$r \setminus s$	10	11	12	13	14	15	16
10	16				20	21	22
11		17					
12							
13				19			
14							
15							
16							23

There is a nonsingular bilinear map $g : \mathbb{H}^3 \times \mathbb{H}^3 \longrightarrow \mathbb{H}^5$ which gives rise to a nonsingular bilinear map $\mathbb{R}^{12} \times \mathbb{R}^{12} \longrightarrow \mathbb{R}^{17}$. For $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$,

$$\begin{aligned} g(a, b) =& (a_1 b_1 + \overline{b_2} a_2 + \overline{b_3} a_3,\ a_2 \overline{b_1} - b_2 a_1,\ a_3 \overline{b_1} - b_3 a_1, \\ & b_2 \overline{a_3} + a_2 \overline{b_3},\ \overline{b_3} a_3 + \overline{a_3} b_3). \end{aligned}$$

11.2 The Hopf construction of a nonsingular bilinear map

Given a nonsingular bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$, consider the Hopf construction $F : \mathbb{R}^r \oplus \mathbb{R}^s \longrightarrow \mathbb{R}^{n+1}$ defined by

$$F(x, y) = (\|x\|^2 - \|y\|^2,\ 2f(x, y)), \quad x \in \mathbb{R}^r,\ s \in \mathbb{R}^s.$$

Restrict to $\mathbb{S}^{r+s-1} \subset \mathbb{R}^r \oplus \mathbb{R}^s$, it does not map into $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. Nevertheless, it maps into $\mathbb{R}^{n+1} \setminus \{0\}$, which is homotopy equivalent to $\mathbb{S}^n$. We therefore regard this Hopf construction as giving rise to a homotopy element of spheres. For example,

[1] K. Y. Lam, Construction of some nonsingular bilinear maps, *Topology*, 6 (1967) 423–426.

nonsingular bilinear map	homotopy element of spheres
$\mathbb{R}^{10} \times \mathbb{R}^{10} \longrightarrow \mathbb{R}^{16}$	$\mathbb{S}^{19} \longrightarrow \mathbb{S}^{16}$
$\mathbb{R}^{12} \times \mathbb{R}^{12} \longrightarrow \mathbb{R}^{17}$	$\mathbb{S}^{23} \longrightarrow \mathbb{S}^{17}$
$\mathbb{R}^{16} \times \mathbb{R}^{16} \longrightarrow \mathbb{R}^{23}$	$\mathbb{S}^{31} \longrightarrow \mathbb{S}^{23}$

Chapter 12

Quadratic forms between euclidean spheres

12.1 Quadratic forms between euclidean spaces

A quadratic form between euclidean spaces is a map $g : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ whose components are homogeneous quadratic functions (of coordinates of points in the domain). Associated with a quadratic form is a symmetric bilinear map $B : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$ satisfying

$$g(av + bv') = a^2 g(v) + b^2 g(v') + 2abB(v, v').$$

As a smooth map, g has differentials $dg_x : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ given by

$$dg_x(y) = 2B(x, y).$$

12.2 Quadratic forms between euclidean spheres

A map $f : \mathbb{S}^m \longrightarrow \mathbb{S}^n$ is quadratic if it is the restriction of a homogeneous quadratic map $F : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}^{n+1}$. Consider two orthogonal unit vectors $v, v' \in \mathbb{S}^m \subset \mathbb{R}^{m+1}$.
(1) If $F(v) = F(v') = q$, then F maps the great circle through v and v' to the point $q \in \mathbb{S}^n$.

(2) If $F(v) \neq F(v')$, then F "wraps" the great circle through v and v' uniformly twice around a circle on $\mathbb{S}^n$ which has $F(v)$ and $F(v')$ as endpoints of a diameter. This follows from

$$
\begin{aligned}
&F(\cos\theta\cdot v+\sin\theta\cdot v')-\frac{F(v)+F(v')}{2}\\
=&\frac{F(v)-F(v')}{2}\cdot\cos 2\theta+B(v,v')\cdot\sin 2\theta.
\end{aligned}
$$

Proposition 12.1. *If $F:\mathbb{S}^m\longrightarrow\mathbb{S}^n$ is a homogeneous quadratic map, then for every $q\in\mathbb{S}^n$, the inverse image $F^{-1}(q)$ is a great sphere in $\mathbb{S}^m$, being the intersection of $\mathbb{S}^m$ with a linear subspace*

$$
W_q=\ker(B_v^*\circ B_v-4p_v)\subset\mathbb{R}^{m+1},\qquad v\in F^{-1}(q).
$$

12.3 Hidden normed bilinear maps

Consider a quadratic form $F:\mathbb{S}^m\longrightarrow\mathbb{S}^n$ with associated bilinear map $B:\mathbb{R}^{m+1}\times\mathbb{R}^{m+1}\longrightarrow\mathbb{R}^{n+1}$. We show that corresponding to every point $q\in\operatorname{Im}F$, there is a restriction of B to a nonsingular bilinear map, whose Hopf construction represents the same homotopy class of spheres as F does. Furthermore, this nonsingular bilinear map 'hidden' at q can be deformed in a canonical way, via nonsingular bilinear maps, into a normed bilinear map, so that we can actually speak of the 'normed bilinear map hidden at' q.

Proposition 12.2. *For any $q\in\mathbb{S}^n$ in the image of F, the restriction of B to*

$$
\tilde{B}_q:W_q\times W_q^{\perp}\longrightarrow\tau_q(\mathbb{S}^n)
$$

is a nonsingular bilinear map.

Proof. The bilinearity of $\tilde{B}_q$ is clear. Let v be a unit vector in W_q and v' a unit vector in $W_q^{\perp}$. By Corollary 1.4, $\tilde{B}_q(v,v')=B(v,v')$ is orthogonal to q. This vector cannot be zero, for otherwise, $F(v')=f(v)$ and $v'\in W_q$, a contradiction. This shows that $\tilde{B}_q$ is nonsingular. ☐

We call this the nonsingular bilinear map *hidden* (inside the quadratic form F) at the point $q\in\mathbb{S}^n$. The Hopf construction of this map can be deformed, by normalization, into a map between spheres, and hence represents a homotopy element of spheres. We show that this must represent the same homotopy class of the quadratic form F. Bypassing details of normalizations, it is enough to prove

Proposition 12.3. *Let q be a point in the image of F. There is a homotopy $H_t : S(V_1) \longrightarrow V_2 \setminus \{0\}$ such that $H_0 = F$ (regarded as a map into $V_2 \setminus \{0\}$) and $H_1 =$ the Hopf construction of $\tilde{B}_q$ with poles $\pm q$.*

Proof. The hidden nonsingular bilinear map $\tilde{B}_q$ is given by

$$\tilde{B}_q(v, v') = \frac{1}{2}(F(v + v') - F(v') - \|v\|^2 q).$$

The Hopf construction of this map with poles $\pm q$ is the map $F_q : \mathbb{S}^m = S(W_q \oplus W_q^{\perp}) \longrightarrow \mathbb{R}^{n+1}$ given by

$$\begin{aligned} F_q(v, v') &= (\|v\|^2 - \|v'\|^2)q + 2\tilde{B}_q(v, v') \\ &= F(v + v') - F(v') - \|v'\|^2 q. \end{aligned}$$

Define a homotopy $H_t : \mathbb{S}^m \longrightarrow \mathbb{R}^{n+1}$, $0 \leq t \leq 1$, by

$$H_t(v, v') = F(v + v') - tF(v') - t\|v'\|^2 q.$$

Note that $H_t(v, v') \neq 0$ for any $t \in [0, 1]$. To see this, observe that $H_t(v, v') = 0$ implies

$$2B(v, v') + (1 - t)F(v') + (\|v\|^2 - t\|v'\|^2)q = 0.$$

Since, by Corollary 1.3, $B(v, v')$ is orthogonal to both q and $F(v')$, such a linear dependence is impossible unless $v = 0$ and $v' = 0$. Thus, each H_t, $t \in [0, 1]$, maps into $V_2 \setminus \{0\}$, and the proposition follows by noting that $H_0 = F$ and $H_1 = F_q$. □

Remark. In fact, the proof above can be slightly modified to show that the homotopy H_t takes place in the orthogonal complement of q.

Let $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ be a bilinear map. For each $x \in \mathbb{R}^r$, let $f_x : \mathbb{R}^s \longrightarrow \mathbb{R}^n$ denote the linear map induced by f. Suppose f is nonsingular. Then, for each nonzero $x \in \mathbb{R}^r$,

(i) the linear map f_x is injective;

(ii) the endomorphism $g_x := f_x^* \circ f_x$ is self - adjoint, and has positive eigenvalues;

(iii) $y \in \mathbb{R}^s$ is an eigenvector of $g_x = f_x^* \circ f_x$ corresponding to an eigenvalue λ if and only if

$$\langle f(x, y),\ f(x, y') \rangle = \lambda \langle y,\ y' \rangle$$

for every $y' \in \mathbb{R}^s$;

(iv) if $y_1, \ldots, y_s$ are orthogonal unit eigenvectors of g_x, then $f(x, y_1), \ldots, f(x, y_s)$ are mutually orthogonal in $\mathbb{R}^n$.

Lemma 12.4. *Let* $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ *be a nonsingular bilinear map. Suppose the endomorphism* g_x *is* independent *of* $x \in \mathbb{S}^{r-1}$. *Then,* f *can be homotoped via nonsingular bilinear maps into a normed bilinear map.*

Proof. Let $\mathcal{E} = (\epsilon_1, \ldots, \epsilon_s)$ be an orthonormal basis of (common) unit eigenvectors of g_x with positive eigenvalues $\lambda_1, \ldots, \lambda_s$. Take an orthonormal basis $E = (e_1, \ldots, e_r)$ of X and tabulate the bilinear map f by the matrix $M_f := (f(e_i, \epsilon_j))$. Note that by the choice of the basis $\mathcal{E}$ of Y, for each $i = 1, \ldots, r$, the vectors $f(e_i, \epsilon_j)$, $j = 1, \ldots, s$ are mutually orthogonal.

For each $t \in [0, 1]$, let M_t be the matrix

$$M_t = \left(\frac{1}{\sqrt{1 - t + t\lambda_j}} f(e_i, \epsilon_j) \right).$$

This tabulates a bilinear map $f_t : X \times Y \longrightarrow Z$ which is nonsingular, since every induced linear map $f_{t,x} : Y \longrightarrow Z$, $x \in X$, is injective, being given by

$$f_{t,x}(\epsilon_j) = \frac{1}{\sqrt{1 - t + \lambda_j}} f(x, \epsilon_j), \qquad 1 \leq j \leq s,$$

and extension by linearity. The family f_t, $0 \leq t \leq 1$, is, therefore, a homotopy of nonsingular bilinear maps. Furthermore,

$$\|f_{1,x}(\epsilon_j)\| = \|\frac{1}{\sqrt{\lambda_j}} f(x, \epsilon_j)\| = 1, \qquad 1 \leq j \leq s.$$

Thus, M_1 tabulates a normed bilinear map, and is obtained simply by *normalizing* each vector in the matrix $M_0 = M_f$. □

Proposition 12.5. *Every hidden nonsingular bilinear map of a quadratic form between spheres can be homotoped via nonsingular bilinear maps into a normed bilinear map.*

Proof. For any q in the image of F, choose an orthonormal basis $w_1, \ldots, w_{m+1-k}$ of $W_q^{\perp}$. By Proposition 2.3,

$$\langle \tilde{B}_q(v, w_i,\ \tilde{B}_q(w_j) \rangle = \langle B_v^* \circ B_v(w_i),\ w_j \rangle = \frac{1}{2}(\langle w_i,\ w_j \rangle - \langle q,\ B(w_i, w_j) \rangle$$

is independent of the choice of $v \in F^{-1}(q) = S(W_q)$. The result now follows from ?. □

We call the normed bilinear map obtained by effecting the above homotopy to the nonsingular bilinear map $\tilde{B}_q$ *the normed bilinear map* $\bar{B}_q$ *hidden* at q. We shall find a simple description of this map in the next section. We summarize the results in the present section by

Theorem 12.6. *Let $F : \mathbb{S}^m \longrightarrow \mathbb{S}^n$ be a nonconstant quadratic form. For each point q in the image of F, there is a hidden normed bilinear map of type $[k, m-k+1, n]$, $m-n+1 \leq k = \dim W_q \leq n$, representing the same homotopy class of F.*

12.4 The hidden normed bilinear maps of a monomial bilinear map

Let $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ be a *monomial* normed bilinear map, with respect to orthonormal bases $e_1, \ldots, e_r$ or $\mathbb{R}^r$, $\epsilon_1, \ldots, \epsilon_s$ of $\mathbb{R}^s$, and $c_1, \ldots, c_n$ of $\mathbb{R}^n$. This means that each $f(e_i,\ \epsilon_j) = c_k$ for some k. Let c be one of these. The hidden normed bilinear at c can easily found from a tabulation of f. Suppose

$$f(e_1, \epsilon_1) = \cdots = f(e_k, \epsilon_k) = c,$$

and f is tabulated by the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then the normed bilinear map hidden at c is the one tabulated by

$$\begin{pmatrix} A & B & C^t \end{pmatrix}.$$

12.5 Appendix: An application

We consider the problem of mapping a euclidean sphere of given dimension via a quadratic form into one of lowest possible dimension. Let

$$q(m) := \min\{n : \exists \text{ nonconstant quadratic form } \mathbb{S}^m \longrightarrow \mathbb{S}^n\}.$$

Analogously, we also consider

$$p(n) := \max\{m : \exists \text{ nonconstant quadratic form } \mathbb{S}^m \longrightarrow \mathbb{S}^n\}.$$

Lemma 12.7. (1) $p(n) \geq n \geq q(n)$.
(2) p *and* q *are increasing functions.*
(3) $q(m) > n$ *if and only if* $m > p(n)$.
(4) $q(2^t + \rho(2^t) - 1) \leq 2^t$.
(5) $p(2^t) \geq 2^t + \rho(2^t) - 1$.

By Wood's theorem,

$$q(2^t) = q(2^t + 1) = \cdots = q(2^t + \rho(2^t) - 1) = 2^t.$$

Here are the beginning values of $q(m)$.

$$\begin{aligned} &q(1) = 1, \\ &q(2) = q(3) = 2, \\ &q(4) = q(5) = q(6) = q(7) = 4, \\ &q(8) = q(9) = \cdots = q(15) = 8, \\ &\vdots \end{aligned}$$

Also, $p(2^t - 1) = 2^t - 1$. Also,

$$p(2^t) = p(2^t + 1) = \cdots = p(2^t + q(\rho(2^t)) - 1) = 2^t + \rho(2^t) - 1.$$

Here are beginning values of $p(n)$.

$$\begin{aligned} &p(1) = 1, \\ &p(2) = p(3) = 3, \\ &p(4) = p(5) = p(6) = p(7) = 7, \\ &p(8) = p(9) = \cdots = p(15) = 15, \\ &\vdots \end{aligned}$$

Theorem 12.8.

$$q(2^t + m) = \begin{cases} 2^t, & 0 \leq m < \rho(2^t), \\ 2^t + q(m), & \rho(2^t) \leq m < 2^t; \end{cases}$$

$$p(2^t + n) = \begin{cases} 2^t + \rho(2^t) - 1, & 0 \leq n < q(\rho(2^t)), \\ 2^t + p(n), & q(\rho(2^t)) \leq n < 2^t. \end{cases}$$

Here is a more explicit expression for $q(m)$. Let $m = \sum_{i=0}^{t-1} m_i \cdot 2^i$, $m_i = 0, 1$, be the dyadic expansion of m, and

$$k = \max\{j : \sum_{i<j} m_i \cdot 2^i < \rho(2^j)\}.$$

Then,

$$q(m) = \sum_{i \geq k} m_i \cdot 2^i.$$

Chapter 13

Spaces of similarities

13.1 The space $\mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^n)$ of linear maps

The space of linear maps $\mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^n)$ has dimension sn. It has a basis $\mu_{i,j} : \mathbb{R}^s \longrightarrow \mathbb{R}^n$, $1 \leq i \leq s$, $1 \leq j \leq n$ defined by

$$\mu_{i,j}(e_k) = \delta_{i,k} e_j,\ 1 \leq k \leq s.$$

We endow it with an inner product by taking $\mu_{i,j}$, $1 \leq i \leq s$, $1 \leq j \leq n$ as an orthonormal basis. Thus, if we represent linear maps $\mathbb{R}^s \longrightarrow \mathbb{R}^n$ by $s \times n$ matrices, the space of matrices $M_{s,n}$ has an inner product given by

$$\langle A,\ B\rangle = \frac{1}{s} \cdot \mathrm{Tr}(AB^t).$$

13.1.1 Similarities

A linear map $\tau : \mathbb{R}^s \longrightarrow \mathbb{R}^n$ is a similarity if there exists $a(\tau) \in \mathbb{R}$ such that

$$\langle \tau(y),\ \tau(y')\rangle = a(\tau)\langle y,\ y'\rangle$$

for all y, $y' \in \mathbb{R}^n$. $a(\tau)$ is called the similarity factor of τ. It follows from the *polarization identity*

$$\langle u,\ v\rangle = \frac{1}{2}\left(||u+v||^2 - ||u||^2 - ||v||^2\right)$$

that τ is a similarity with factor b^2 if $||\tau(y)|| = b||y||$ for every $y \in \mathbb{R}^n$. An isometry is a similarity with factor 1.

A linear map $\tau : \mathbb{R}^s \longrightarrow \mathbb{R}^n$ has a *dual* $\tau^* : \mathbb{R}^n \longrightarrow \mathbb{R}^s$ defined by

$$\langle x,\ \tau^*(y)\rangle = \langle \tau(x),\ y\rangle, \quad x \in \mathbb{R}^s,\ y \in \mathbb{R}^n.$$

$\tau \in \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^n)$ is a similarity if and only if $\tau^* \circ \tau = k\iota$ for some $k \in \mathbb{R}$.

Let $\tau : \mathbb{R}^s \longrightarrow \mathbb{R}^n$ be a similarity with factor k represented by an $s \times n$ matrix A. The matrix A satisfies $AA^t = kI_s$, and is called a similarity matrix.

13.2 Spaces of similarities and normed bilinear maps

Let $f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ be a bilinear map. Its *adjoint* is the linear map $f^\# : \mathbb{R}^r \longrightarrow \mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^n)$ given by

$$f^\#(x)(y) = f(x, y).$$

If f is *normed*, then

$$||f^\#(x)(y)|| = ||f(x, y)|| = ||x||||y||,\ x \in \mathbb{R}^r,\ y \in \mathbb{R}^s.$$

It follows that each $f^\#(x)$ is a similarlity (with factor $||x||^2$). Thus, a normed bilinear map $\mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$ is equivalent to an r-dimensional *subspace of similarities* of $\mathrm{Hom}(\mathbb{R}^s, \mathbb{R}^n)$. In such a subspace, we may choose an orthonormal basis consisting of r similarities, which are represented by $s \times n$ matrices A_1, A_2, ..., A_r. These matrices satisfy

$$A_i A_i^t = I_s, \quad 1 \le i \le r,$$
$$A_i A_j^t + A_j A_i^t = 0, \quad 1 \le i, j \le r;\ i \ne j.$$

Chapter 14

Isoclinic n-planes in $\mathbb{R}^{2n}$

14.1 Angles between two n-planes in $\mathbb{R}^{2n}$

Consider $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$, and write elements of $\mathbb{R}^{2n}$ as (x, y) with $x, y \in \mathbb{R}^n$. Let $\mathcal{O}$ be the n-plane $y = 0$, and $\mathcal{O}^\perp$ the one with equation $x = 0$. An n-plane $\mathbb{A}$ can be represented by an equation $y = xA$ for an $n \times n$ matrix A if and only if $\mathbb{A} \cap \mathcal{O}^\perp = \{0\}$.

Consider two n-planes with equations $\mathbb{A}: \ y = xA$ and $\mathbb{B}: \ y = xB$ for $n \times n$ matrices A and B. The orthogonal projection of a vector $(u, uA) \in \mathbb{A}$ in the n-plane $\mathbb{B}$ is the vector (v, vB) with

$$v = u(I + AB^t)(I + BB^t)^{-1}.$$

Consequently, the angle $\theta(u)$ between $(u, uA) \in \mathbb{A}$ and its orthogonal projection in $\mathbb{B}$ is given by

$$\cos^2\theta = f(u) := \frac{uPu^t}{uQu^t},$$

where $P = (I+AB^t)(I+BB^t)^{-1}(I+BA^t)$ and $Q = I+AA^t$. There are n critical values of $f(u)$, which are the roots of the polynomial equation

$$\det(P - \lambda Q) = 0.$$

The corresponding values of $\theta \in \left[0, \frac{\pi}{2}\right]$ are the angles between the two n-planes $\mathbb{A}$ and $\mathbb{B}$.

The two n-planes are said to be *isoclinic* if these angles are all equal to θ, equivalently, $P = \lambda Q$ for $\lambda = \cos^2\theta$.

14.2 Maximal sets of isoclinic n-planes in $\mathbb{R}^{2n}$

Consider a set of mutually isoclinic n-planes in $\mathbb{R}^{2n}$. Without loss of generality we consider one of these to be $\mathcal{O}$. The n-plane $\mathcal{O}^\perp$ is clearly isoclinic to $\mathcal{O}$, and every n-plane isoclinic to $\mathcal{O}$ is also isoclinic to $\mathcal{O}^\perp$. Let $\mathbb{A} : y = xA$ be one such n-plane, then

The only n-plane isoclinic to $\mathcal{O}$ that cannot be written in the form $y = xA$ is $\mathcal{O}^\perp$. The matrix A satisfies $AA^t = \sigma I$ for some $\sigma \in \mathbb{R}$, *i.e.*, A is a similarity matrix.

Let $\mathbb{A} : \ y = xA$ and $\mathbb{B} : \ y = xB$ be both isoclinic to $\mathcal{O}$. Then $AA^t = \lambda_1 I$ and $BB^t = \lambda_2 I$ for $\lambda_1, \ \lambda_2 \in \mathbb{R}$. The n-planes $\mathbb{A}$ and $\mathbb{B}$ are themselves isoclinic if and only if $(A + B)(A + B)^t = \lambda I$ for some λ. Equivalently, every linear combination of A and B is a similarity matrix.

Theorem 14.1 (Wong). *A maximal set of mutually isoclinic n-planes in $\mathbb{R}^{2n}$ is equivalent to a subspace of similarities of $\mathbb{R}^n$.*

Chapter 15

Hurwitz-Radon Theorem

15.1 Normed bilinear maps and systems of anticommuting skew symmetric matrices

A normed bilinear map $f : \mathbb{R}^r \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is equivalent to a subspace of similarities (in the euclidean space of linear transformations) of $\mathbb{R}^n$. For a given integer n, we determine the largest possible dimension of a subspace of similarities of $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$.

Let V be an r-dimensional subspace of similarities of $\mathbb{R}^n$. A family $f_1, \dots, f_r \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ form an orthonormal basis of V if and only if

$$\begin{aligned} &f_i^* \circ f_i = \iota, \quad 1 \leq i \leq r, \\ &f_i^* \circ f_j + f_j^* \circ f_i = 0, \quad 1 \leq i, j \leq r,\ i \neq j. \end{aligned}$$

If $\iota \in V$, and $\tau_1 = \iota, \tau_2, \dots, \tau_r$ form an orthonormal basis of V, then

$$\begin{aligned} \tau_i^* &= -\,\tau_i, \quad 2 \leq i \leq r, \\ \tau^2 &= -\,\iota, \quad 2 \leq i \leq r, \\ \tau_i\tau_j &= -\,\tau_j\tau_i, \quad i \neq j,\ 2 \leq i, j \leq r. \end{aligned}$$

Therefore, a normed bilinear map $\mathbb{R}^r \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is equivalent to a system of $r - 1$ mutually anticommutating skew symmetric matrices (of order n) each squaring to $-I_n$. We determine the Hurwitz-Radon function

$$\rho(n) := \max\{r : \exists \text{ normed bilinear map } \mathbb{R}^r \times \mathbb{R}^n \longrightarrow \mathbb{R}^n\}.$$

15.2 (r, s)-families of similarities

Two subspaces V, W of similarities are said to be *amicable* if $\tau^* \circ \varphi = \varphi^* \circ \tau$ for $\tau \in V$ and $\varphi \in W$.

Lemma 15.1. *If V and W are amicable and have orthonormal bases $\iota, \tau_2, \ldots, \tau_r$ for V and*
$\varphi_1, \ldots, \varphi_s$ for W, then
(i) *$\tau_2, \ldots, \tau_r$ are skew and $\tau_i^2 = -\iota$ for $2 \leq i \leq r$;*
(ii) *$\varphi_1, \ldots, \varphi_s$ are symmetric, and $\varphi^2 = \iota$ for $1 \leq j \leq s$;*
(iii) *$\tau_2, \ldots, \tau_r, \varphi_1, \ldots, \varphi_s$ mutually anticommute.*

We shall call a collection of isometries

$$\iota, \tau_2, \ldots, \tau_r;\ \varphi_1, \ldots, \varphi_s$$

satisfying these conditions an (r, s)-family (of isometries on $\mathbb{R}^n$).

Lemma 15.2. (a) *If $\tau : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear map satisfying $\tau^2 = -\iota$, then n must be even.*

(b) *If $\tau_2,\ \tau_3 : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are anticommuting linear maps satisfying $\tau_2^2 = \tau_3^2 = -\iota$, then n must be divisible by 4.*

Proof. (a) Let $V_+ := \{x + \tau(x) : x \in \mathbb{R}^n\}$ and $V_- := \{x - \tau(x) : x \in \mathbb{R}^n\}$. It is easy to check that τ maps V_+ into V_- and V_- into V_+. Since $\tau^2 = -\iota$, the composites $V_+ \longrightarrow V_- \longrightarrow V_+$ and $V_- \longrightarrow V_+ \longrightarrow V_-$ are both $-\iota$. This shows that V_+ and V_- are isomorphic. Since $V_+ \oplus V_- = \mathbb{R}^n$, n must be even. [1]

(b) Let $V_+ := \{x + \tau_2(x) : x \in \mathbb{R}^n\}$ and $V_- := \{x - \tau_2(x) : x \in \mathbb{R}^n\}$. From (a), τ_2 interchanges V_+ and V_-. It is easy to check that τ_3 also interchanges V_+ and V_-. It follows that $\tau := \tau_2\tau_3$ is an endomorphism on V_+. Note that $\tau = \tau_1\tau_2$ satisfies $\tau^2 = -\iota$. This means that $\dim V_+$ is even, and n must be divisible by 4.

□

Lemma 15.3 (Construction lemma). *Given an (r, s)-family*

$$\iota, \tau_2, \ldots, \tau_r;\ \varphi_1, \ldots, \varphi_s$$

[1] Alternatively, the determinant of an odd order skew symmetric matrix is zero.

on $\mathbb{R}^n$, there is an $(r+1, s+1)$-family on $\mathbb{R}^{2n}$

$$\begin{pmatrix} \iota & \\ & \iota \end{pmatrix}, \begin{pmatrix} \tau_2 & \\ & \tau_2 \end{pmatrix}, \ldots, \begin{pmatrix} \tau_r & \\ & \tau_r \end{pmatrix}, \begin{pmatrix} & -\iota \\ \iota & \end{pmatrix};$$
$$\begin{pmatrix} \varphi_1 & \\ & \varphi_1 \end{pmatrix}, \ldots\ldots, \begin{pmatrix} \varphi_s & \\ & \varphi_s \end{pmatrix}, \begin{pmatrix} & \iota \\ \iota & \end{pmatrix}.$$

Lemma 15.4 (Subspace lemma). *If there is an (r, s)-family on $\mathbb{R}^n$ with $r \geq 2$, $s \geq 1$, then n is even, say $n = 2m$, and there is an $(r-1,\ s-1)$-family on $\mathbb{R}^m$.*

Lemma 15.5 (Shift lemma). *Suppose there is an (r, s)-family on $\mathbb{R}^n$.*
If $r > 4$, then there is an $(r-4,\ s+4)$-family on $\mathbb{R}^n$,
If $s \geq 4$, then there is an $(r+4,\ s-4)$-family on $\mathbb{R}^n$.

Lemma 15.6 (Expansion lemma). *Suppose there is an (r, s)-family on $\mathbb{R}^n$.*
If $r - s \equiv 3 \bmod 4$, then this can be expanded into an $(r+1,\ s)$-family on $\mathbb{R}^n$,
If $r - s \equiv 1 \bmod 4$, then this can be expanded into an $(r,\ s+1)$-family on $\mathbb{R}^n$.

Lemma 15.7 (Reduction lemma). *If $\rho(n) \geq 9$, then n is divisible by 16, and $\rho\left(\frac{n}{16}\right) \geq \rho(n) - 8$.*

Proposition 15.8 (Upper bound theorem). *Let $n = 2^t(2c+1)$.*
(a) $\rho(n) \leq 2t + 2$.
(b) *The maximum possible value of $r + s$ for an (r, s)-family on $\mathbb{R}^n$ is $2t + 2$.*

15.2.1 Proof of subspace lemma

Let

$$\iota, \tau_2, \ldots, \tau_r;\ \varphi_1, \ldots, \varphi_s$$

be an (r, s)-family on $\mathbb{R}^n$. Consider $h := \tau_r\varphi_s$. Note that
(i) $h^2 = \iota$,
(ii) $h\tau_r = -\tau_r h$, and
(iii) h commutes with each of $\tau_2, \ldots, \tau_{r-1}, \varphi_1, \ldots, \varphi_{s-1}$. From (i), there is an orthogonal decomposition

$$\mathbb{R}^n = V_+ \oplus V_-,$$

where

$$V_+ := \{x + h(x) : \ x \in \mathbb{R}^n\},$$
$$V_- := \{x - h(x) : \ x \in \mathbb{R}^n\}.$$

In fact, V_+ and V_- are respectively the eigenspaces of h corresponding to the eigenvalues $+1$ and -1. We show that τ_r interchanges V_+ and V_-. For $x \in V_+$,

$$\tau_r(x) = \tau_r(h(x)) = -h\tau_r(x).$$

Therefore, $\tau_r(x) \in V_-$. Similarly, $\tau_r(x) \in V_+$ for $x \in V_-$. It follows that $\dim V_+ = \dim V_-$, and n must be even. We write $n = 2m$.

From (iii), each of $\tau_2, \ldots, \tau_{r-1}$, and $\varphi_1, \ldots, \varphi_{s-1}$ preserves V_+ (and V_-). Identifying V_+ with $\mathbb{R}^m$, we have an $(r-1, s-1)$-family on $\mathbb{R}^m$.

15.2.2 Proof of shift and expansion lemmas

Let A be a noncommutative ring with unit. Consider in A distinct pairwise anticommuting elements $a_1, \ldots, a_h$. Put $a = a_1 \cdots a_h$.

(1) a anticommutes with each of $a_1, \ldots, a_h$ if and only if h is even.

(2) Suppose $a_i = \varepsilon = \pm 1$. Then $a^2 = (-1)^{\frac{1}{2}h(h-\varepsilon)}$. In other words, (i) if each $a_i^2 = -1$, then

$$a^2 = \begin{cases} 1, & h \equiv 0, 3 \bmod 4, \\ -1, & h \equiv 1, 2 \bmod 4; \end{cases}$$

(ii) if each $a_i^2 = 1$, then

$$a^2 = \begin{cases} 1, & h \equiv 0, 1 \bmod 4, \\ -1, & h \equiv 2, 3 \bmod 4. \end{cases}$$

(3) If a_1, a_2, a_3, a_4 are anticommuting elements of A and $a = a_1a_2a_3a_4$. Put $a_i' = aa_i$ for $i = 1, 2, 3, 4$. Then a_1', a_2', a_3', a_4' are anticommuting, and each ${a_i'}^2 = -1$ if and only if each $a_i^2 = 1$.

(4) Suppose $a_1, \ \ldots, a_h$ and $b_1, \ \ldots, b_k$ are all distinct and mutually anticommute. Let $a = a_1 \cdots a_h$ and $b = b_1 \cdots b_k$. Then $ab = (-1)^{hk}ba$.

15.2.3 Proof of shift lemma

Let

$$\iota, \tau_2, \ldots, \tau_r; \ \varphi_1, \ldots, \varphi_s$$

be an (r, s)-family on $\mathbb{R}^n$. If $r > 4$, let $\tau = \tau_{r-3}\tau_{r-2}\tau_{r-1}\tau_r$, and

$$\varphi_{s+1} = \tau\tau_{r-3},\ \varphi_{s+2} = \tau\tau_{r-2},\ \varphi_{s+3} = \tau\tau_{r-1},\ \varphi_{s+4} = \tau\tau_r.$$

We have an $(r - 4,\ s + 4)$-family

$$\iota, \tau_2, \ldots, \tau_{r-4};\ \varphi_1, \ldots, \varphi_s, \varphi_{s+1}, \varphi_{s+2}, \varphi_{s+3}, \varphi_{s+4}.$$

The proof for an $(r + 4, s - 4)$-family given $s \geq 4$ is similar.

15.2.4 Proof of expansion lemma

Let

$$\iota, \tau_2, \ldots, \tau_r;\ \varphi_1, \ldots, \varphi_s$$

be an (r, s)-family on $\mathbb{R}^n$. Consider

$$z = \tau_2 \cdots \tau_r \varphi_1 \cdots \varphi_s.$$

Note that this contains $r + s - 1$ factors. By (1), z anticommutes with each of $\tau_2, \ldots, \tau_r,\ \varphi_1, \ldots, \varphi_s$ if and only if $r + s - 1$ is even. Write

$$\tau = \tau_2 \cdots \tau_r, \quad \varphi = \varphi_1 \cdots \varphi_s.$$

Note that

$$\begin{aligned} z^2 =& (\tau\varphi)^2 \\ =& (-1)^{(r-1)s}\tau^2\varphi^2 \\ =& (-1)^{(r-1)s + \frac{1}{2}(r-1)r + \frac{1}{2}s(s-1)}\iota \\ =& (-1)^{\frac{1}{2}(r+s-1)^2 + \frac{1}{2}(r-s-1)}\iota. \end{aligned}$$

Assuming $r + s - 1$ even, we easily see that

$$z^2 = \begin{cases} \iota, & \text{if } r - s - 1 \equiv 0 \bmod 4, \\ -\iota, & \text{if } r - s - 1 \equiv 2 \bmod 4. \end{cases}$$

Therefore, we obtain an $(r + 1, s)$-family

$$\iota, \tau_2, \ldots, \tau_r, z;\ \varphi_1, \ldots, \varphi_s$$

if $r - s \equiv -1 \bmod 4$, and an $(r, s + 1)$-family

$$\iota, \tau_2, \ldots, \tau_r;\ z, \varphi_1, \ldots, \varphi_s$$

if $r - s \equiv 1 \bmod 4$.

15.2.5 Proof of reduction lemma

Let $r \geq 9$ and there be an $(r, 0)$-family on $\mathbb{R}^n$.

$$
\begin{aligned}
&(r,0) - \text{family on } \mathbb{R}^n \\
\Rightarrow &(r-4,\ 4) - \text{family on } \mathbb{R}^n \\
\Rightarrow &(r-8,\ 0) - \text{family on } \mathbb{R}^{\frac{n}{16}}
\end{aligned}
$$

15.2.6 Proof of upper bound theorem

(a) For $t \geq 4$, $\rho(2^t(2c+1)) \geq 2t+3 \Rightarrow \rho(2^{t-4}(2c+1)) \geq 2(t-4)+3$ by the reduction lemma. We need only consider $t \leq e3$. For $t = 0, 1$, this follows from Lemma 15.2. In fact, $\rho(2(2c+1)) \leq 2$.

For $t = 2$, if $\rho(4(2c+1)) \geq 7$, we have by restriction a $(5, 0)$-family. By the expansion lemma, there is a $(5, 1)$-family on $\mathbb{R}^{4(2c+1)}$. By the subspace lemma, there is a $(4, 0)$-family on $\mathbb{R}^{2(2c+1)}$, contradicting $\rho(2(2c+1)) \leq 2$.

For T$t = 3$, if $\rho(8(2c+1)) \geq 9$, we have a $(9, 0)$-family on $\mathbb{R}^{8(2c+1)}$. By the expansion lemma, there is a $(9, 1)$-family. By the subspace lemma, there is a $(8, 0)$ family on $\mathbb{R}^{4(2c+1)}$, contradicting $\rho(4(2c+1)) \leq 6$.

(b) Beginning with a $(2, 2)$-family on $\mathbb{R}^2$,

$$
\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}; \quad \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}, \begin{pmatrix} 1 & \\ & -1 \end{pmatrix},
$$

on $\mathbb{R}^2$, we easily construct a $(t+1,\ t+1)$-family on $\mathbb{R}^{2^t}$, and so on $\mathbb{R}^{2^t(2c+1)}$.

To show that $r + s \leq 2t + 2$ we use induction. This is clearly true for $t = 1$, and for $n = 2(2c+1)$ more generally.

Suppose there is an (r, s)-family on $\mathbb{R}^n$ with $r + s = 2t + 3$. By (a), $r \leq 2t + 2$. It follows that $s \geq 1$, and $r \geq 2$. By the subspace lemma there is an $(r-1,\ s-1)$-family on $\mathbb{R}^{2^{t-1}(2c+1)}$ with $(r-1) + (s-1) = 2(t-1) + 3$. Continuing, we obtain an (r, s)-family on $\mathbb{R}^{2(2c+1)}$ with $r + s = 5$. This is a contradiction since $r \leq 2$. If $r = 1$, from a $(1, 4)$-family, we obtain a $(5, 0)$-family, an impossibility by the shift lemma. If $r = 2$, from a $(2, 3)$-family, we obtain a $(1, 2)$-family on $\mathbb{R}^{2c+1}$. This is an impossibility since the existence of anticommuting φ_1 and φ_2 satisfying $\varphi_1^2 = \varphi_2^2 = \iota$ requires the dimension to be even.

15.3 Hurwitz-Radon theorem

Theorem 15.9 (Hurwitz-Radon). *For a given integer $n = 2^t(2c+1)$, the largest possible value of r for which there exists a normed bilinear map $\mathbb{R}^r \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is*

$$\rho(n) = \begin{cases} 2t+1, & t \equiv 0 \bmod 4, \\ 2t, & t \equiv 1, 2 \bmod 4, \\ 2t+2, & t \equiv 3 \bmod 4. \end{cases}$$

Proof. By the upper bound theorem (Proposition 15.8) $\rho(n) \leq 2t+2$. Indeed, if an $(r, 0)$-family on $\mathbb{R}^n$ is extended to an (r, s)-family for some s, $r + s \leq 2t + 2$. Note that there is a $(t+1,\ t+1)$-family on $\mathbb{R}^n$. We examine 4 cases.

$t \equiv 3 \bmod 4$

If $t \equiv 3 \bmod 4$, by the shift lemma, there is a $(2t+2,\ 0)$-family on $\mathbb{R}^n$. Therefore, there is a $(2t+2,\ 0)$-family, and

$$\rho(n) = 2t+2 \quad \text{if } t \equiv 3 \bmod 4.$$

$t \equiv 0 \bmod 4$

By the shift lemma, there is a $(2t+1, 1)$-family on $\mathbb{R}^n$. We claim that $\rho(n) = 2t+1$, for if $\rho(n) \geq 2t+2$, then by the reduction lemma, $\rho(2c+1) = \rho\left(\frac{n}{2^t}\right) \geq 2$, a contradiction.

$t \equiv 1 \bmod 4$

By the shift lemma, there is a $(2t, 2)$-family on $\mathbb{R}^n$. We claim that $\rho(n) = 2t$, for if $\rho(n) \geq 2t+1$, then by the reduction lemma, $\rho(2(2c+1)) = \rho\left(\frac{n}{2^{t-1}}\right) \geq (2t+1) - (2t-2) = 3$, a contradiction.

$t \equiv 2 \bmod 4$

We need only consider $t > 4$. By the shift lemma, there is a $(2t-1, 3)$-family. By restriction we have a $(2t-1, 0)$-family. Since $(2t-1) - 0 \equiv 3 \bmod 4$, by the expansion lemma, we can extend this to a $(2t, 0)$-family.

We claim that $\rho(n) = 2t$, for if $\rho(n) \geq 2t + 1$, then by the reduction lemma, $\rho(4(2c+1)) = \rho\left(\frac{n}{2^{t-2}}\right) \geq (2t+1) - 2(t-2) = 5$, a contradiction.

$\square$

Chapter 16

Fundamental group

16.1 The loop space $\Omega(X, x_0)$ and the fundamental group $\pi_1(X, x_0)$

Let X be a topological space with a basepoint $x_0 \in X$. The space of paths in X emanating from x_0 is the space

$$P(X, x_0) := \{\omega \in X^I : \ \omega(0) = x_0\}.$$

The space of loops based at x_0 is the subspace

$$\Omega(X, x_0) := \{\omega \in X^I : \ \omega(0) = \omega(1) = x_0\}.$$

There is a natural addition of loops: for ω_1, $\omega_2 \in \Omega(X, x_0)$, their concatenation $\omega_1 \cdot \omega_2 = \omega$ is given by

$$\omega(t) = \begin{cases} \omega_1(2t), & 0 \leq t \leq \frac{1}{2}, \\ \omega_2(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

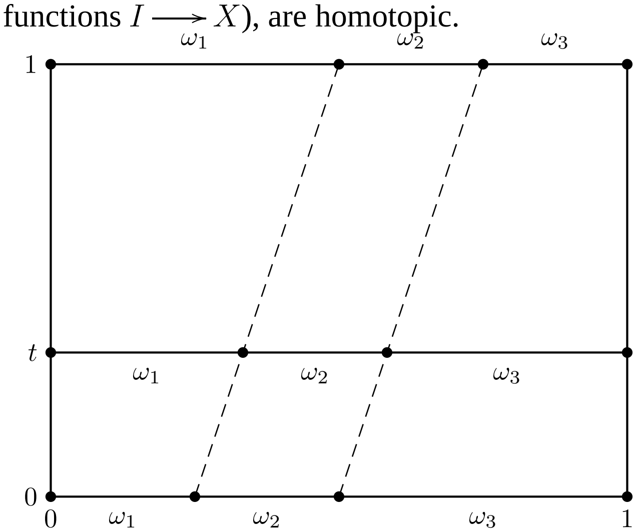
Two loops ω_0 and ω_1 are homotopic if there is a path in $\Omega(X, x_0)$ connecting ω_0 to ω_1, ***i.e.***, a homotopy $H : I \times I \longrightarrow X$ satisfying

$$\begin{array}{ll} H(t, 0) = \omega_0(t), & H(t, 1) = \omega_1(t); \\ H(0, s) = x_0, & H(1, s) = x_0. \end{array}$$

The set of homotopy classes of loops in X with basepoint x_0 is denoted by $\pi_1(X, \ x_0)$. The following proposition justifies calling this the **fundamental group** of X (with basepoint x_0).

Proposition 16.1. *The loop addition induces a group structure on* $\pi_1(X,\ x_0)$.

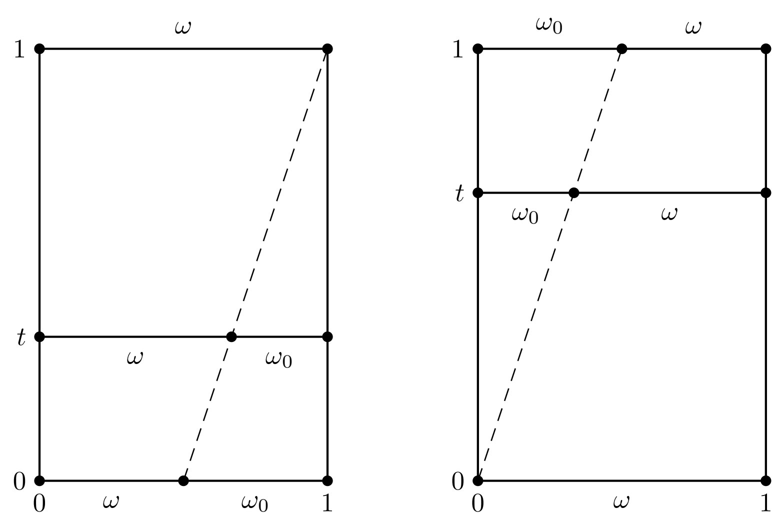
Proof. (1) Associativity. $(\omega_1 \cdot \omega_2) \cdot \omega_3$ and $\omega_1 \cdot (\omega_2 \cdot \omega_3)$, though not strictly equal (as functions $I \longrightarrow X$), are homotopic.



(2) Identity. The constant path $\omega_0 : I \longrightarrow X$ (with $\omega_0(t) = x_0$ for every $t \in I$) satisfies

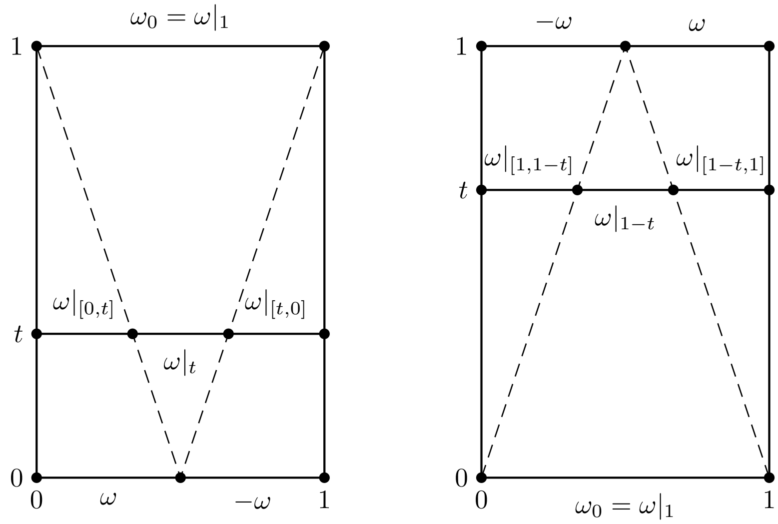
$$\omega \cdot \omega_0 \sim \omega \sim \omega_0 \cdot \omega$$

for every $\omega \in \Omega(X, x_0)$.



(3) Inverse. For $\omega : I \longrightarrow X$, let $-\omega$ be the loop defined by $-\omega(t) = \omega(1-t)$, $t \in I$. Then

$$\omega \cdot (-\omega) \sim \omega_0 \sim (-\omega) \cdot \omega.$$



□

Proposition 16.2. $\pi_1(-)$ *is a covariant functor from the category of spaces (with basepoints and basepoint preserving maps) to the category of groups (and homomorphisms).*

If $f, g : (X, x_0) \longrightarrow (Y, y_0)$ are homotopic rel endpoints, then $f_* = g_*$.

Proposition 16.3. *If* $f : (X, x_0) \longrightarrow (Y, y_0)$ *is a homotopy equivalence, then* $f_* : \pi_1(X, x_0) \longrightarrow \pi_1(Y, y_0)$ *is an isomorphism.*

16.1.1 Conjugacy of fundamental groups

Let $x_0,\ x_1 \in X$ be in the same path component, *i.e.*, there is a path $\alpha : I \longrightarrow X$ with $\alpha(0) = x_0$ and $\alpha(1) = x_1$. Such a path induces an isomorphism $\alpha_* : \pi_1(X,\ x_0) \longrightarrow \pi_1(X,\ x_1)$, namely, $\alpha_*([\omega]) = [\omega']$, with

$$\omega(t) = \begin{cases} \alpha(1-3t), & 0 \le t \le \frac{1}{3}, \\ \omega(3t-1), & \frac{1}{3} \le t \le \frac{2}{3}, \\ \alpha(3t-2), & \frac{2}{3} \le t \le 1. \end{cases}$$

We shall simply write $\pi_1(X)$ for $\pi_1(X,\ x_0)$ when there is no danger of confusion of the basepoint, or when only the isomorphism class of the fundamental group is relevant.

A connected space X is *simply connected* if $\pi_1(X) = 0$. Clearly, a contractible space is simply connected. Also, $\mathbb{S}^n$ is simply connected when $n \geq 2$.

16.2 The fundamental group of the circle

The first nontrivial example of fundamental group is that of the circle. We take as basepoint of $\mathbb{S}^1$ the complex number 1.

Theorem 16.4. $\pi_1(\mathbb{S}^1,\ 1)$ *is isomorphic to the additive group of integers.*

Proof. Consider the **exponential map** $\exp : \mathbb{R} \longrightarrow \mathbb{S}^1$ given by

$$\exp(t) = e^{2\pi i t},\ t \in \mathbb{R}.$$

Clearly, $\exp^{-1}(1) = \mathbb{Z}$.

A loop in $\Omega(\mathbb{S}^1,\ 1)$ is given by a map $\omega : I \longrightarrow \mathbb{S}^1$ satisfying $\omega(0) = \omega(1) = 1$. Such a loop is the image under $\exp$ of a unique path $\widetilde{\omega} : I \longrightarrow \mathbb{R}$ with initial point $\widetilde{\omega}(0) = 0$. Note that $\widetilde{\omega}(1)$ is an integer, which we denote by $\deg(\omega)$, and call the *degree* of ω.

(1) $\deg(\omega)$ depends only on the homotopy class of ω. In other words, if $\omega' \in \Omega(\mathbb{S}^1,\ 1)$ is homotopic to ω rel endpoints, then $\deg(\omega') = \deg(\omega)$. Therefore we have a function $\deg : \pi_1(\mathbb{S}^1,\ 1) \longrightarrow \mathbb{Z}$.

(2) Clearly, every integer n is equal to $\deg(\omega)$ for some loop ω in $\mathbb{S}^1$. If $n \neq 0$, the path $f_n : I \longrightarrow \mathbb{R}$ given by $f_n(t) = nt$ clearly projects under $\exp$ into a loop in $\mathbb{S}^1$ (which winds around the circle n times, counterclockwise or clockwise according as n is positive or negative). If $n = 0$, we simply take the constant loop. This shows that $\deg$ is surjective.

(3) On the other hand, $\deg$ is injective. This follows from the fact that any two paths in $\mathbb{R}$ with the same endpoints are homotopic rel endpoints. Therefore, $\deg : \pi_1(\mathbb{S}^1,\ 1) \longrightarrow \mathbb{Z}$ is a bijection.

(4) Consider two loops ω_1 and ω_2 in $\Omega(\mathbb{S}^1,\ 1)$. If these are "covered" under $\exp$ by paths $\widetilde{\omega_1}, \widetilde{\omega_2}$ with $\widetilde{\omega_1}(1) = m$ and $\widetilde{\omega_2}(1) = n$, then the loop $\omega_1 \cdot \omega_2$ is covered by the path which is the "concatenation" of $\widetilde{\omega_1}$ and $\widetilde{\omega_2}$:

$$\widetilde{\omega}(t) = \begin{cases} \widetilde{\omega_1}(2t), & 0 \leq t \leq \frac{1}{2}, \\ m + \widetilde{\omega_2}(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Clearly,

$$\deg([\omega_1]\cdot[\omega_2]) = \deg([\omega_1 \cdot \omega_2]) = \widetilde{\omega}(1) = m+n = \deg([\omega_1]) + \deg([\omega_2]).$$

This means that $\deg : \pi_1(\mathbb{S}^1,\ 1) \longrightarrow \mathbb{Z}$ is a homomorphism. Since $\deg$ is bijective, it is an isomorphism. □

16.3 Fundamental theorem of algebra

Theorem 16.5. *Every nonconstant polynomial with complex coefficients has a zero in* $\mathbb{C}$.

Proof. We deduce a contradiction by assuming a polynomial

$$p(z) = z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$$

without zero. Considered the polynomial as a map $p : \mathbb{S}^1 \longrightarrow \mathbb{C} - \{0\}$.

Write $p(z) = z^n + f(z)$. We show that for sufficiently large $r > 0$, if $||z|| = r$, $z^n + tf(z)$ is nonzero for $t \in [0, 1]$. This follows from

$$\begin{aligned} ||z^n + tf(z)|| &> ||z||^n - t||f(z)|| \\ &> ||z||^n - ||f(z)|| \\ &> ||z||^n - \sum_{k=1}^{n} ||a_k z^{n-k}|| \\ &> ||z||^n - \left(\sum_{k=1}^{n} ||a_k||\right) ||z||^{n-1} \qquad \text{assuming } ||z|| > 1 \\ &= \left(r - \sum_{k=1}^{n} ||a_k||\right) r^{n-1}, \end{aligned}$$

which is nonzero if we choose $r = \max(\sum_{k=1}^{n} ||a_k||, 1)$.

Thus, via the homotopy $p_t : \mathbb{S}^1 \times [0, 1] \longrightarrow \mathbb{C} - \{0\}$ given by $p_t(z) = (rz)^n + tf(rz)$, the maps $z \mapsto p(rz)$ and $z \mapsto (rz)^n$ are homotopic. Regarded as a map $\mathbb{S}^1 \longrightarrow \mathbb{S}^1$, the latter map clearly has degree n. This is a contradiction since the map $z \mapsto p(rz)$ is homotopic to the constant map $z \mapsto a_n$ via the homotopy $z \mapsto p(trz)$. □

Chapter 17

Covering spaces

17.1 Covering maps

The exponential map $\exp : \mathbb{R} \longrightarrow \mathbb{S}^1$ is an example of a covering map. We say that a map $p : E \longrightarrow B$ is a covering map if each $b \in B$ has a neighborhood U such that $p^{-1}(U)$ is the disjoint union of open sets each of which is mapped homeomorphically onto U by p.

$$\begin{array}{ccc} p^{-1}(U) & \xrightarrow{\approx} & U \times p^{-1}(b) \\ & {\scriptstyle p} \searrow \quad \swarrow {\scriptstyle \pi} & \\ & U & \end{array}$$

Here are some examples of covering maps.

(1) The exponential map $\exp : \mathbb{R} \longrightarrow \mathbb{S}^1$: $\exp(t) = e^{2\pi i t}$.

(2) The plane $\mathbb{R}^2$ covering the torus: $\mathbb{R}^2 \longrightarrow \mathbb{S}^1 \times \mathbb{S}^1$.

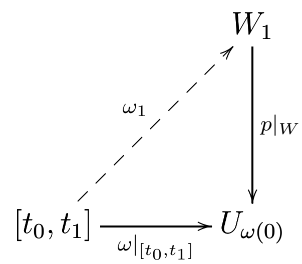
(3) $\mathbb{S}^n \longrightarrow \mathbb{R}\mathrm{P}^n$ is a double covering.

Proposition 17.1 (Path lifting property). *Let ω be a path in B with initial point b and $e \in E$ covering b,* i.e, *$p(e) = b$. There is a unique lifting $\widetilde{\omega}$ of ω with initial point $e \in E$.*

Proposition 17.2 (Homotopy lifting property). *Let $h_t : I \longrightarrow B$ be a homotopy of free paths in B, and $\widetilde{\omega}$ a path in E covering theititial points $h_t(0)$. Then there is a unique homotopy $\widetilde{h}_t$ covering h_t.*

Proof of path lifting property

Let $\{U_b : b \in B\}$ be a family of coordinate neighborhoods. Consider a path $\omega : I \longrightarrow B$. The family $\{\omega^{1}(U_b) : b \in B\}$ is an open cover of I, say with Lebesque number ε. Choose n so that $\frac{1}{n} < \varepsilon$, and subdivide I equally into n equal subintervals $[t_k, t_{k+1}]$ for $0 \leq k \leq n-1$. Each $[t_k, t_{k+1}]$ is contained in some $\omega^{-1}(U_b)$. In particular, $[t_0, t_1] \subseteq \omega^{-1}(U_{\omega(0)})$. There is a unique neighborhood W_1 of e homeomorphic to $U_{\omega(0)}$, and subsequently a unique lifting of $\omega|_{[t_0, t_1]}$ to W_1:



Iterating, we obtain a unique lifting over each $[t_k, t_{k+1}]$ for $k = 1, \ldots, n-1$. These together give a unique lifting of ω over I. □

17.2 Covering spaces and fundamental groups

Proposition 17.3. (a) Monodromy. *If ω_1 and ω_2 are homotopic paths (rel endpoints), and $\widetilde{\omega_1}$ and $\widetilde{\omega_2}$ their liftings with the same initial point, then $\widetilde{\omega_1}$ and $\widetilde{\omega_2}$ have the same endpoint.*

(b) *If B is path-connected, all $p^{-1}(b)$ have the same cardinality.*

(c) *Let $b_0 \in B$ and $F = p^{-1}(b_0)$. The fundamental group $\pi_1(B, b_0)$ acts as a group of permutations of F.*

Proof. (a) Let h_t be a homotopy of paths $\omega_1 \sim \omega_2$ rel endpoints b_0 and b_1. By the homotopy lifting property, there is a homotopy $\widetilde{h}_t$ covering h_t. The path $\widetilde{h}_1$ has image in the discrete set $p^{-1}(b_1)$. It must be constant. This means $\widetilde{\omega_1}$ and $\widetilde{\omega_2}$ have the same endpoint.

(b) Let ω be a path in B connecting b_0 and b_1. By the unique path lifting property, this induces a map $\omega_* : p^{-1}(b_0) \longrightarrow p^{-1}(b_1)$ given by

$$\omega_*(e) = \widetilde{\omega}_e(1),$$

where $\widetilde{\omega}_e$ is the unique lifting of ω with initial point $e \in p^{-1}(b_0)$. This is a bijection with inverse ω_*^{-1} similarly defined.

(c) Let $F = p^{-1}(b_0)$ and ω be a loop with $\omega(0) = \omega(1) = b_0$. By (a), for each $e \in F$, $\widetilde{\omega}_e(1)$ depends only on the homotopy class of ω in $\pi_1(B, b_0)$. There is a map $\phi : F \times \pi_1(B, b_0) \longrightarrow F$ defined by

$$\phi(e, [\omega]) = \widetilde{\omega}_e(1).$$

This satisfies
(i) $\phi(e, [\omega_1] \cdot [\omega_2]) = \phi(\phi(e, [\omega_1]), [\omega_2])$,
(ii) $\phi(e, [\omega_0]) = e$ for the constant path ω_0. □

In particular, if $e_0 \in F$, the map $\phi_0 : \pi_1(B, b_0) \longrightarrow F$ given by $\phi_0([\omega]) = \phi(e_0, [\omega])$ is onto. This is because for every $e_1 \in F$ and a path $\widetilde{\omega}$ in E joining e_0 to e_1, if we put $\omega = p \circ \widetilde{\omega}$, this is a loop at b_0 and clearly

$$\phi_0([\omega]) = \phi(e_0, [\omega]) = \widetilde{\omega}_{e_0}(1) = e_1.$$

17.3 Universal covering

A covering map $p : E \longrightarrow B$ is universal if the total space E is simply connected.

Proposition 17.4. *If* $p : E \longrightarrow B$ *is a universal covering, there is a one-to-one correspondence* $\pi_1(B, b_0) \longrightarrow F$.

Proof. If E is simply connected, we show that ϕ_0 is one-to-one. Suppose $\phi_0([\omega_0]) = \phi_0([\omega_1])$. This means that the liftings of ω_0 and ω_1 (with common initial point e_0) have the same endpoint e_1. Since E is simply connected, there is a homotopy $f_t : I \longrightarrow E$ satisfying $f_t(0) = e_0$ and $f_t(1) = e_1$ for $t \in [0, 1]$. It is clear that $p \circ f_t$ is a homotopy of ω_0 and ω_1. This shows that ϕ_0 is a one-to-one correspondence. □

Remark. If E is simply connected, the one-to-one correspondence ϕ_0 makes F into a group with identity e_0. For $e_1,\ e_2 \in F$, let ω be a loop corresponding to e_1. Find a lifting $\widetilde{\omega}$ with initial point e_2. The end point of this lifting is the product $e_1 * e_2$.

Proposition 17.5. (a) $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \oplus \mathbb{Z}$.
(b) $\pi_1(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}_2$ *for* $n \geq 2$.

17.4 Borsuk-Ulam theorem for $\mathbb{S}^2$

Theorem 17.6 (Borsuk-Ulam). *There is no continuous map* $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^1$ *satisfying* $f(-x) = -f(x)$.

Proof. We deduce a contradiction by assuming an antipodal map $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^1$. Such a map induces $g : \mathbb{R}\mathrm{P}^2 \longrightarrow \mathbb{S}^1$ making the diagram

$$\begin{array}{ccc} \mathbb{S}^2 & \xrightarrow{\ f\ } & \mathbb{S}^1 \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_1} \\ \mathbb{R}\mathrm{P}^2 & \xrightarrow[\ g\ ]{} & \mathbb{S}^1 \end{array}$$

commutative.

Let ω be a path in $\mathbb{S}^2$ whose endpoints are antipodal. Since f is antipodal, the endpoints of $f \circ \omega$ are antipodal on $\mathbb{S}^1$. Note that $p_2 \circ \omega$ and $p_1 \circ f \circ \omega$ are loops in $\mathbb{R}\mathrm{P}^2$ nd $\mathbb{S}^1$ respectively. Now $[p_2 \circ \omega]$ and $[p_1 \circ f \circ \omega]$ are nontrivial elements in the fundamental groups since they acts nontrivially on $p_2^{-1}(x_0)$ and $p_1^{-1}(y_0)$. This contradicts the commutativity of the diagram above since $g_* : \pi_1(\mathbb{R}\mathrm{P}^2) \longrightarrow \pi_1(\mathbb{S}^1)$ is trivial. $\square$

Corollary 17.7. (a) *Let* $f : \mathbb{S}^2 \longrightarrow \mathbb{R}^2$ *be a continuous map satisfying* $f(-x) = -f(x)$. *Then there exists* $x \in \mathbb{S}^2$ *such that* $f(x) = 0$.

(b) *Let* $f : \mathbb{S}^2 \longrightarrow \mathbb{R}^2$ *be a continuous map. There exists* $x \in \mathbb{S}^2$ *such that* $f(x) = f(-x)$.

Chapter 18

Calculation of fundamental groups

18.1 Van Kampen theorem

Consider a space X with basepoint x_0. Suppose $X = X_1 \cup X_2$, both X_1, X_2 containing the basepoint x_0.

Theorem 18.1 (van Kampen theorem). *If $X_1 \cap X_2$ is path-connected and $X = \text{int}\, X_1 \cup \text{int}\, X_2$, then*

$$\begin{array}{ccc} \pi_1(X_1 \cap X_2, x_0) & \xrightarrow{i_{2*}} & \pi_1(X_2, x_0) \\ \downarrow{\scriptstyle i_{1*}} & & \downarrow{\scriptstyle j_{2*}} \\ \pi_1(X_1, x_0) & \xrightarrow[j_{1*}]{} & \pi_1(X, x_0) \end{array}$$

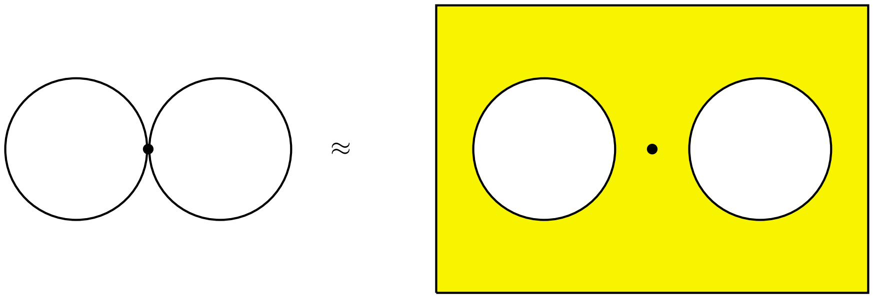
is a pushout diagram.

This means $\pi_1(X, x_0)$ is isomorphic to the quotient F/R where
(i) F is the *free product* of $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$, and
(ii) R is the normal subgroup of F generated by words of the form $i_{1*}(x)i_{2*}(x)^{-1}$, $x \in \pi_1(X_1 \cap X_2, x_0)$.

In particular, if $X_1 \cap X_2$ is simply connected, then $\pi_1(X, x_0)$ is the *free product* of $\pi_1(X_1, x_0)$ and $\pi_1(X_2, x_0)$.

Examples

(1) Consider the one-point union of two circles, $\mathbb{S}^1 \vee \mathbb{S}^1$. The fundamental group $\pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)$ is the free group on 2 generators.



(2) $\pi_1(\mathbb{S}^n) = 0$ for $n \geq 2$.

18.2 Fundamental group from triangulations

Chapter 19

Group structures on $[X, Y]$

Let X and Y be given spaces with basepoints x_0 and y_0. We shall write simply $[X, Y]$ for the set of homotopy classes of basepoint preserving maps $f : (X, x_0) \longrightarrow (Y, y_0)$. We consider the possibility of endowing with $[X, Y]$ a natural group structure.

19.1 Topological groups

A topological group is a group with a topology in which the multiplication and inversion maps are continuous. Examples abound.

1. The real line $\mathbb{R}$.

2. The spheres $\mathbb{S}^n$ for $n = 1, 3$. $\mathbb{S}^7$ is not a topological group; it is an H-space.

3. The general linear groups $GL(n)$.

4. The orthogonal groups $O(n)$ and special groups $SO(n)$.

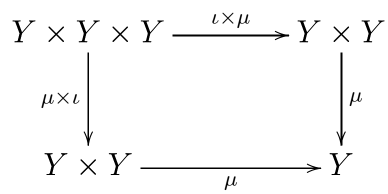
If Y is a topological group, then for every X, the function space Y^X is a topological group: $(f \cdot g)(x) = f(x) \cdot g(x)$ for $x \in X$. This induces a group operation on $[X, Y]$:

$$[f] \cdot [g] = [f \cdot g].$$

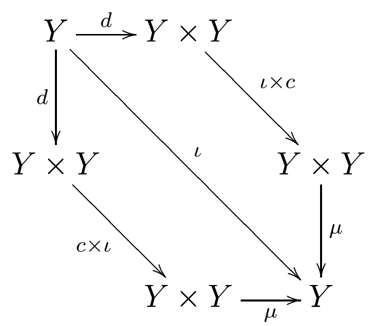
19.2 H-spaces

Let (Y, y_0) be a space with basepoint. Denote by $d : Y \longrightarrow Y \times Y$ the diagonal map $d(x) = (x,\ x)$. We say that Y is an H-space if there are

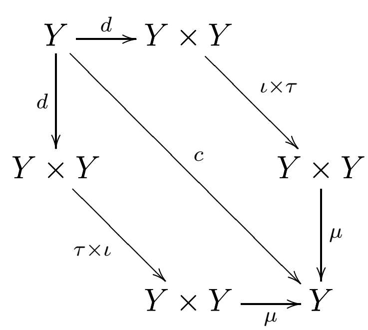
(i) a multiplication $\mu : Y \times Y \to Y$ and,
(ii) an inverse map $\tau : Y \to Y$ such that
(1) μ is homotopy associative, *i.e.*, the diagram



is homotopy commutative;
(2) the constant map $c : X \longrightarrow Y$ (mapping into y_0) is a homotopy identity, *i.e.*, the diagram



is homotopy commutative;
(3) τ is a homotopy inverse map, *i.e.*, the diagram



is homotopy commutative.

An H-space Y (with multiplication μ and inverse map τ) is homotopy abelian if, with with $\phi : Y \times Y \longrightarrow Y \times Y$ defined by $\phi(y_1, y_2) = (y_2, y_1)$, the diagram

$$\begin{array}{ccc} Y \times Y & \xrightarrow{\phi} & Y \times Y \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\mu} & \\ & Y & \end{array}$$

is homotopy commutative. If Y is homotopy abelian, then the group $[X, Y]$ is abelian.

Let (Y_1, μ_1) and (Y_2, μ_2) be H-spaces. A map $f : Y_1 \longrightarrow Y_2$ is called an H-map if the diagram

$$\begin{array}{ccc} Y_1 \times Y_1 & \xrightarrow{f \times f} & Y_2 \times Y_2 \\ \Big\downarrow{\scriptstyle \mu_1} & & \Big\downarrow{\scriptstyle \mu_2} \\ Y_1 & \xrightarrow[f]{} & Y_2 \end{array}$$

is homotopy commutative.

Proposition 19.1. *Let Y be a space with basepoint y_0. Then $[-,\ Y]$ is a functor into the category of groups if and only if Y is an* H-*space.*

Proof. It is enough to prove the necessity part. Assume that the functor $[-,\ Y] : \mathbf{Top} \longrightarrow \mathbf{Set}$ lifts into $\mathbf{Grp}$, *i.e.*, there is a natural group structure on each $[X, Y]$. We show this assumption uniquely determines an H-space structure on Y. Let $p_1,\ p_2 : Y \times Y \longrightarrow Y$ be the projections from the first and second factors respectively. Choose
(i) $\mu : Y \times Y \longrightarrow Y$ such that

$$[\mu] = [p_1] + [p_2]$$

in $[Y \times Y,\ Y]$, [1]
(ii) $\tau : Y \longrightarrow Y$ such that $[\tau] = -[\iota_Y]$.

We show that for any space X with basepoint x_0, the group structure on $[X, Y]$ is determined by μ:

$$\begin{aligned} [\mu(u \times v)] &= (u \times v)^*([p_1] + [p_2]) \\ &= (u \times v)^*([p_1]) + (u \times v)^*([p_2]) \\ &= [u] + [v]. \end{aligned}$$

[1] Note that the choice of $[\mu]$ giving rise to a fixed operation is unique since $[p_1] + [p_2] = [\mu(p_1 \times p_2)] = [\mu]$.

Homotopy associativity: let $q_1,\ q_2,\ q_3 : Y \times Y \times Y \longrightarrow Y$ be the projection maps.

$$\begin{aligned}[\mu(\iota \times \mu)] =& [\mu(q_1 \times \mu(q_2 \times q_3))] \\ =& [q_1] + ([q_2] + [q_3]) \\ =& ([q_1] + [q_2]) + [q_3] \\ =& [\mu(\mu(q_1 \times q_2) \times q_3)] \\ =& [\mu(\mu \times \iota)].\end{aligned}$$

Homotopy identity:

$$[\mu(\iota \times c) = [\iota] + [c] = [\iota].$$

Homotopy inverse:

$$[\mu(\iota \times \tau)] = [\iota] + [\tau] = [\iota] + (-[\iota]) = 0.$$

□

19.2.1 The space of loops

Proposition 19.2. *The loop space $\Omega(Y, y_0)$ is an H-space.*

Proof. The multiplication is loop concatenation, and the inverse map is given by $\tau(\omega) = -\omega$ defined by $-\omega(t) = \omega(1-t)$, $t \in I$. □

If $f : Y_1 \longrightarrow Y_2$ is a basepoint preserving map, the induced map $\Omega f : \Omega Y_1 \longrightarrow \Omega Y_2$ is an H-map.

Therefore, for pointed spaces (X, x_0) and (Y, y_0), there is a natural group structure on $[(X, x_0),\ \Omega(Y, y_0)]$.

19.3 Co-H-spaces and suspensions

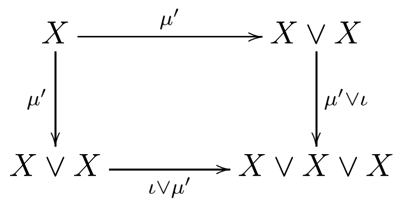
Let (X, x_0) be a space with basepoint. Consider

$$X \vee X := \{(x_1, x_2) : x_1 = x_0 \text{ or } x_2 = x_0\}$$

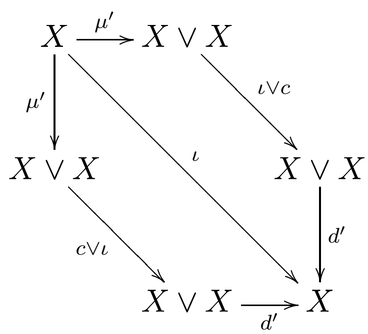
with the folding map $d' : X \longrightarrow X \vee X$ given by $d'(x, x_0) = d(x_0, x) = x$. We say that X is a co-H-space if there are

(i) a comultiplication map $\mu' : X \to X \vee X$,

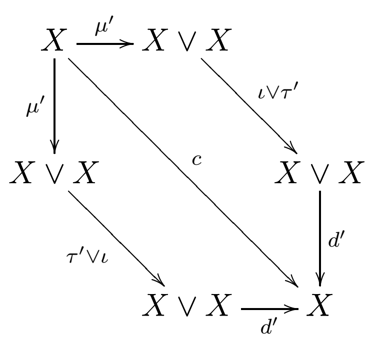
(ii) an inverse map $\tau' : X \longrightarrow X$ such that
(1) μ' is homotopy associative, *i.e.*, the diagram



is homotopy associative;
(2) the constant map $c : X \longrightarrow X$ is a homotopy identity, *i.e.*, the diagram



is homotopy commutative;
(iii) τ' is a homotopy inverse, *i.e.*, the diagram



is homotopy commutative.

19.3.1 The reduced suspension

Let (X, x_0) be a pointed space. The *reduced suspension* of X is the space

$$\Sigma X := X \times I/(X \times \{0,1\} \cup x_0 \times I).$$

It has basepoint $[x_0, t] = [x, 0] = [x, 1]$, $t \in [0, 1]$. ΣX has a natural comultiplication

$$\mu'([x,t]) = \begin{cases} [x, 2t], *, & 0 \le t \le \frac{1}{2}, \\ *, [x, 2t-1], & \frac{1}{2} \le t \le 1. \end{cases}$$
$$\tau'[x,t] = [x, 1-t].$$

This makes ΣX into a co-H-space.

Proposition 19.3. *For every space Y, there is a natural group structure on $[\Sigma X,\ Y]$.*

Proposition 19.4. $[\Sigma X,\ Y]$ *and* $[X,\ \Omega Y]$ *are naturally isomorphic.*

Proof. There is a homeomorphism $\Theta :\ Y^{\Sigma X} \longrightarrow (\Omega Y)^X$ given by

$$\Theta(f)(x)(t) = f[x, t].$$

This induces an isomorphism $[\Sigma X,\ Y] \longrightarrow [X,\ \Omega Y]$. □

Proposition 19.5. *If X is co-H-space and Y an H-space, then the group structures induced on $[X,\ Y]$ are the same, and are abelian.*

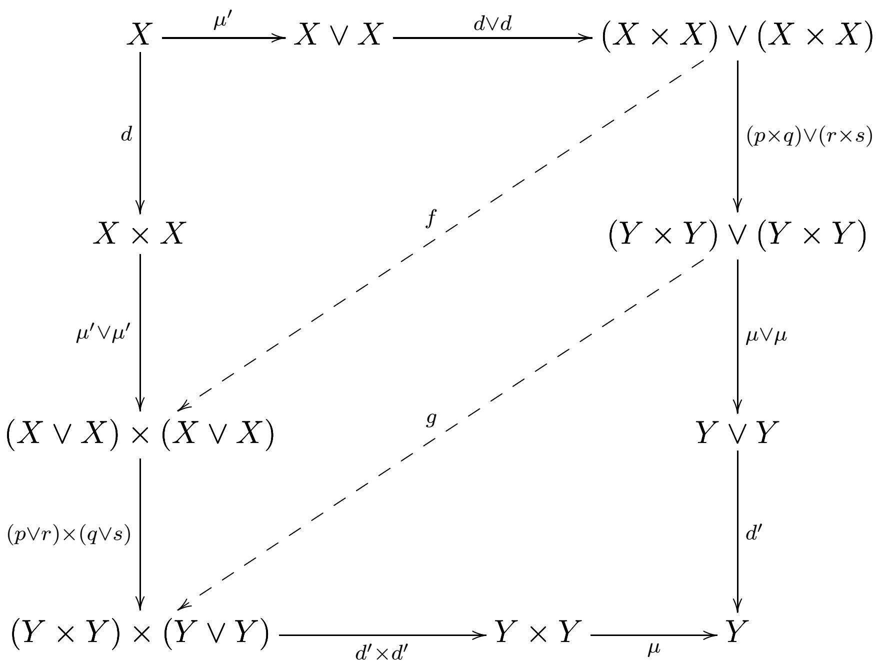
Proof. Denote by $+$ the group operation on $[X, Y]$ induced by the H-space structure of Y, and by $+'$ that induced by the co-H-space structure of X. Let $p, q, r, s :\ X \longrightarrow Y$ be arbitrary maps.

$$\begin{aligned} ([p] + [q]) +' ([r] + [s]) &= d'(\mu(p \times q)d \vee \mu(r \times s)d)\mu', \\ &= d'(\mu \vee \mu)((p \times q) \vee (r \times s)(d \vee d)\mu'; \\ ([p] +' [r]) + ([q] +' [s]) &= \mu(d'(p \vee r)\mu' \times d'(q \vee s)\mu')d, \\ &= \mu(d' \times d')((p \vee r) \times (q \vee s))(\mu' \vee \mu')d. \end{aligned}$$

The result

$$([p] + [q]) +' ([r] + [s]) = ([p] +' [r]) + ([q] +' [s])$$

follows from the homotopy commutativity of the diagram



which is clear from the (strict) commutativity of the "parallelogram" with the map f given by

$$f((x_1, x_2), *) =((x_1, *),\ (x_2, *)),$$
$$f(*,\ (x_3, x_4)) =f((*, x_3),\ (*, x_4)),$$

and g similarly defined. In particular,

$$[p] +' [s] = ([p] + [c]) +' ([c] + [s]) = ([p] +' [c]) + ([c] +' [s]) = [p] + [s].$$

Also, it is abelian since

$$[p] +' [s] = ([c] + [p]) +' ([s] + [c]) = ([c] + [s]) +' ([p] + [c]) = [s] +' [p].$$

□

19.4 Homotopy groups

The n-th homotopy group of (X, x_0) is defined as $\pi_n(X) := [\mathbb{S}^n,\ X]$. If $n \geq 2$, we have $[\mathbb{S}^n,\ X] \approx [\Sigma\mathbb{S}^{n-1}, \Omega X]$, and the group $\pi_n(X)$ is abelian.

Examples

(1) If X is contractible, then $\pi_n(X) = 0$ for every $n \geq 1$.

(2) If $m < n$, then $\pi_m(\mathbb{S}^n) = 0$.

Chapter 20

The homotopy groups

20.1 Higher homotopy groups

We give an alternative definition of homotopy groups which allow more convenient description of the addition of homotopy elements.

Consider $\mathbb{S}^1 = I/\{0,1\}$ with base point $[0] = [1]$. Its reduced suspension is

$$\Sigma\mathbb{S}^1 = I \times I/(I \times \{0,\ 1\} \cup \{0,\ 1\} \times I) = I^2/\partial I^2 \equiv \mathbb{S}^2.$$

More generally, $\mathbb{S}^n \equiv I^n/\partial I^n$, and

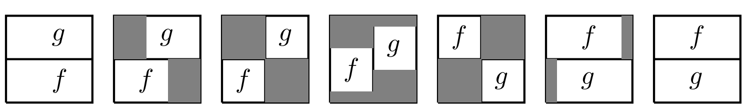
$$\Sigma\mathbb{S}^n \equiv I^n \times I/(I^n \times \{0,1\} \cup \partial I^n \times I) \equiv I^{n+1}/\partial I^{n+1} \equiv \mathbb{S}^{n+1}.$$

For $n \geq 2$, a map $\mathbb{S}^n \longrightarrow X$ can be replaced by a map of pairs $(I^n,\ \partial I^n) \longrightarrow (X,\ x_0)$. In this form, the addition of homotopy elements in $\pi_n(X)$ is given by $[f] + [g] = [h]$, where

$$h(t_1, \ldots, t_n) = \begin{cases} f(t_1, \ldots, 2t_n), & 0 \leq t \leq \frac{1}{2}, \\ g(t_1, \ldots, 2t_n - 1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Proposition 20.1. *For $n \geq 2$, $\pi_n(X)$ is abelian.*

Proof.



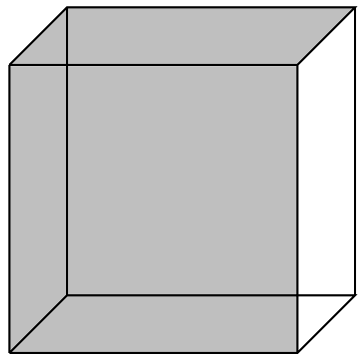
□

20.2 Relative homotopy groups

For $n \geq 2$, we regard

$$\partial I^n = (\{1\} \times I^{n-1}) \cup J^{n-1},$$

where $J^{n-1} = \{0\} \times I^{n-1} \cup I \times \partial I^{n-1}$.



Let (X, x_0) be a space with basepoint x_0, and A a subspace of X containing x_0. The n-th *relative homotopy group* $\pi_n(X, A)$ is the group of homotopy classes of maps

$$(I^n,\ \partial I^n,\ J^{n-1}) \longrightarrow (X, A, x_0)$$

with addition defined by $[f] + [g] = [h]$, where

$$h(t_1, \ldots, t_{n-1},\ t_n) = \begin{cases} f(t_1, \ldots, t_{n-1},\ 2t_n), & \text{if } 0 \leq t_n \leq \frac{1}{2}, \\ g(t_1, \ldots, t_{n-1},\ 2t_n - 1), & \text{if } \frac{1}{2} \leq t_n \leq 1. \end{cases}$$

Since a map $(I^n, \partial I^n) \longrightarrow (X, x_0)$ can be regarded as $(I^n, \partial I^n,\ J^{n-1}) \longrightarrow (X,\ x_0,\ x_0)$, and conversely, we have $\pi_n(X) = \pi_n(X, x_0)$.

20.3 Alternative description of homotopy groups

The relative homotopy group $\pi_n(X, A)$ can also be described as consisting of homotopy classes of maps $f : (\mathbb{D}^n,\ \mathbb{S}^{n-1}) \longrightarrow (X, A)$ satisfying $f(\mathbb{S}^{n-1}) \subset A$. We shall consider all spheres imbedded in euclidean space, and with basepoint $1 = (1, \ldots)$.

The sum of homotopy elements can be determined as follows. Let $\mathbb{E}^n_+$ and $\mathbb{E}^n_-$ be the northern and southern hemispheres respectively. There are homotopies $\iota \sim \theta_\pm : \mathbb{D}^n \longrightarrow \mathbb{D}^n$ such that

$$\begin{aligned} \theta_+(x_1, \ldots, x_n) =& 1, \text{ if } x_n \leq 0; \\ \theta_-(x_1, \ldots, x_n) =& 1, \text{ if } x_n \geq 0. \end{aligned}$$

Then for $f, g : (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (X, A)$, the homotopy class $[f] + [g]$ is represented by $h : (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (X, A)$, where

$$h(x_1, \ldots, x_n) = \begin{cases} f \circ \theta_+(x_1, \ldots, x_n), & \text{if } x_n \leq 0; \\ f \circ \theta_-(x_1, \ldots, x_n), & \text{if } x_n \geq 0. \end{cases}$$

Lemma 20.2. *A map $f : \mathbb{S}^n \longrightarrow X$ is nullhomotopic if and only if it can be extended to $g : \mathbb{D}^{n+1} \longrightarrow X$.*

Lemma 20.3. *A map $f : (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (X, A)$ represents the zero element in $\pi_n(X, A)$ if and only if there is a homotopy $f_t : \mathbb{D}^n \longrightarrow X$ such that $f_0 = f$ and $f_1(\mathbb{D}^n) \subset A$.*

Proof. ($\Leftarrow$) It is enough to assume $f(\mathbb{D}^n) \subset A$. Since $\mathbb{D}^n$ is contractible, there is a homotopy $g_t : \mathbb{D}^n \longrightarrow A$ with $g_0 = f$ and $g_1 = c$ constant map. This can be regarded as a homotopy $g_t : (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (X, A)$.

($\Rightarrow$) We begin with a homotopy $f \sim c : (\mathbb{D}^n,\ \mathbb{S}^{n-1}) \longrightarrow (X, A)$ as a map defined on the cone with base $\mathbb{D}^n \times 0$ and vertex $(0, 1) \in \mathbb{R}^n \times I$. We may assume that the "surface" of the cone is mapping into A. This map can be extended to the cyclinder $\mathbb{D}^n \times I$ by "laying radii along generators" to give a homotopy satisfying the requirement. □

20.4 The homotopy exact sequence of a pair

Given a pair (X, A), we consider the inclusion maps
$i : A \longrightarrow X$ and $j : (X,\ x_0) \longrightarrow (X,\ A)$. These induce, for each $n \geq 0$, homomorphisms
$i_* : \pi_n(A) \longrightarrow \pi_n(X)$ and $j_* : \pi_n(X) \longrightarrow \pi_n(X, A)$.

There is also a family of boundary homomorphisms
$\partial_n : \pi_n(X, A) \longrightarrow \pi_{n-1}(A)$ defined geometrically by

$$\partial_n([f]) := [f|_{1 \times I^{n-1}}] \quad \text{for every } f : (I^n, \partial I^n, J^{n-1}) \longrightarrow (X, A, x_0),$$

or

$$\partial_n([f]) = [f|_{\mathbb{S}^{n-1}}]$$

if f is represented by $(\mathbb{D}^n,\ \mathbb{S}^{n-1}) \longrightarrow (X, A)$.

Theorem 20.4. *In the sequence*

$$\cdots \longrightarrow \pi_n(A) \xrightarrow{i_*} \pi_n(X) \xrightarrow{j_*} \pi_n(X,A) \xrightarrow{\partial_n} \pi_{n-1}(A) \xrightarrow{i_*} \pi_{n-1}(X) \longrightarrow \cdots ,$$

(1) $\ker j_* = \operatorname{Im} i_*$,
(2) $\ker \partial_n = \operatorname{Im} j_*$,
(3) $\ker i_* = \operatorname{Im} \partial_n$.

This is summarized by saying the sequence above is exact. We call this the **homotopy exact sequence** of the pair (X, A).

Proof. (1a) First note that $j_* \circ i_* = 0$ by Lemma 20.3 since for every $f : (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (A, x_0)$, $j_* i_*[f] = [j \circ i \circ f]$ is represented by a map

$$(\mathbb{D}^n,\ \mathbb{S}^{n-1}) \longrightarrow (A, x_0) \longrightarrow (X, x_0) \longrightarrow (X, A)$$

whose image is in A. Therefore, $\operatorname{Im} j_* \subset \ker i_*$.

(1b) Suppose $f : (\mathbb{D}^n,\ \mathbb{S}^{n-1}) \longrightarrow (X, x_0)$ satisfies $j_*([f]) = 0 \in \pi_n(X, A)$. There is a homotopy $g_t : (\mathbb{D}^n,\ \mathbb{S}^{n-1}) \longrightarrow (X, A)$ with
(i) $g_1(\mathbb{D}^n) \subset A$,
(ii) $g_t|_{\mathbb{S}^{n-1}} = j \circ f|_{\mathbb{S}^{n-1}}$ for every $t \in I$, *i.e.*, $g_t(\mathbb{D}^n) = x_0$. Therefore, there is a factorization

$$g_1 : (\mathbb{D}^n,\ \mathbb{S}^{n-1}) \xrightarrow{g} (A, x_0) \xrightarrow{i} (X, x_0) \xrightarrow{j} (X, A).$$

Now, $g_1 \sim j \circ f \Rightarrow i \circ g \sim f \Rightarrow i_*([g]) = [f]$. This means that $\ker j_* \subset \operatorname{Im} i_*$.

(2a) For any $g : (\mathbb{D}^n,\ \mathbb{S}^{n-1}) \longrightarrow (X, x_0)$, clearly $j \circ g|_{\mathbb{S}^{n-1}}$ is the constant map. This means $\partial_n(j_*([g])) = 0$ and $\operatorname{Im} j_* \subset \ker \partial_n$.

(2b) Suppose $f : (\mathbb{D}^n,\ \mathbb{S}^{n-1}) \longrightarrow (X, A)$ satisfies $[f|_{\mathbb{S}^{n-1}}] = 0$. There is a map $H : \mathbb{D}^n \times 0 \cup \mathbb{S}^{n-1} \times I \longrightarrow X$ for which $H(\mathbb{S}^{n-1} \times 1) = x_0$. This can be extended to a map $\hat{H} : \mathbb{D}^n \times I \longrightarrow X$ with

$$\hat{H}_1 : (\mathbb{D}^n,\ \mathbb{S}^{n-1}) \xrightarrow{h} (X, x_0) \xrightarrow{j} (X, A).$$

Clearly, $[f] = [\hat{H}_1] = j_*([h])$. This shows that $\ker \partial_n \subset \operatorname{Im} j_*$.

(3a) Consider $f : (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (A, A)$. The homotopy class $i_*\partial_n([f])$ is represented by the composite

$$(\mathbb{S}^{n-1}, 1) \xrightarrow{f|_{\mathbb{S}^{n-1}}} (A, x_0) \xrightarrow{(} X, x_0).$$

This is nullhomotopic since it can be extended to $\mathbb{D}^n$. This shows that $\mathrm{Im}\,\partial_n \subset \ker i_*$.

(3b) Let $g : \mathbb{S}^{n-1} \longrightarrow A$ be a map with $i \circ g : \mathbb{S}^{n-1} \longrightarrow A \longrightarrow X$ nullhomotopic. Extend this to $f : \mathbb{D} \longrightarrow X$, which can be regarded as a map $f : (\mathbb{D}^n, \mathbb{S}^{n-1}) \longrightarrow (X, A)$. Clearly, $\partial_n([f]) = [g]$. This shows that $\ker i_* \subset \mathrm{Im}\,\partial_n$. □

Example 20.1. $\pi_m(\mathbb{D}^n, \mathbb{S}^{n-1})$.

Since $\mathbb{D}^{n+1}$ is contractible and $\pi_m(\mathbb{D}^{n+1}) = 0$ for every $m \geq 0$, from the homotopy exact sequence of the pair $(\mathbb{D}^{n+1}, \mathbb{S}^n)$, we have an isomorphism : $\pi_m(\mathbb{D}^{n+1}, \mathbb{S}^n) \xrightarrow[\approx]{} \pi_{m-1}(\mathbb{S}^n)$.

20.5 Appendix: calculations with exact sequences

A sequence of (abelian) groups and homomorphisms

$$(\mathcal{E}) : \quad \cdots \xrightarrow[\alpha_{n+2}]{} A_{n+1} \xrightarrow[\alpha_{n+1}]{} A_n \xrightarrow[\alpha_n]{} A_{n-1} \xrightarrow[\alpha_{n-1}]{} A_{n-2} \longrightarrow \cdots$$

is exact if for every n, $\mathrm{Im}\,\alpha_{n+1} = \ker \alpha_n$.

Proposition 20.5. *In the exact sequence $(\mathcal{E})$, the following are equivalent.*
(i) α_{n+1} is an epimorphism.
(ii) $\alpha_n = 0$.
(iii) α_{n-1} is a monomorphism.

Proposition 20.6. *Consider a short exact sequence*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0.$$

The following statements are equivalent.
(i) $B \approx A \oplus C$.
(ii) g has a right inverse.
(iii) f has a left inverse.

Proposition 20.7 (The five-lemma). *Consider the following commutative diagram in which the top and bottom rows are exact sequences:*

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & A_4 & \xrightarrow{\alpha_4} & A_5 \\
{\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
B_1 & \xrightarrow[\beta_1]{} & B_2 & \xrightarrow[\beta_2]{} & B_3 & \xrightarrow[\beta_3]{} & B_4 & \xrightarrow[\beta_4]{} & B_5
\end{array}$$

(i) If f_2 and f_4 are epic and f_5 is monic, then f_3 is epic.

(ii) If f_2 and f_4 are monic and f_1 is epic, then f_3 is monic.

(iii) Therefore, if f_2 and f_4 are iso, f_1 is epic, and f_5 is monic, then f_3 is iso.

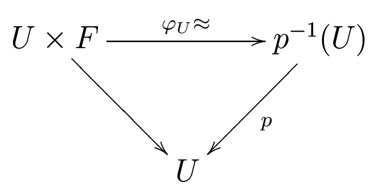
(iv) If f_1, f_2, f_4, f_5 are iso, then so is f_3.

Chapter 21

Homotopy exact sequence

21.1 Locally trivial bundles

A locally trivial bundle with fiber F is a map $p : E \longrightarrow B$ such that for every $b_0 \in B$ there is a neighborhood U and a homeomorphism $\varphi_U : U \times F \longrightarrow p^{-1}(U)$ for which



is commutative.

Examples

If $E = B \times F$, this is a trivial bundle.

If F is discrete, this is a covering map.

$\mathbb{S}^{2n+1} \longrightarrow \mathbb{C}\mathrm{P}^n$ is a locally trivial bundle with fiber $\mathbb{S}^1$.

$\mathbb{S}^{4n+3} \longrightarrow \mathbb{H}\mathrm{P}^n$ is a locally trivial bundle with fiber $\mathbb{S}^3$.

Here are some important facts about locally trivial bundles.

(1) A locally trivial bundle has the homotopy lifting property for any compact Hausdorff space, in particular for I^n, $n \geq 0$.

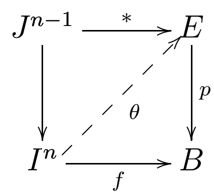
(2) If $p : E \longrightarrow B$ is a Serre fibration, *i.e.*, if it has the homotopy lifting property for I^n, $n \geq 0$, then $p_* : \pi_n(E, F) \longrightarrow \pi_n(B, *)$ is a one-to-one correspondence.

21.2 Homotopy exact sequence of a locally trivial bundle

If $p : E \longrightarrow B$ is a locally trivial bundle with fiber F, there is an exact sequence

$$\cdots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \cdots$$
$$\cdots \longrightarrow \pi_1(B) \xrightarrow{\partial} \pi_0(F) \longrightarrow \pi_0(E) \longrightarrow \pi_0(B).$$

The boundary homomorphism $\Delta : \pi_n(B) \longrightarrow \pi_{n-1}(F)$ is defined as follows. Given a map $f : (I^n, \partial I^n) \longrightarrow (B, b_0)$, there is a lifting $\theta : I^n \longrightarrow E$ making the diagram



commutative. Define

$$\Delta([f]) = [\theta|_{1 \times I^{n-1}}].$$

21.3 Applications of the homotopy exact sequence

21.3.1 $\pi_n(\mathbb{S}^1) = 0$ for $n > 1$

Theorem 21.1. $\pi_n(\mathbb{S}^1) = 0$ *for* $n > 1$.

21.3.2 $\pi_m(\mathbb{R}\mathrm{P}^n) \approx \pi_m(\mathbb{S}^n)$ for $m \geq 2$

Theorem 21.2. *If* $p : E \longrightarrow B$ *is a covering map with discrete fiber* F*, then* $\pi_n(E, e_0) \approx \pi_n(B, b_0)$ *for* $n \geq 2$.

Corollary 21.3. $\pi_m(\mathbb{R}\mathrm{P}^n) \approx \pi_m(\mathbb{S}^n)$ *for* $m > 1$.

Theorem 21.4. $\pi_n(X \times Y, (x_0, y_0)) \approx \pi_n(X, x_0) \oplus \pi_n(Y, y_0)$.

Proof. The homotopy exact sequence of the trivial bundle
$Y \longrightarrow X \times Y \longrightarrow X$ splits. □

Chapter 22

Some homotopy groups of spheres

22.1 Stable homotopy groups of spheres

THEOREM (Freudenthal suspension theorem). If X is an $(n-1)$-connected space, *i.e.*, $\pi_k(X) = 0$ for $k \leq n-1$, the suspension homomorphism

$$E : \pi_r(X) \longrightarrow \pi_{r+1}(\Sigma X)$$

is an isomorphism for $r < 2n-1$ and an epimorphism for $r = 2n-1$.

Corollary 22.1. $\pi_{r+k}(\mathbb{S}^r) \approx \pi_{r+k+1}(\mathbb{S}^{r+1})$ *for* $r \geq k+2$.

Therefore, we can speak of the *stable homotopy groups of spheres*. The stable k-stem is $\pi_k^s := \pi_{r+k}(\mathbb{S}^r)$ for $r \geq k+2$. Thus, $\pi_m(\mathbb{S}^n)$ is stable if $m \leq 2n-2$, otherwise it is said to be an unstable group.

22.2 The 0-stem

The 0-stem π_0^s is isomorphic to $\pi_2(\mathbb{S}^2)$. This latter can be determined from the lower end of the homotopy exact sequence of the Hopf fibration $\mathbb{S}^1 \longrightarrow \mathbb{S}^3 \xrightarrow{\eta} \mathbb{S}^2$. From the exactness of

$$\begin{array}{ccccccc} \pi_2(\mathbb{S}^3) & \to & \pi_2(\mathbb{S}^2) & \to & \pi_1(\mathbb{S}^1) & \to & \pi_1(\mathbb{S}^3) \\ \| & & & & & & \| \\ 0 & & & & & & 0 \end{array}$$

we conclude that $\pi_2(\mathbb{S}^2) = \mathbb{Z}$. Consequently, $\pi_n(\mathbb{S}^n) = \mathbb{Z}$ for every positive integer n.

22.3 $\pi_3(\mathbb{S}^2) = \mathbb{Z}$

The unstable group $\pi_3(\mathbb{S}^2)$ is the first nontrivial homotopy group of spheres from a higher dimension to a lower dimension. It follows from the exactness of

$$\begin{array}{ccccccc} \pi_3(\mathbb{S}^1) & \to & \pi_3(\mathbb{S}^3) & \xrightarrow{\eta_*} & \pi_3(\mathbb{S}^2) & \to & \pi_2(\mathbb{S}^1) \\ \| & & \| & & & & \| \\ 0 & & \mathbb{Z} & & & & 0 \end{array}$$

that $\pi_3(\mathbb{S}^2)$ is infinite cyclic, and is generated by $[\eta]$.

By the Freudenthal suspension theorem, $E : \pi_3(\mathbb{S}^2) \longrightarrow \pi_4(\mathbb{S}^3)$ is an epimorphism. By more sophisticated methods, this is shown to be isomorphic to $\mathbb{Z}_2$. It follows that the 1-stem is $\pi_1^s = \mathbb{Z}_2$.

Chapter 23

Smooth manifolds

23.1 Smooth maps

Let $X \subset \mathbb{R}^k$, $Y \subset \mathbb{R}^n$ and $f : X \longrightarrow Y$ be a continuous map.

(1) If X is open in $\mathbb{R}^k$, we say that f is smooth if all partial derivatives exist and are continuous.

(2) More generally, if X is not necessarily open in $\mathbb{R}^k$, we say that f is smooth at $x \in X$ if it is the restriction of a smooth map on an open set $W_x \subset \mathbb{R}^k$, *i.e.*, there exists an open set $W_x \subset \mathbb{R}^k$ and a smooth map $F : W_x \longrightarrow \mathbb{R}^n$ that $F|_{W \cap X} = f|_{W_x \cap X}$. Finally, f is smooth if it is smooth at each $x \in X$.

$f : X \longrightarrow Y$ is called a *diffeomorphism* if
(i) it is a homeomorphism,
(ii) both f and f^{-1} are smooth maps.

$X \subset \mathbb{R}^k$ and $Y \subset \mathbb{R}^n$ are diffeomorphic if there is a diffeomorphism between them.

Diffeomorphism is an equivalence relation.

23.2 Smooth manifolds

$M \subset \mathbb{R}^k$ is called a smooth manifold of dimension m if each point $x \in M$ has a neighborhood diffeomorphic to an open subset of $\mathbb{R}^m$, *i.e.*, for each $x \in M$, there are
(i) an open set $W_x \subset \mathbb{R}^k$ with $x \in W_x \cap M$,
(ii) an open set $U \subset \mathbb{R}^m$, and
(iii) a diffeomorphism $\varphi_x : U \longrightarrow W_x \cap M$ called a local parametrization at $x \in M$.

Suppose there are two local parametrizations $\varphi_1 : U_1 \longrightarrow W_{x,1} \cap M$, $\varphi_2 : U_2 \longrightarrow W_{x,2} \cap M$ at $x \in M$, and $0 \in U_1 \cap U_2$ with $\varphi_1(0) = \varphi_2(0) = x$. The composite $\varphi_2^{-1} \circ \varphi_1$ is a diffeomorphism of appropriate neighborhoods of $0 \in \mathbb{R}^m$.

An open set of $\mathbb{R}^m$ is a smooth manifold of dimension m.

23.3 Examples of smooth manifolds

23.3.1 Smooth connected 1-manifolds

A connected smooth 1-manifold is diffeomorphic to one of $\mathbb{S}^1$, $[0,1]$, $(0,1)$, and $(0,1]$.

23.3.2 The sphere

The sphere $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : ||x||1\}$ is a smooth n-manifold. Let $N = (0,\dots,0,1)$ and $S = (0,\dots,0,-1) \in \mathbb{S}^n$ be the north and south poles respectively. For $x \in \mathbb{S}^n - \{N\}$, stereographic projection from N onto the "equatorial plane" $x_{n+1} = 0$ gives a local parametrization $\varphi : \mathbb{R}^n \longrightarrow \mathbb{S}^n - \{N\}$:

$$\varphi(x) = \frac{1}{1+||x||^2}\left(2x,\ ||x||^2 - 1\right).$$

Similarly, for $x \in \mathbb{S}^n - \{S\}$, there is a local parametrization $\psi : \mathbb{R}^n \longrightarrow \mathbb{S}^n - \{S\}$ given by

$$\psi(x) = \frac{1}{1+||x||^2}\left(2x,\ 1 - ||x||^2\right).$$

The composite $\psi^{-1} \circ \varphi : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}^n \setminus \{0\}$ is inversion with respect to the unit sphere of the equatorial plane:

$$\psi^{-1} \circ \varphi(x) = \frac{x}{||x||^2}, \qquad x \neq 0.$$

23.3.3 The real projective spaces

The real projective space $\mathbb{R}\mathrm{P}^n$ can be regarded as the space of (puctured) lines through the origin of $\mathbb{R}^{n+1}$. As such it is covered by $n+1$ open sets

$$V_j := \{[x_1, \ldots, x_{n+1}] : \; x_j \neq 0\}.$$

with local parametrizations $\varphi_j : \mathbb{R}^n \longrightarrow V_j$ given by

$$\varphi_j(x_1, \ldots, x_n) = [x_1, \ldots, x_{j-1}, 1, x_{j+1}, \ldots, x_n].$$

For distinct indices $j, k = 1, \ldots, n+1$, $\varphi_k^{-1} \cdot \varphi_j : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$ is given by

$$\varphi_k^{-1} \circ \varphi_j(x) = x',$$

where x' is obtained from x by
(i) inserting 1 in the j-th position,
(ii) deleting x_k or x_{k-1} according as $k < j$ or $k > j$, and
(iii) dividing the resulting vector by the same element in (ii).

23.3.4 The Grassman manifolds

Let $k < n$. The Grassmann manifold $G_{n,k}$ consists of the (punctured) k-planes in $\mathbb{R}^n$. If a k-plane does not intersect the subspace

$$x_{n-k+1} = \cdots = x_n = 0,$$

then it can be represented by an equation

$$y := \begin{pmatrix} x_{n-k+1} & \cdots & x_n \end{pmatrix} = \begin{pmatrix} x_1 & \cdots & x_{n-k} \end{pmatrix} A$$

for an $(n-k) \times k$ matrix A. The collection of such k-planes forms one of the $\binom{n}{k}$ open sets covering the Grassmann manifold $G_{n,k}$, which has dimension $k(n-k)$.

23.3.5 Spaces of matrices of constant rank

For given positive integers r and s, we shall regard the space $M(r, s)$ of all $r \times s$ matrices over $\mathbb{R}$ as diffeomorphic to $\mathbb{R}^{rs}$. Let $k \leq \min(r, s)$. The set $M(r, s; k)$ of all real $r \times s$ with rank k is a smooth manifold of dimension

$$rs - (r-k)(s-k) = k(r+s-k).$$

23.4 Appendix: Milnor's proof of the fundamental theorem of algebra

Consider the sphere $\mathbb{S}^2$ with stereographic projections φ and ψ from the north and south poles respectively. Let $P : \mathbb{C} \longrightarrow \mathbb{C}$ be a polynomial of degree n.

Conjugating P by φ, we obtain a smooth map $f : \mathbb{S}^2 \longrightarrow \mathbb{S}^2$. Further conjugating f by ψ^{-1} we have a map g smooth in a neighborhood of 0. Indeed,

$$g(z) = \frac{z^n}{\overline{a_0} + \overline{a_1}z + \cdots + \overline{a_n}z^n}.$$

Therefore, f is smooth in a neighborhood of the north pole.

Now, f has a finite number of critical points. The set of regular values of f is $\mathbb{S}^2$ with a finite number of points removed. It is connected. Since $\#f^{-1}(y)$ is locally constant for the regular values, it is actually constant. Since this number cannot be zero everywhere, it is zero nowhere. This means that f is surjective and P must have a zero.

Chapter 24

Mappings between manifolds

24.1 Tangent space

Let $M \subset \mathbb{R}^k$ be a smooth manifold of dimension m. Consider $x \in M$. For every local parametrization $\varphi_x : U \longrightarrow W_x \cap M \subset \mathbb{R}^k$, the image of the differential $d\varphi_x : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ is independent of the choice of U and φ, and is defined to be the tangent space of M at x, denoted $\tau_x(M)$. Note that $\dim \tau_x(M) = m$.

Let $M^m \subset \mathbb{R}^k$ and $N^n \subset \mathbb{R}^\ell$ be smooth manifolds, and $f : M \longrightarrow N$ a smooth map. For every $x \in M$, there is an induced map

$$df_x : \tau_x(M) \longrightarrow \tau_{f(x)}(N)$$

called the *differential* of f at x, defined as follows. Choose an open set $W_x \in \mathbb{R}^k$ and a smooth map $F : W_x \longrightarrow \mathbb{R}^\ell$ agreeing with f on $W_x \cap M$, and set $df_x(u) = dF_x(u)$ for every $u \in \tau_x(M)$.

(i) $df_x(u) \in \tau_{f(x)}(N)$ and is independent of the choice of W_x and F.

(ii) $df_x : \tau_x(M) \longrightarrow \tau_{f(x)}(N)$ is a linear map.

Proposition 24.1 (The Chain rule). *If $f : M \longrightarrow N$ and $g : N \longrightarrow P$ are smooth maps between smooth manifolds, then for every $x \in M$,*

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Theorem 24.2 (Inverse function theorem). *Let $f : M \longrightarrow N$ a smooth map. If $df_x : \tau_x(M) \longrightarrow \tau_{f(x)}(N)$ is an isomorphism, then f is a local diffeomorphism.*

Remark. Let $\varphi : U \longrightarrow M$ and $\psi : V \longrightarrow N$ be local parametrizations at $x \in M$ and $f(x) \in N$ respectively with $\varphi(0) = x$ and $\psi(0) = f(x)$. df_x

is an isomorphism if and only if the Jacobian of $\psi \circ f \circ \psi^{-1} : U \longrightarrow V$ has nonzero determinant at $0 \in U$.

Theorem 24.3 (Inverse image theorem). *Let $f : M \longrightarrow N$ be a smooth map. If $y \in N$ is a regular value in the sense that $df_x : \tau_x(M) \longrightarrow \tau_{f(x)}(N)$ is surjective for every $x \in f^{-1}(y)$ (assumed nonempty), then $f^{-1}(y)$ is a submanifold of M, of dimension $m - n$.*

24.2 Approximation of continuous map by a smooth one

1. Every compact manifold N can be given a Riemannian metric, *i.e.*, a positive definite inner product on each tangent space, which induces a metric ρ on N.

2. For compact manifolds M, N, the function space $\mathcal{C}(M, N)$ is a metric space:

$$\rho(f, g) = \max_{x \in M} \rho(f(x), g(x))$$

3. Let M, N be compact manifolds. Given a metric on N, there exists $\varepsilon > 0$ such that two continuous maps $f, g : M \longrightarrow N$ are homotopic whenever $\rho(f, g) < \varepsilon$.

4. Given a continuous map $f : M \longrightarrow N$ and $\varepsilon > 0$, there exists a smooth map $g : M \longrightarrow N$ such that $\rho(f, g) < \varepsilon$.

Theorem 24.4. (A) *Every continuous map $f : M \longrightarrow N$ between compact, smooth manifolds is homotopic to some smooth map $g : M \longrightarrow N$.*

(B) *If $f, g : M \longrightarrow N$ are homotopic smooth maps, they are smoothly homotopic.*

24.3 Regular and critical values

24.3.1 Sets of measure zero

A cube in $\mathbb{R}^n$ is a product of intervals $\prod_{i=1}^n [a_n,\ b_n]$, and has volume (measure) $\prod_{i=1}^n (b_i - a_i)$. A subset $A \subset \mathbb{R}^n$ has measure zero if it can be covered by a countable collection of cubes having arbitrarily small total volume.

Let M^m be a smooth manifold of dimension m. A subset $A \subset M$ has measure zero if for any local parametrization $\varphi : U \longrightarrow M$, $\varphi^{-1}(A)$ has measure zero in $\mathbb{R}^m$.

24.3.2 Sard's theorem

Let $f : M \longrightarrow N$ be a smooth map.
(i) $x \in M$ is critical point f if $df_x : \tau_(M) \longrightarrow \tau_{f(x)}(N)$ is not surjective.
(ii) $y \in N$ a critical value of f if $y = f(x)$ for some critical point $x \in M$.

THEOREM (Sard). The set of critical values a smooth map has measure zero.

Corollary 24.5. *If $m \geq n$, then every surjective smooth map $f : M^m \longrightarrow N^n$ must have regular values. In fact, regular values are dense.*

Chapter 25

Hopf's theorem

25.1 Orientable manifolds

25.1.1 Orientation of a vector space

Let X be a vector space X of dimension n. An orientation of X is determined by an ordered basis. We say that two basis $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ determine the same orientation if the linear map $f : X \longrightarrow X$ given by $f(u_i) = v_i$, $1 \leq i \leq n$, has positive determinant; otherwise the orientations are opposite.

25.1.2 Orientation of a manifold

A smooth manifold M^n is orientable if it is possible to choose an orientation of each tangent space $\tau_x(M)$ and also of $\mathbb{R}^n$ (containing the coordinate neighborhoods) such that these orientations correspond under all local parametrizations $\varphi : U \longrightarrow W_x \cap M$. Otherwise, the manifold is non-orientable.

If a manifold is non-orientable, there is an epimorphism $\pi_1(M) \longrightarrow \{\pm 1\}$. Therefore, a simple connected space is always orientable.

25.1.3 The oriented spheres

Let $v_1 \wedge \cdots \wedge v_n$ be an oriented basis of the tangent space $\tau_x(\mathbb{S}^n)$, so that $x \wedge v_1 \wedge \cdots \wedge v_n$ is a basis of $\tau_x(\mathbb{R}^{n+1})$. Transporting this oriented basis along a path joining x to $-x$ on the sphere induces an orientation on each tangent space along the path. In particular at the antipodal point $-x$, the tangent space has orientation $-v_1 \wedge \cdots \wedge v_n$.

25.1.4 Orientability of $\mathbb{R}\mathrm{P}^n$

An orientation of $\mathbb{R}\mathrm{P}^n$ comes from one on $\mathbb{S}^n$, namely, $\wedge v_1 \wedge \cdots \wedge v_n$, which satisfies

$$(-x) \wedge (-v_1) \wedge \cdots \wedge (-v_n) = x \wedge v_1 \wedge \cdots \wedge v_n.$$

This is possible if and only if n is odd.

25.2 The degree of a map between oriented manifolds of the same dimension

Let $f : M \longrightarrow N$ be a smooth map between oriented manifolds of the same dimension. We assume M compact, N connected, and both without boundary.

Let $y \in N$ be a regular value. Then $f^{-1}(y)$ is a finite set. For each $x \in f^{-1}(y)$, the differential $df_x : \tau_x(M) \longrightarrow \tau_y(N)$ is an isomorphism and must have nonzero determinant. We define the degree of f to be the integer

$$\deg f := \sum_{x \in f^{-1}(y)} \mathrm{sgn} df_x.$$

This degree does not depend on the choice of the regular value $y \in N$.

THEOREM (Hopf). If M^n is a compact oriented manifold, two maps $f, g : M \longrightarrow \mathbb{S}^n$ are homotopic if and only if $\deg f = \deg g$.

THEOREM. $\deg : \pi_n(\mathbb{S}^n) \longrightarrow \mathbb{Z}$ is an isomorphism.

25.2.1 The degree of the antipodal map $\mathbb{S}^n \longrightarrow \mathbb{S}^n$

Consider the antipodal map $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ given by $f(x) = -x$. Let $v_1, \ldots, v_n$ be an orthonormal basis of $\tau_x(\mathbb{S}^n)$. Correspondingly, $-v_1, \ldots, -v_n$ is an orthonormal basis of the $\tau_{-x}(\mathbb{S}^n)$. Now, if $\tau_x(\mathbb{S}^n)$ has orientation $v_1 \wedge v_2 \cdots \wedge v_n$, $\tau_{-x}(\mathbb{S}^n)$ should have orientation $-v_1 \wedge v_2 \cdots \wedge v_n$. It follows that $\deg f = (-1)^{n+1}$.

25.2.2 The degree of $p_{n+1} \circ \rho : \mathbb{S}^n \longrightarrow \mathbb{S}^n$

Let $f = p_{n+1} \circ \rho : \mathbb{S}^n \longrightarrow O(n+1) \longrightarrow \mathbb{S}^n$ be the composite of the hyperplane reflection map $\rho : \mathbb{S}^n \longrightarrow O(n+1)$ and the projection p_{n+1} :

$O(n+1) \longrightarrow \mathbb{S}^n$ which selects the "last" vector from an orthonormal basis. Thus,

$$f(x) = e_{n+1} - 2\langle x,\ e_{n+1}\rangle x.$$

The differential Df_x is given by

$$Df_x(v) = -2(\langle x,\ e_{n+1}\rangle v + \langle v,\ e_{n+1}\rangle x), \qquad v \in \tau_x(\mathbb{S}^n).$$

To compute the degree of f, we choose $y = -e_{n+1}$ with an oriented basis $v_1 \wedge \cdots \wedge v_n$ of $\tau_y(\mathbb{S}^n)$. Note that $f^{-1}(y) = \{\pm y\}$, and τ_{-y} has orientation $-v_1 \wedge \cdots \wedge v_n$.

Since $Df_y(v) = 2v$, for every $v \in \tau_y(\mathbb{S}^n)$, we have

$$\text{sgn}(df_y) = 1.$$

On the other hand, from $Df_{-y}(v) = -2v$ for each $v \in \tau_{-y}(\mathbb{S}^n)$, we have

$$\text{sgn}(df_{-y}) = (-1)^{n+1}.$$

It follows that

$$\deg f = 1 + (-1)^{n+1}.$$

Chapter 26

Vector fields on spheres

26.1 The tangent and normal bundles of an imbedded manifold

Let M be a smooth m-manifold imbedded in $\mathbb{R}^k$. Consider the *tangent bundle*

$$\tau(M) := \{(x, v) \in M \times \mathbb{R}^k : \; v \in \tau_x(M)\}.$$

(1) This is an m-dimensional vector bundle over M.
(2) $\tau(M)$ is a smooth $2m$-manifold.
We also consider the *normal bundle* of the imbedding $M^n \subset \mathbb{R}^k$

$$\nu(M) := \{(x, v) \in M \times \mathbb{R}^k : \; v \in \tau_x(M)^{\perp}\}.$$

This is a $(k - m)$-dimensional vector bundle.

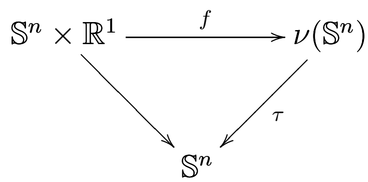
There is a basic relation between the tangent and normal bundles. Their Whitney sum, formed by taking the orthogonal direct sum of the tangent space and the normal space at each $x \in M$, is a trivial bundle, in the sense that the total space of $\tau(M) \oplus \nu(M)$ is simply the product $M \times \mathbb{R}^k$. We call this latter the trivial k-plane bundle on M, and write

$$\tau(M) \oplus \nu(M) = k\varepsilon.$$

26.2 The normal bundle of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$

The sphere $\mathbb{S}^n$ being imbedded in $\mathbb{R}^{n+1}$, the tangent space $\tau_x(\mathbb{S}^n)$ is the orthogonal complement of the 1-dimensional subspace spanned by x. There is an isomorphism $f : \mathbb{S}^n \times \mathbb{R}^1 \longrightarrow \nu(\mathbb{S}^n)$ given by

$$f(x, \lambda) = (x, \lambda x).$$



The normal bundle $\nu(\mathbb{S}^n)$ of $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is a trivial line bundle over $\mathbb{S}^n$.

26.3 The tangent bundle of $\mathbb{S}^n$

26.3.1 Spheres which are parallelizable

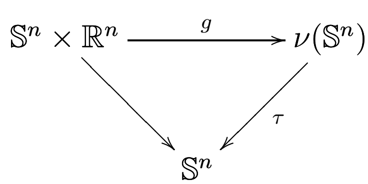
The tangent bundle of a manifold is in general not a trivial bundle. In case it is, the manifold is said to be *parallelizable*.

The tangent bundle of $\mathbb{S}^n$ is characterized by

$$\tau(\mathbb{S}^n) \oplus \varepsilon = (n+1)\varepsilon.$$

It is, however, not a trivial bundle except when $n = 1, 3, 7$, thanks to the existence of normed bilinear maps $\mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^{n+1}$.

For $n = 1, 3, 7$, there is a set of anticommuting skew symmetric linear maps $\tau_1, \ldots, \tau_n$ on $\mathbb{R}^{n+1}$. These provide n sections of the tangent bundles:



$$g(x, \{t_1, \dots, t_n\}) := t_1\tau_1(x) + \cdots + t_n\tau_n(x).$$

The nontriviality of $\tau(\mathbb{S}^n)$ is a deep theorem proved in the late 1950's.

THEOREM (Kervaire-Milnor). The sphere $\mathbb{S}^n$ is parallelizable if and only if $n = 1, 3, 7$.

26.3.2 The hairy ball theorem

Theorem 26.1. *If n is even, there is no nonzero vector field on $\mathbb{S}^n$.*

Proof. Suppose $\mathbb{S}^n$ admits a nonzero vector field. We may assume it is a unit tangent vector field $\tau(x)$, so that $(x, \tau(x))$ are orthogonal unit vectors for every $x \in \mathbb{S}^n$. Consider the homotopy $h_t : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ given by

$$h_t(x) = x \cdot \cos t\pi + \tau(x) \cdot \sin t\pi.$$

Clearly, h_0 is the identity map on $\mathbb{S}^n$. On the other hand, h_1 is the antipodal map. These two maps being homotopic, they should have the same degree. It follows that $1 = (-1)^{n+1}$ and n must be odd. □

26.3.3 Linearly independent tangent vector fields on $\mathbb{S}^n$

For odd values of n, the tangent bundle of $\mathbb{S}^n$ may contain trivial subbundles.

THEOREM (Adams). The maximum number of linearly independent tangent vector fields on $\mathbb{S}^n$ is $\rho(n+1) - 1$.

Chapter 27

Some fibre bundles over $\mathbb{S}^n$

27.1 The bundle p_{n+1}

Consider the map $p_{n+1} : O(n+1) \longrightarrow \mathbb{S}^n$ which selects the last vector from an orthonormal basis $\mathbb{R}^{n+1}$ (or the last row from an orthogonal matrix of order $n+1$). This is a locally trivial bundle with fibre $O(n)$.

27.1.1 Stability of $\pi_k(O(n))$

From the homotopy exact sequence of the fibration

$$O(n) \xrightarrow{i} O(n+1) \xrightarrow{p_{n+1}} \mathbb{S}^n$$

we easily conclude that

$$i_* : \pi_k(O(n)) \longrightarrow \pi_k(O(n+1))$$

is an isomorphism for $k+1 < n$. Therefore, we can speak of the homotopy group of the infinite orthogonal group $O = \lim_{n->\infty} O(n)$, namely, $\pi_k(O) \approx \pi_k(O(k+2))$.

THEOREM (Bott periodicity theorem). The homotopy groups of the infinite orthogonal group are as follows.

$k \bmod 8$	0	1	2	3	4	5	6	7
$\pi_k(O)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

In particular, for $n \geq 3$, $\pi_1(SO(n)) \approx \pi_1(SO(3)) = \pi_1(\mathbb{R}\mathrm{P}^3) = \mathbb{Z}_2$. This means that there is a simply-connected space doubly covering $SO(n)$ for $n \geq 2$. [1]

[1] This is the group $\mathrm{Spin}(n)$.

27.1.2 Cross sections of p_{n+1}

A cross section of the bundle p_{n+1} is a map $s : \mathbb{S}^n \longrightarrow O(n+1)$ such that $s \circ p_{n+1} = \iota_{\mathbb{S}^n}$. This is equivalent to a continuous orthonormal frame of the tangent bundle of $\mathbb{S}^n$. If such a framing is possible, we say that the sphere $\mathbb{S}^n$ is *parallelizable*. [2]

27.2 Stiefel manifolds

The Stiefel manifold $V_{n,k}$ is the space of orthonormal k-frames in $\mathbb{R}^n$:

$$V_{n,k} := \{(v_1, \ldots, v_k) : \ v_1, \ldots, v_k \text{ are mutually orthogonal unit vectors in } \mathbb{R}^n\}.$$

It can also be regarded as the subspaces of $k \times n$ real matrices A satisfying $AA^t = I_k$.

There is a fibre bundle

$$V_{n,\,k} \longrightarrow V_{n+1,\,k+1} \xrightarrow{p_{n+1,k+1}} \mathbb{S}^n$$

in which the map $p_{n+1,k+1}$ selects the last vector from an orthonormal $(k+1)$-frame in $\mathbb{R}^{n+1}$.

From the homotopy exact sequence, we have isomorphisms

$$i_* : \pi_m(V_{n,\,k}) \longrightarrow \pi_m(V_{n+1,\,k+1})$$

for $m + 1 < n$.

A cross section of the bundle $p_{n+1,k+1}$ is equivalent to k linearly independent tangent vector fields on $\mathbb{S}^n$. [3]

27.3 Characteristic element of a bundle over $\mathbb{S}^n$

Consider a bundle $p : E \longrightarrow \mathbb{S}^n$ with fibre F. From the homotopy exact sequence, we easily conclude that the injection $i_* : \pi_k(F) \longrightarrow \pi_k(E)$ is an isomorphism for $k < n - 1$ and an epimorphism for $k = n - 1$. [4] From

$$\cdots \longrightarrow \pi_n(\mathbb{S}^n) \xrightarrow{\Delta_*} \pi_{n-1}(F) \xrightarrow{i_*} \pi_{n-1}(E) \longrightarrow 0,$$

[2] See the Kervaire-Milnor theorem (Theorem 26.3.1) in §26.

[3] See Adams' theorem (Theorem 26.3.3) in §26

[4] This fact is often summarized by saying that $i : F \longrightarrow E$ is an n-equivalence.

the kernel of $i_* : \pi_{n-1}(F) \longrightarrow \pi_{n-1}(E)$ is the cyclic subgroup generated by $\omega_{n-1} := \Delta_*(\iota_n) \in \pi_{n-1}(F)$. This is called the **characteristic element** of the bundle. Since $p_* : \pi_n(E, F, e_0) \longrightarrow \pi_n(\mathbb{S}^n)$ is an isomorphism, there is a map $f : (\mathbb{E}^n_+,\ \mathbb{S}^{n-1}) \longrightarrow (E, F)$ such that $p \circ f \sim \iota_n$. This is called a **sectional element** for p. With this, we have $\omega_{n-1} = [f|_{\mathbb{S}^{n-1}}]$.

27.4 Characteristic element of p_{n+1}

The sectional element of the fibre bundle

$$O(n) \longrightarrow O(n+1) \xrightarrow{p_{n+1}} \mathbb{S}^n,$$

is given by the hyperplane reflection map

$$\rho : (\mathbb{E}^n_+,\ \mathbb{S}^{n-1}) \longrightarrow (O(n+1),\ O(n)),$$

where $\rho(x) = \rho_x : O(n+1) \longrightarrow O(n+1)$ is the reflection in the hyperplane $x^\perp$. The map ρ is the restriction of $\rho_{n+1} : \mathbb{S}^n \longrightarrow O(n+1)$, and it is clear that

$$\begin{array}{ccc} \mathbb{S}^{n-1} & \xrightarrow{\rho_n} & O(n) \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbb{S}^n & \xrightarrow[\rho_{n+1}]{} & O(n+1) \end{array}$$

is commutative.

The characteristic element of the bundle p_{n+1} is therefore $[\rho_n] \in \pi_{n-1}(O(n))$.

From the naturality,

$$\begin{array}{ccccc} O(n) & \longrightarrow & O(n+1) & \longrightarrow & \mathbb{S}^n \\ {\scriptstyle p_{n,k}}\downarrow & & \downarrow{\scriptstyle p_{n+1,k+1}} & & \downarrow{\scriptstyle =} \\ V_{n,\,k} & \longrightarrow & V_{n+1,\,k+1} & \longrightarrow & \mathbb{S}^n \end{array}$$

the characteristic element of the fibre bundle

$$V_{n,\,k} \longrightarrow V_{n+1,\,k+1} \xrightarrow{p_{n+1,k+1}} \mathbb{S}^n$$

is the homotopy class of

$$\rho_{n,k} = p_{n,k} \circ \rho_n : \mathbb{S}^{n-1} \longrightarrow V_{n,\, k}$$

in $\pi_{n-1}(V_{n,\, k})$.

Proposition 27.1. *The sphere $\mathbb{S}^n$ admits k linearly independent tangent vector fields if and only if $\rho_{n,k} : \mathbb{S}^{n-1} \longrightarrow V_{n,\, k}$ is trivial.*

Chapter 28

The Stiefel manifold $V_{n+1,2}$

28.1 The unit tangent bundle of the sphere

The unit tangent bundle of $\mathbb{S}^n$ is a locally trivial bundle with total space

$$V_{n+1,2} := \{(x, y) : \ x, y \text{ are orthogonal unit vectors in } \mathbb{R}^{n+1}\}$$

and fibre $\mathbb{S}^{n-1}$. Denote the projection by $p_{n+1,2} : V_{n+1,2} \longrightarrow \mathbb{S}^n$.

The characteristic element of the bundle $p_{n+1,2}$ is the homotopy class $[p_{n,1} \circ \rho_n]$ of the map $f : \mathbb{S}^{n-1} \longrightarrow O(n) \longrightarrow \mathbb{S}^{n-1}$ given by

$$f(x) = \rho_x(e_n) = e_n - 2\langle x,\ e_n\rangle x.$$

This map has degree $1 + (-1)^n$.

Theorem 28.1.

$$\pi_{n-1}(V_{n+1,\,2}) = \begin{cases} \mathbb{Z}, & n \text{ odd}, \\ \mathbb{Z}_2, & n \text{ even}. \end{cases}$$

Theorem 28.2. (a) *If n is odd, $\pi_i(V_{n+1,\,2}) = \pi_i(\mathbb{S}^{n-1}) \oplus \pi_i(\mathbb{S}^n)$. In particular,*

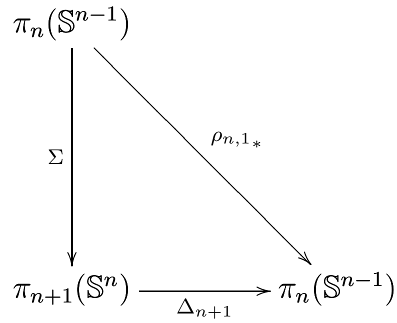
$$\pi_n(V_{n+1,\,2}) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & n = 3; \\ \mathbb{Z}_2 \oplus \mathbb{Z}, & n \geq 3. \end{cases}$$

(b) *If n is even and ≥ 4, then $\pi_n(V_{n+1,\,2}) = \mathbb{Z}_2$.*

Proof. (b) From the homotopy exact sequence

$$\cdots \longrightarrow \pi_{n+1}(\mathbb{S}^n) \xrightarrow{\Delta_{n+1}} \pi_n(\mathbb{S}^{n-1}) \longrightarrow \pi_n(V_{n+1,\,2}) \longrightarrow \pi_n(\mathbb{S}^n) \xrightarrow{2} \pi_{n-1}(\mathbb{S}^{n-1}) \longrightarrow \cdots$$

the map $\pi_n(V_{n+1,\,2}) \longrightarrow \pi_n(\mathbb{S}^n)$ is trivial. Therefore, $\pi_n(V_{n+1,\,2}) = \operatorname{coker}(\Delta_{n+1})$. If $n \geq 4$, $\pi_{n+1}(\mathbb{S}^n)$ and $\pi_n(\mathbb{S}^n)$ are both $\mathbb{Z}_2$, and $\Delta_{n+1} = 0$ or ι. From the Freudenthal suspension theorem,



Note that the map $(\rho_{n,1})_* : \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ is induced by a map which has degree 2. As such, it is trivial. Therefore, $\Delta_{n+1} = 0$ and $\pi_n(V_{n+1,\,2}) = \mathbb{Z}_2$. $\square$