which we have put $B^j \equiv B_j$. The first symbolic treatment of the Bernoulli numbers is credited to the inventor of umbral calculus, John Blissard (1803–1875); see [**2**] for a historical account. More on umbral equivalences and some other examples of identities that have a succinct representation in umbral form can be found in [**5**]. Seen in the light of umbral calculus, our identity of symmetry takes the form

$$\frac{1}{a}\big[a\,B + b\,\sigma(a-1)\big]^m \equiv \frac{1}{b}\big[b\,B + a\,\sigma(b-1)\big]^m, \tag{7}$$

where the symbols $B$ and $\sigma$ are to be treated as "umbrae", with the symbolic equivalences $B^j \equiv B_j$ and $\sigma^j \equiv \sigma_j$. Beauty and what constitutes elegance is, of course, a matter of taste, but one cannot deny that (7) is more memorable than either (5) or (6), and definitely serves as a better mnemonic.

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# Heronian Triangles Are Lattice Triangles


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A *Heronian triangle* has integer sides and integer area. It is possible to put the vertices of a Heronian triangle at lattice points, provided the triangle has an integer height. There are, however, Heronian triangles with no integer heights. For example, the triangle with sides 25, 34, and 39 has area 420, but none of its heights is an integer [**1**]. The paper [**4**] gives a complete characterization, and construction, of such triangles. The Heronian triangle $(25, 34, 39; 420)$ can nevertheless be realized with vertices at the origin and at the lattice points $(16, 30)$, $(36, 15)$. Here is the main result of this note.

**Theorem 1.** *Every Heronian triangle can be realized as a lattice triangle.*

It is enough to consider primitive Heronian triangles, i.e., those with side lengths $a$, $b$, $c$ satisfying $\gcd(a, b, c) = 1$. The area of such a triangle is given by Heron's formula

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \tag{1}$$

where $s := \frac{1}{2}(a+b+c)$ is the semiperimeter. Inspection of (1) shows that this formula can be rewritten in terms of the side lengths as

$$(4\Delta)^2 = -a^4 - b^4 - c^4 + 2a^2b^2 + 2b^2c^2 + 2c^2a^2. \tag{2}$$

From this, it is clear that exactly two of the side lengths are odd. We therefore assume $c$ odd, and $a$, $b$ of different parity. Such primitive Heronian triangles therefore correspond to the *positive* integer solutions of the quartic equation

$$(4W)^2 = -x^4 - y^4 - z^4 + 2x^2y^2 + 2y^2z^2 + 2z^2x^2, \qquad z \text{ odd}. \tag{3}$$

We study this quartic equation by viewing it as a quadratic equation in $x^2$, $y^2$, and $z^2$. Putting

$$X = x^2, \qquad Y = y^2, \qquad Z = z^2, \tag{4}$$

and replacing $4W$ by $2W'$, we write (3) as

$$(2W')^2 = -X^2 - Y^2 - Z^2 + 2XY + 2YZ + 2ZX, \tag{5}$$

and reorganize it into a quadratic equation in $Z$:

$$Z^2 - 2(X+Y)Z + (X-Y)^2 + (2W')^2 = 0. \tag{6}$$

This has integer solutions if and only if the discriminant $(X+Y)^2 - (X-Y)^2 - (2W')^2$ is the square of an even integer, say $2T$. In other words,

$$(X+Y)^2 = (X-Y)^2 + (2T)^2 + (2W')^2, \qquad \gcd(X+Y, X-Y, T) = 1. \tag{7}$$

The general integer solutions of (7) can be found in [**3**, p. 14]. With a minor change of notation, we have

$$X = p^2 + q^2, \qquad Y = u^2 + v^2, \qquad T = pu + qv, \qquad W' = pv - qu, \tag{8}$$

for integers $p$, $q$, $u$, $v$. With these, the quadratic equation (6) has integer roots

$$Z = (X+Y) \pm 2T = (p \pm u)^2 + (q \pm v)^2. \tag{9}$$

Now, a Heronian triangle $(a, b, c; \Delta)$ corresponds to a solution (8) of the form

$$(X, Y, Z, W') = (a^2, b^2, c^2; 2\Delta). \tag{10}$$

It follows that

$$a^2 = p^2 + q^2, \qquad b^2 = u^2 + v^2, \qquad c^2 = (p \pm u)^2 + (q \pm v)^2, \tag{11}$$

for an appropriate choice of signs. The Heronian triangle $(a, b, c; \Delta)$ can therefore be realized with vertices at the origin and at the lattice points $(p, q)$, $(u, v)$.

Our reasoning actually applies to a wider class of triangles. Conway et al. [**2**] consider geodetic angles. An angle $\theta$ is said to be geodetic if $\sin^2\theta$ is a rational number. If the angles of a triangle are all geodetic, then by the law of sines, the similarity class of such a triangle contains a unique one whose sides are $\sqrt{X}$, $\sqrt{Y}$, $\sqrt{Z}$, for positive integers $X$, $Y$, $Z$ satisfying $\gcd(X, Y, Z) = 1$. Let us say that a triangle is geodetic if it has side lengths $\sqrt{X}$, $\sqrt{Y}$, $\sqrt{Z}$ for integers $X$, $Y$, $Z$, *and* rational area (which is necessarily an integer or a half integer). Without assuming condition (4), our proof shows that every geodetic triangle can be realized as a lattice triangle.

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