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and let $\zeta_1, \ldots, \zeta_n$ be the (not necessarily distinct) eigenvalues of N. We leave the proof of the following assertion as an exercise:

$$|M| = \prod_{r=1}^{n} p(\zeta_r).$$

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Mixtilinear Incircles

Paul Yiu

L. Bankoff [**1**] has coined the term *mixtilinear incircles* of a triangle for the three circles each tangent to two sides and to the circumcircle internally. Consider a triangle ABC and its mixtilinear incircle in the angle A, with center K_A, and radius ρ_A. Bankoff has established the fundamental formula

$$r = \rho_A \cdot \cos^2 \frac{\alpha}{2}, \tag{1}$$

where r is the inradius of the triangle, and α is the magnitude of the angle at A. This formula had appeared earlier as an exercise in [**2**, p. 23]. It leads to a simple construction of the mixtilinear incircle. Denote by I the incenter of triangle ABC, and let the perpendicular through I to the bisector of angle A intersect the sides AC, AB at Y_1 and Z_1, respectively. The perpendiculars at these points to their respective sides intersect again on the angle bisector, at the mixtilinear incenter K_A. The circle with center K_A, passing through Y_1 (and Z_1), is the mixtilinear incircle in angle A; see Figure 1.

In this note, we demonstrate the usefulness of the notion of barycentric coordinates in discovering remarkable geometric properties relating to the mixtilinear incircles of a triangle. To keep the note self-contained, we refrain from using (1), except for the remarks at the end.

Denote by A' the point of contact of the mixtilinear incircle in angle A with the circumcircle. For convenience, we denote K_A by K, and ρ_A by ρ when there is no danger of confusion; see Figure 2. The center K lies on the bisector of angle A, and $AK : KI = \rho : -(\rho - r)$. In terms of barycentric coordinates,

$$K = \frac{1}{r}[-(\rho - r)A + \rho I]. \tag{2}$$

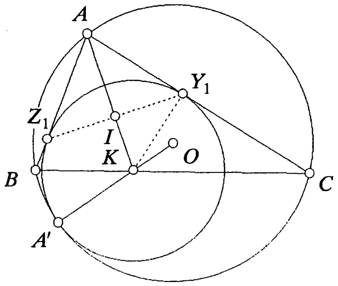

Figure 1

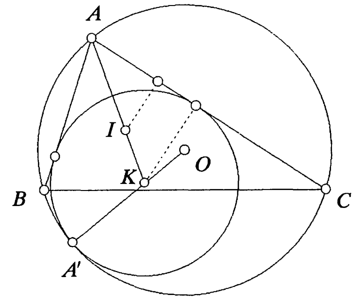

Figure 2

Also, since the circumcircle $O(A')$ and the mixtilinear incircle $K(A')$ touch each other at A', we have $OK : KA' = R - \rho : \rho$, where R is the circumradius. From this,

$$K = \frac{1}{R}[\rho O + (R - \rho)A']. \tag{3}$$

Comparing (2) and (3), we obtain, by rearranging terms,

$$\frac{RI - rO}{R - r} = \frac{R(\rho - r)A + r(R - \rho)A'}{\rho(R - r)}. \tag{4}$$

We note some interesting consequences of this formula. First of all, it gives the intersection of the lines joining AA' and OI. Note that the point P on the line OI represented by the left hand side of (4) is the external center of similitude of the circumcircle and the incircle of the given triangle. This, by definition, is the point dividing the segment OI externally in the ratio of the radii of the circles. As such, it can be constructed as the intersection of the lines OI and MD, where M is the intersection of the bisector of angle A with the circumcircle, and D the point of contact of the incircle with the side BC; see Figure 3.

The same reasoning applied to the other two mixtilinear incircles shows that each of the lines AA', BB', CC' passes through the same point P on the line OI; see Figure 4.

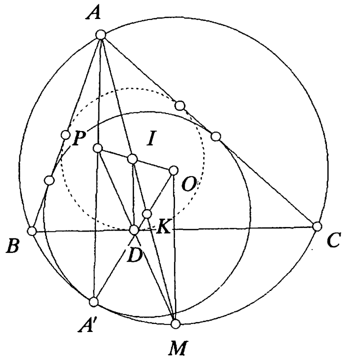

Figure 3

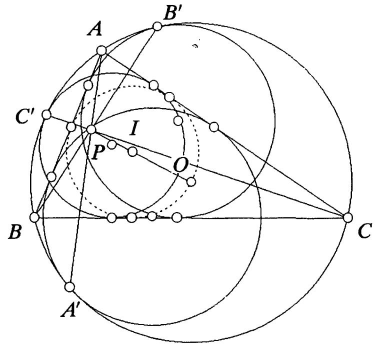

Figure 4

Theorem 1. *The three lines each joining a vertex to the point of contact of the circumcircle with the mixtilinear incircle in the angle of the vertex are concurrent at the external center of similitude of the circumcircle and the incircle.*

Equation (4) also leads to an alternative construction of the mixtilinear incircle, without the use of (1).

Construction 2. Given a triangle ABC, let P be the external center of similitude of the circumcircle (O) and incircle (I). Extend AP to intersect the circumcircle at A'. The intersection of AI and $A'O$ is the center K_A of the mixtilinear incircle in angle A.

Theorem 1 means that the triangles ABC and $A'B'C'$ are in perspective. By Desargues' Theorem, the intersections of the three pairs of lines BC, $B'C'$; CA, $C'A'$, and AB, $A'B'$ are collinear. The intersection X of the lines BC and $B'C'$ is indeed the external center of similitude of the mixtilinear incircles (K_B) and (K_C). This is clear from the following lemma, whose proof we omit.

Lemma 3. *If two distinct circles are tangent to a third circle, both internally or both externally, then the line joining the points of contact passes through the external center of similitude of the two circles.*

If one of the tangencies is internal and the other is external, then the line joining the points of contact passes through the internal center of similitude of the two circles; see Figure 5.

It is easy to determine the barycentric coordinates of X with respect to B and C. In fact,

$$X = \frac{\rho_C \cdot K_B - \rho_B \cdot K_C}{\rho_C - \rho_B} = \frac{-\left(1 - \frac{r}{\rho_B}\right)B + \left(1 - \frac{r}{\rho_C}\right)C}{\left(\frac{1}{\rho_B} - \frac{1}{\rho_C}\right)r}.$$

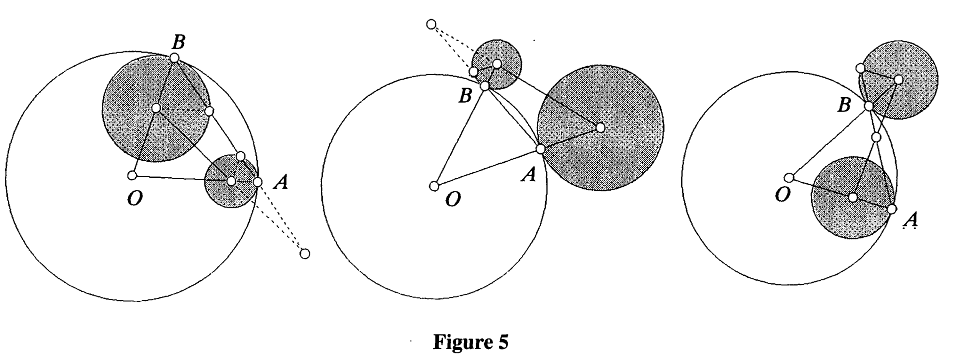

Figure 5

Here, we have made use of analogues of (2). Similarly, the external centers of similitude of the pairs of circles (K_C), (K_A), and (K_A), (K_B) are

$$Y = \frac{-\left(1-\frac{r}{\rho_C}\right)C + \left(1-\frac{r}{\rho_A}\right)A}{\left(\frac{1}{\rho_C}-\frac{1}{\rho_A}\right)r} \quad \text{and} \quad Z = \frac{-\left(1-\frac{r}{\rho_A}\right)A + \left(1-\frac{r}{\rho_B}\right)B}{\left(\frac{1}{\rho_A}-\frac{1}{\rho_B}\right)r}.$$

These three points X, Y, Z all lie on the line

$$\frac{x}{1-\frac{r}{\rho_A}} + \frac{y}{1-\frac{r}{\rho_B}} + \frac{z}{1-\frac{r}{\rho_C}} = 0. \tag{5}$$

Indeed, the triangles ABC, $A'B'C'$, and $K_AK_BK_C$ are pairwise in perspective, with line (5) as common axis of perspective.

We close with a few remarks. Since the points X, Y, Z are the external centers of similitude of pairs of circles from (K_A), (K_B), (K_C), their collinearity also follows from the famous Desargues Three-Circle Theorem **[5]**. If we make use of (1), this axis of perspective has equation

$$\frac{x}{\sin^2\frac{\alpha}{2}} + \frac{y}{\sin^2\frac{\beta}{2}} + \frac{z}{\sin^2\frac{\gamma}{2}} = 0.$$

Finally, we note another interesting consequence of (1). The Gergonne point of a triangle is the point of intersection of the three cevians joining each vertex to the point of contact of the incircle with the opposite side. This is the point X_7 of **[4]**, and has trilinear coordinates

$$\sec^2\frac{\alpha}{2} : \sec^2\frac{\beta}{2} : \sec^2\frac{\gamma}{2}.$$

As such, this is the unique point whose distances to the sides are proportional to the radii of the mixtilinear incircles in the respective angles.

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