Counting Lattice Paths that have Infinite Step Sets that can not be Reinterpreted as Weighted, Finite Step Sets

Katherine Humphreys and Heinrich Niederhausen
Florida Atantic University
Boca Raton, FL 33431

Abstract

A lattice path in the $\mathbb{N}_0 \times \mathbb{N}_0$ plane with an infinite step set $\mathfrak{S}$ can go to infinitely many lattice points within the boundary but can only come from finitely many points. In previous work, we have shown explicit solutions to counting lattice paths from $(0,0)$ to (n,m) in the first quadrant, above a boundary line $y = a\,(x-\ell)+b$ where a, ℓ, and b are nonnegative integers, or also a path that has a special access to the boundary using an additional set of (privileged) step vectors $\mathfrak{P}$. We find closed form solutions via Sheffer sequences and functionals using results of the Umbral Calculus. Examples of infinite step sets in previous work could be interpreted as step sets with a finite number of weighted steps. In this paper, we discuss infinite step sets that can not be reinterpreted as weighted, finite step sets and show that it is possible to use our methods to find explicit path counts for similar lattice path problems.

Keywords: Umbral Calculus, Sheffer sequences, infinite step set, lattice path counting, not rational generating function
AMS Classification: 05140, 05A15

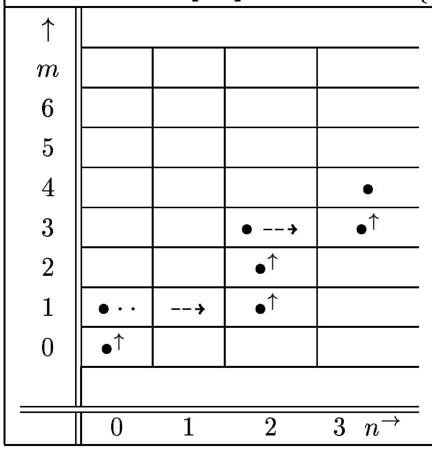
1 Introduction

Denote by $D(n,m)$ the number of lattice paths from the origin to the integer lattice point (n,m) and let $\mathfrak{S}$ be the set of steps the path can take. A step vector $\langle i,j \rangle \in \mathfrak{S}$ denotes a path step from (n,m) to $(n+i, m+j)$. An infinite step set has steps by which the path can go to infinitely many points assuming no boundary restrictions apply. Each point on the path has only finitely many predecessors. For this work, a boundary may consist of two pieces, a weak boundary at the horizontal axis for n from 0 to $\ell - 1$

intersected with a *restricted (half) line* $a(n-\ell)+b$ where a, b, and ℓ are nonnegative integers. We allow paths to step onto the restricted line for $n \geq \ell$ if they arrive there by *privileged access* step vectors from a special set $\mathfrak{P}$ as introduced in [4].

For example, a path with step set $\mathfrak{S} = \{\uparrow, \langle i, 0\rangle \mid i \geq 1, i \in \mathbb{Z}\}$ can go up or to infinitely many places to the right. Table 1 shows a sample path and the path counts $D(n,m)$ when there is only a horizontal boundary and no privileged access step set $\mathfrak{P}$.

Table 1: Sample path with $\mathfrak{S} = \{\uparrow, \langle i, 0\rangle \mid i \geq 1, i \in \mathbb{Z}\}$ and $D(n,m)$



$\uparrow$ m					
6	1	7	35	147	553
5	1	6	27	104	363
4	1	5	20	70	225
3	1	4	14	44	129
2	1	3	9	25	66
1	1	2	5	12	28
0	1	1	2	4	8
-1	**1**	**0**	**0**	**0**	**0**
	0	1	2	3	$n^{\rightarrow}$

With Umbral Calculus we find an explicit equation for the path counts from a polynomial sequence $(d_n(x))_{n\geq 0}$ where $D(n,m) = d_n(m)$. For our example, the polynomial solution for the columns of Table 1 starts off with $d_0(x) = 1$, $d_1(x) = x+1$, $d_2(x) = \frac{1}{2}(x+1)(x+4)$.

Notice from the table that for $m > 0$,

$$D(n,m) = D(n,m-1) + 2D(n-1,m) - D(n-1,m-1).$$

Instead of an infinite step set, this lattice path problem could be restated with a finite step set with weights, $\mathfrak{S}_W = \{\langle 0,1\rangle_1, \langle 1,0\rangle_2, \langle 1,1\rangle_{-1}\}$ where a step vector $\langle i,j\rangle_\omega \in \mathfrak{S}_W$ has weight ω.

In [3] we solve lattice path problems using infinite step sets. Like our introductory example, the step sets in that work can be reinterpreted as finite, weighted step sets. In this paper we discuss *irreducible* infinite step sets that can not be reinterpreted as weighted, finite step sets, and show that it is also possible to use our methods to find explicit path counts for such lattice path problems. In Section 2, we construct irreducible infinite step sets. In Section 3, we go over some necessary Umbral Calculus to understand the Functional Expansion Theorem and its components. In Section 4, we discuss the solution methods and work through an example for three types of problems and in Section 5, we give the results we found for our new infinite step sets. In Section 6 we list more examples with their solutions.

The support of D is the region supp (D) where the path counts $D(n, m)$ are positive. In this paper, we will show solutions to problems where $d_n(m) = D(n, m)$ in all of the support of D. This restriction is not necessary to our solution methods. Some solutions may be eventually polynomial: for every n there is an m_n such that $d_n(m) = D(n, m)$ for $m \geq m_n$; a few points at each n may need special manipulation. Lattice path problems with eventually polynomial solutions as well as a variety of other types of lattice path problems with privileged access are addressed in [3]. Lattice path problems with more than a two piece boundary are solved in [7] and a more generalized Umbral Calculus solving a variety of combinatorics problems is found in [9] and [10].

Notation throughout our work defines $\mathbb{N}_0$ as nonnegative integers and $\mathbb{N}_1$ as positive integers.

2 Infinite Step Sets or not

Why are the path counts for $\mathfrak{S} = \{\uparrow, \langle i, 0\rangle \mid i \in \mathbb{N}_1\}$ the same as those for $\mathfrak{S}_W = \{\langle 0,1\rangle_1, \langle 1,0\rangle_2, \langle 1,1\rangle_{-1}\}$? The step set defines the path count recursion. The polynomials $(d_n(m))_{n\geq 0}$ will follow the same recursion as that of the path counts $D(n, m)$. We show by our introductory example how the operator equation of the polynomial recursion can redefine the step set.

2.1 The Operator Equation

Two common linear operators on polynomials are the backwards difference operator ∇ where $\nabla d_n(x) = d_n(x) - d_n(x-1)$ and the shift by a operator E^a where $E^a d_n(x) = d_n(x+a)$. Shift invariant operators are those that commute with the shift operator E^a for all a. Notice that ∇ is a degree reducing operator.

If the backwards difference of a function is a polynomial of degree $n-1$, then the function is a polynomial of degree n. For our example, the recursion for the path counts

$$D(n, m) = D(n, m-1) + \sum_{i\geq 1} D(n-i, m)$$

expressed as

$$\nabla D(n, m) = \sum_{i\geq 1} D(n-i, m)$$

shows that $D(n,m)$ can be *extended to a polynomial* $d_n(m)$ of degree n for almost all m where $D(n-1,m)$ already has a polynomial extension. The polynomial extension (d_n) follows the same recursion

$$d_n(x) = d_n(x-1) + \sum_{i\geq 1} d_{n-i}(x).$$

We define a linear operator, say W, on polynomials which maps $d_n(x)$ to $d_{n-1}(x)$. Then we rewrite the polynomial recursion to give an operator equation $Q = \phi(W)$, a power series in W, where $\phi(0) = 0$, Q is a known operator, in this case $Q = \nabla$, and the coefficients of W are in the ring of shift invariant operators.

For the introductory example, the polynomial recursion shows that the operator equation is

$$\nabla = \sum_{i\geq 1} W^i$$

which simplifies to

$$\nabla = \frac{W}{1-W} = 2W - E^{-1}W.$$

This simplification $\nabla = 2W - E^{-1}W$ translates the polynomials back to path counts as

$$\begin{aligned}\nabla D(n,m) &= 2WD(n,m) - E^{-1}WD(n,m) \Rightarrow \\ D(n,m) &= D(n,m-1) + 2D(n-1,m) - D(n-1,m-1).\end{aligned}$$

Here we see a weighted, finite step set $\{\langle 0,1\rangle_1, \langle 1,0\rangle_2, \langle 1,1\rangle_{-1}\}$ for this path count recursion. This reduction to finiteness will occur in all cases the operator equation can simplify to a ratio of polynomials. If such a reduction does not exist, we call the infinite step set *irreducible.*

2.2 Constructing Irreducible Infinite Step Sets

We construct irreducible, infinite step sets two ways. In the first two examples, we define a step set and then show the operator equation $Q = \phi(W)$ is not the ratio of two polynomials. In the other examples we use the following generating functions in the construction of the lattice path step set. The generating function for the central binomial numbers $\binom{2i}{i}$

$$\sum_{i=0}^{\infty} \binom{2i}{i} t^i = \frac{1}{\sqrt{1-4t}} \tag{1}$$

can be verified by comparing terms. And the well known Catalan numbers have generating function

$$\sum_{i=0}^{\infty} \frac{1}{1+i} \binom{2i}{i} t^i = \frac{1-\sqrt{1-4t}}{2t} = \frac{2}{1+\sqrt{(1-4t)}}. \tag{2}$$

Poisson Set

A **Poisson** infinite step set uses the example above adjoining the weight 1 to $\uparrow$ and the weight $\lambda^i/i!$ to $\langle i, 0\rangle$ where $\lambda > 0$. For $\mathfrak{S}_\lambda = \left\{\uparrow, \langle i, 0\rangle_{\lambda^i/i!} \mid i \in \mathbb{N}_1\right\}$, the path recursion

$$D(n, m) = D(n, m-1) + \sum_{i \geq 1} \frac{\lambda^i}{i!} D(n-i, m)$$

gives the operator equation

$$\nabla = \sum_{i \geq 1} \lambda^i / i! W^i = e^{\lambda W} - 1.$$

If a positive number i of arrivals during an hour in a Poisson process is modelled by a jump of i units to the right, and "no arrival" is represented by an upwards unit step, then $e^{-\lambda} D(n, m)$ is the probability of n arrivals (in at most $n+m$ hours) and m hours without arrivals.

Power Set

The **power** step set is $\mathfrak{S}_p = \left\{\uparrow, \langle \beta^i, \gamma\rangle \mid i \geq 0\right\}$ where $\beta > 1$ and $\beta, \gamma \in \mathbb{Z}$. The path recursion

$$D(n, m) = D(n, m-1) + \sum_{i \geq 0} D(n-\beta^i, m-\gamma)$$

translates to the operator equation

$$E^\gamma \nabla = \sum_{i \geq 0} W^{\beta^i}$$

with $Q = E^\gamma \nabla$ a known operator.

Lemma 1 *If $\beta \in N_1$ then $\phi(W) = \sum_{i \geq 0} W^{\beta^i}$ is not a rational generating function.*

Proof. Assume $\sum_{i \geq 0} t^{\beta^i} = \sum_{n \geq 0} f_n t^n = \frac{p(t)}{q(t)}$ where $q(t) = \sum_{k=0}^{d} \alpha_k t^k$ with $\alpha_0 = 1$ and $\alpha_d \neq 0$ and $\deg(p(t)) < d$. From the characterization of rational generating functions in [13, p. 202], we have that for all $n \geq 0$ holds

$$\alpha_0 f_{n+d} + \alpha_1 f_{n+d-1} + \cdots + \alpha_d f_n = 0. \tag{3}$$

Let $n = \beta^j$ where $j = (\log_2 d) + 1$. Hence $f_n = 1$, but since $\beta^{j+1} - \beta^j \geq \beta^j \geq 2^j > d$, the term $n + d$ is less than β^{j+1} so all the coefficients $f_{n+1}, f_{n+2}, \ldots, f_{n+d}$ are zero, thus by (3) $\alpha_d = 0$. This contradicts our assumption and we conclude that $\sum_{i \geq 0} t^{\beta^i}$ is not a rational generating function. ■

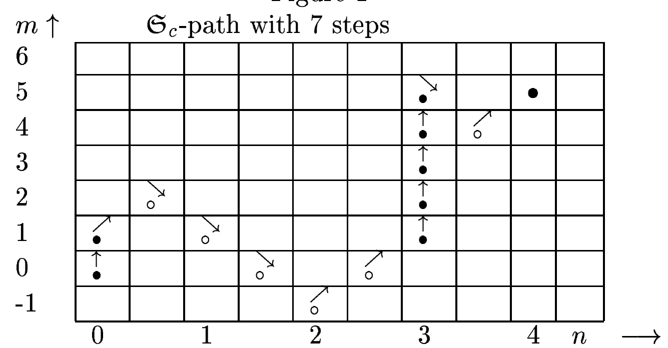
Central Pathlets

We define a *central pathlet* P as a $\{\langle \frac{1}{2}, 1\rangle, \langle \frac{1}{2}, -1\rangle\}$-path that steps in the half integer lattice $\frac{1}{2}\mathbb{N}_0 \times \mathbb{Z}$ of the xy-plane and starts and ends on the same value of m. Each path step must end in the nonnegative integer lattice, although a central pathlet may fall below the x-axis. (A central pathlet may also be defined as any $\{\langle \frac{1}{2}, 0, 1\rangle, \langle \frac{1}{2}, 0, -1\rangle\}$-path in the xz plane with steps "coming out of or going into the page" starting and ending at $z = 0$ and remaining on the same value of m. Then any boundary in the xy-plane could be strictly enforced.) There are $\binom{2i}{i}$ central pathlets P from (n, m) to $(n + i, m)$, $i \in \mathbb{N}_1$. Denote by $\mathbf{C}$ the set of all central pathlets. Consider each central pathlet as a single step and define the **central pathlet** infinite step set

$$\mathfrak{S}_c = \{\uparrow\} \cup \mathbf{C}.$$

Figure 1 shows a sample $\mathfrak{S}_c$-path with seven steps; two of the steps are pathlets of six and two "steps" respectively. Open dots between solid, endpoint dots designate a central pathlet step.

Figure 1
$\mathfrak{S}_c$-path with 7 steps



The path counts,

$$D(n,m) = D(n,m-1) + \sum_{P\in\mathbf{C}} D(n - \frac{1}{2}length(P), m)$$
$$= D(n,m-1) + \sum_{i\geq 1} \binom{2i}{i} D(n-i,m).$$

Making use of the generating function (1), the operator equation becomes

$$\nabla = \sum_{i\geq 1} \binom{2i}{i} W^i = \frac{1}{\sqrt{1-4W}} - 1.$$

Dyck Pathlets

Along the same vein, we define a Dyck pathlet P_a as a $\{\uparrow, \rightarrow\}$-path that starts and ends on a diagonal and stays weakly above the same diagonal and a Dyck pathlet P_b as a $\{\downarrow, \rightarrow\}$-path that starts and ends on a second diagonal while staying weakly below it.

Definition 2 *A* diagonal *is any line parallel to $y = x$ and a* second diagonal *is any line parallel to $y = -x$.*

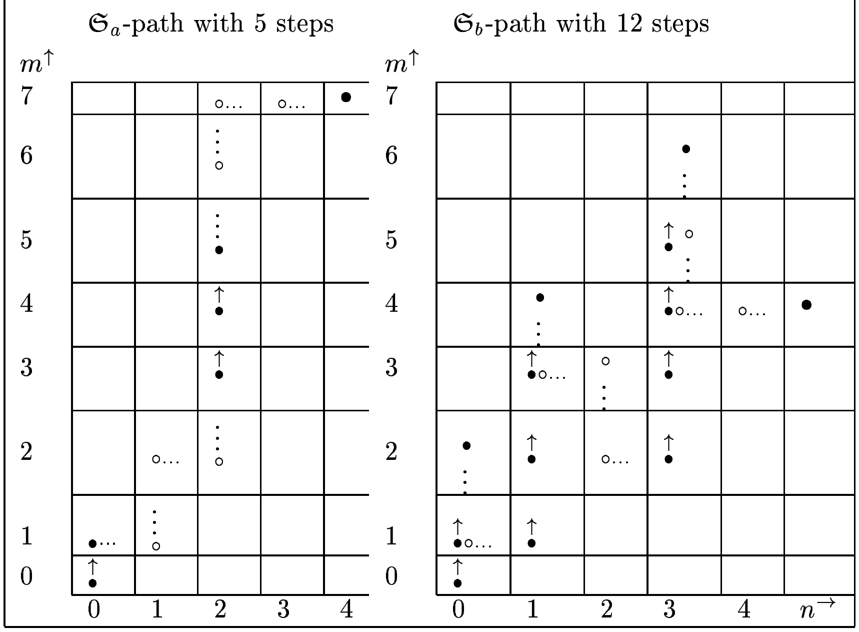
There are $\frac{1}{1+i}\binom{2i}{i}$ Dyck pathlets P_a from (n,m) to $(n+i, m+i)$ and the same for pathlets P_b from (n,m) to $(n+i, m-i)$. Let $\mathbf{S}_a$ be the set of all Dyck pathlets P_a and consider each pathlet $P_a \in \mathbf{S}_a$ as a single step. Define the **Dyck pathlet** infinite step set

$$\mathfrak{S}_a = \{\uparrow\} \cup \mathbf{S}_a$$

and similarly define $\mathfrak{S}_b$. Figure 2 shows a sample path from each step set. Hollow dots ($\circ \ldots \circ \ldots$) between solid, endpoint dots denote a Dyck pathlet;

remember the pathlets are considered as single steps in $\mathfrak{S}_a$ and $\mathfrak{S}_b$.

Figure 2



Making use of the generating function (2), the path recursion for the number of paths $D(n,m)$ from the origin to (n,m) with infinite step set $\mathfrak{S}_b$

$$D(n,m) = D(n,m-1) + \sum_{i\geq 1} \frac{1}{1+i}\binom{2i}{i} D(n-i,m+i)$$

translates into the operator equation

$$\begin{aligned}\nabla &= \sum_{i\geq 1} \frac{1}{1+i}\binom{2i}{i}(WE)^i \\ &= \frac{1-\sqrt{1-4WE}}{2WE} - 1.\end{aligned}$$

The operator equation for step set $\mathfrak{S}_a$ is

$$\nabla = \frac{1-\sqrt{1-4WE^{-1}}}{2WE^{-1}} - 1.$$

3 Theory

Our goal in this paper is to utilize the methods of Umbral Calculus as in [3] to find an explicit expression for the number of paths $D(n,m)$ for paths that have infinite step sets that do not reduce to finite, weighted step sets. Our solution is in the form of polynomials $d_0, d_1, \ldots$, where $d_n(m) = D(n,m)$. The polynomial d_n is of degree n; we set $d_n(x) = 0$ for all x iff $n < 0$. In this section we first discuss linear operators and functionals on polynomials of Umbral Calculus (as developed in [6], [11], and [12]) needed to explain our method of solving recursions. We state the expansion theorem we use to solve the polynomials and the transfer formula we use to find the basic sequences. We will use the word 'operator' as a synonym for 'linear operator on polynomials', and do the same for 'functionals'.

3.1 Linear Operators

It has been shown in [12] that the shift invariant operators are isomorphic to formal power series; the invertible shift invariant operators correspond to power series with nonzero constant term. In contrast, a power series $\phi(t)$ is a *delta series* iff $\phi(0) = 0$ and $\phi'(0) \neq 0$. The corresponding operator $\phi(\mathcal{D})$, where $\mathcal{D}$ is the derivative operator, is a *delta operator*. Examples are $\mathcal{D}$, of course and the backwards difference operator $\nabla = 1 - e^{-\mathcal{D}} = 1 - E^{-1}$. Delta series have compositional inverses, and this makes it possible to express every delta operator as a delta series in any other delta operator as we did for the operator equation $\nabla = \phi(W)$ in the last section.

Delta operators reduce the degree of polynomials by exactly 1. Suppose B is a delta operator. A *B-Sheffer sequence* $(s_n)_{n\geq 0}$ is a polynomial sequence such that $Bs_n(x) = s_{n-1}(x)$ for all $n \in \mathbb{N}_0$. In applications, the combinatorial recursion generates this system of operator equations. For example, the binomial coefficients $\binom{n+x}{n}$ are ∇-Sheffer polynomials because

$$\nabla \binom{n+x}{n} = \binom{n+x}{n} - \binom{n+x-1}{n} = \binom{n-1+x}{n-1}.$$

Different B-Sheffer sequences solve the same recursion, but satisfy different initial conditions. There exists a unique B-Sheffer sequence $(b_n)_{n\geq 0}$, the *B-basic sequence*, which has the initial values $b_n(0) = \delta_{n,0}$ for all $n \in \mathbb{N}_0$. The basic sequence serves as the basis for expanding Sheffer sequences, and carries the recursion information of the combinatorial problem. The ∇-basic sequence is $\left(\binom{n-1+x}{n}\right)_{n\geq 0}$.

3.2 Functionals

We denote the action of a linear functional L on a polynomial $p(x)$ by $\langle L \mid p(x)\rangle$. The functional eval_a is defined as *evaluation at a,*

$$\langle \text{eval}_a \mid p(x)\rangle = p(a).$$

We will use functionals L to describe the initial conditions of the polynomial sequence. In this paper such a functional could be as simple as evaluation at -1, asking that $\langle L \mid d_n \rangle = \langle \mathrm{eval}_{-1} \mid d_n(x) \rangle = d_n(-1)$ for all $n \geq 0$, or as in a problem with privileged access to the boundary, we define L so that $\langle L \mid d_n \rangle$ on the boundary line is zero for $n \geq \ell$.

With every functional L comes a unique shift invariant operator

$$\mu(L) := \sum_{n \geq 0} \langle L \mid b_n \rangle B^n. \tag{4}$$

It is shown in [11] that the operator $\mu(L)$ is invariant under the choice of the delta operator B and its basic sequence (b_n). For example, if $L = \mathrm{eval}_a$ it is convenient to choose the pair $\mathcal{D}$ and $(x^n/n!)$ to show that

$$\mu(\mathrm{eval}_a) = \sum_{n \geq 0} \frac{a^n}{n!} \mathcal{D}^n = e^{a\mathcal{D}}, \tag{5}$$

the shift operator E^a. If $\langle L \mid 1 \rangle \neq 0$ then $\mu(L)$ is invertible, as in the case of shift operators.

3.3 Expansion Theorem

To find the explicit polynomial solutions to the lattice path problems, we use the Functional Expansion Theorem [9]:

Theorem 3 *Suppose $(s_n)_{n \in \mathbb{N}_0}$ is a B-Sheffer sequence and L a functional such that $\langle L \mid 1 \rangle \neq 0$. The polynomials $s_n(x)$ can be expanded in terms of the B-basic sequence $(b_n)_{n \in \mathbb{N}_0}$ as*

$$s_n(x) = \sum_{k=0}^{n} \langle L \mid s_k \rangle \mu(L)^{-1} b_{n-k}(x). \tag{6}$$

For the generating function holds

$$\sum_{n \geq 0} s_n(x) t^n = \frac{\sum_{k \geq 0} \langle L \mid s_k \rangle t^k}{\sum_{j \geq 0} \langle L \mid b_j \rangle t^j} \sum_{n \geq 0} b_n(x) t^n.$$

The *Binomial Theorem for Sheffer Sequences* [12],

$$s_n(x) = \sum_{k=0}^{n} s_k(a) b_{n-k}(x-a) \tag{7}$$

is a special case of the Functional Expansion Theorem if we choose $L = \mathrm{Eval}_a$.

After the recursion and initial values of a polynomial solution have been extracted from the combinatorial problem, two obstacles remain: finding $\mu(L)^{-1}$ and the basic sequence (b_n) explicitly. For $\mu(L)$ we can use the expansion (4), which has the following easy to prove corollary.

Corollary 4 *If the functional L acts on polynomials p as $\langle L \mid p\rangle = \langle \mathrm{Eval}_a \mid B^k p\rangle$ for some delta operator B and $k \in \mathbb{N}_0$, then*

$$\mu(L) = E^a B^k.$$

To find basic sequences, we use the following *transfer formula* (8) as shown in [8], based on results in [12]. Delta operators can be expanded as delta series ϕ with coefficients in the ring of shift invariant operators. Suppose $Q = \phi(B)$ where operator Q has a known basic sequence $(q_n(x))$. With the help of Lagrange-Bürmann inversion it can be shown that

$$b_n(x) = \sum_{j=0}^{n-1} x\alpha_{n,j} \frac{1}{x} q_{n-j}(x) \tag{8}$$

for positive n, where $\alpha_{n,j}$ is the coefficient of B^n in $\phi(B)^{n-j}$. For an example, if $Q = E^a B$, then for all $n > 0$

$$\begin{aligned} b_n(x) &= \sum_{j=0}^{n-1} x\left([B^n]\left(E^a B\right)^{n-j}\right) \frac{1}{x} q_{n-j}(x) \\ &= xE^{an}\frac{1}{x} q_n(x) = \frac{x}{x+an} q_n(x+an). \end{aligned} \tag{9}$$

4 Solution Methods

The path counts $D(n,m)$ from the origin to (n,m) follow the recursion $D(n,m) =$

$$\begin{cases} \sum_{\langle i,j\rangle \in \mathfrak{S}} D(n-i, m-j) & \text{if } (n,m) \in \operatorname{supp}(D) \setminus \{(n, a(n-\ell)+b)\} \\ \sum_{\langle p,q\rangle \in \mathfrak{P}} D(n-p, m-q) & \text{if } n \geq \ell \text{ and } m = a(n-\ell)+b \\ 0 & \text{if } (n,m) \notin \operatorname{supp}(D) \end{cases}$$

with initial value $D(0,0) = 1$. If the privileged step set $\mathfrak{P}$ is empty then $(n, a(n-\ell)+b) \notin \operatorname{supp}(D)$. Recursion formulas for the numbers $D(n,m)$ always assume that $D(n,m) = 0$ outside $\operatorname{supp}(D)$, without stating this as an initial condition. For polynomial sequences (d_n) the only such implicit assumption is $d_n(m) = 0$ for $n < 0$, the half plane inaccessible to our lattice paths independent of any imposed restrictions. Thus we define the following enlargement of the support.

Definition 5 *The* allowed region *of a lattice path problem is the support,* supp (D)*, extended to the left half plane* $\{(n,m)\in\mathbb{Z}^2 : n<0\}$.

We define W to be the delta operator for the path count W-Sheffer sequence $(d_n(x))_{n\geq 0}$ with basic sequence $(w_n(x))_{n\geq 0}$ in all of our problems. We discuss Umbral Calculus solution methods and the main challenges for three different problem settings.

To demonstrate each setting, we use the Dyck pathlet infinite step set $\mathfrak{S}_b=\{\uparrow\}\cup\mathbf{S}_b$ as defined in Subsection 2.2. See Figure 2 for a sample path. Table 2 shows the path counts and the polynomial extension when the **paths stay in the first quadrant**. In Table 3, we place a **boundary** at $n-2$. In Table 4, the paths have the standard step set $\mathfrak{S}=\{\uparrow,\rightarrow\}$ but we allow the paths to go to the boundary via the **privileged** step set $\mathfrak{P}_b=\mathfrak{S}_b\backslash\{\uparrow\}=\mathbf{S}_b$.

4.1 Paths Staying in the First Quadrant

We restrict the paths to the first quadrant by imposing a weak boundary at the horizontal axis. No step comes from below the horizontal axis, and as $d_0(x)=1$, the initial conditions for the polynomials, in bold in Table 2, are $d_n(-1)=\delta_{n,0}$. We use the Binomial Theorem for Sheffer Sequences (7), specifically with

$$d_n(x)=\sum_{k=0}^{n} d_k(-1)\,w_{n-k}(x+1)=w_n(x+1)\,. \tag{10}$$

After determining the operator equation $Q=\phi(W)$ and the initial conditions, the challenge in this problem is finding the basic sequence $(w_n(x))$ using (8). Looking at $w_n(x)=\sum_{i=0}^{n-1} x\alpha_{n,i}\frac{1}{x}q_{n-i}(x)$, we must be able to describe in an explicit, finite sum, the coefficients $\alpha_{n,i}$. If the coefficients include operators, we need to know how the operator acts on $\frac{1}{x}q_{n-i}(x)$.

For our example step set, the operator equation is $\nabla=\sum_{i\geq 1}\frac{1}{1+i}\binom{2i}{i}(WE)^i=\frac{1-\sqrt{1-4WE}}{2WE}-1$ and the basic sequence is

$$\begin{aligned}
w_n(x) &= \sum_{j=0}^{n-1} x\left([W^n]\left(\frac{1-\sqrt{1-4WE}}{2WE}-1\right)^{n-j}\right)\frac{1}{x}\binom{n-j-1+x}{n-j}\\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-j}\binom{-k}{n-j-k}\binom{2n-1+k}{n}\frac{x}{k+n}\binom{2n-j-1+x}{n-j-1}
\end{aligned}$$

where we use the Catalan polynomials identity $\sum_{n=0}^{\infty}\frac{x}{x+n}\binom{2n-1+x}{n}t^n=$

$\left(\frac{1-\sqrt{(1-4t)}}{2t}\right)^x$. The solution for the total path count using (10) is

$$D(n,m) = \sum_{j=0}^{n-1}\sum_{k=0}^{n-j}\binom{-k}{n-j-k}\binom{2n-1+k}{n}\frac{m+1}{k+n}\binom{2n-j+m}{n-j-1}.$$

Table 2 shows the path counts and the polynomial extension below the boundary.

Table 2 : Path Counts for $\mathfrak{S}_b$-paths staying in First Quadrant

$m^{\uparrow}$							
6	1	7	49	343	2415	17129	
5	1	6	39	260	1767	12198	
4	1	5	30	190	1245	8361	
3	1	4	22	132	833	5440	
2	1	3	15	85	516	3276	
1	1	2	9	48	280	1728	
0	1	1	4	20	112	672	
−1	**1**	**0**	**0**	**0**	**0**	**0**	
	0	1	2	3	4	5	$\to n$

4.2 Boundary Problems

Boundary Problems have a boundary that consists of two pieces: a weak boundary at the horizontal axis for n from 0 to $\ell-1$ and a strong boundary $a(n-\ell)+b$ where a, b, and ℓ are nonnegative integers. When our boundary partly consists of the line $a(n-\ell)+b$ it is helpful to introduce the $E^{-a}W$-Sheffer sequence $\left(\hat{d}_n\right)$, where

$$\hat{d}_n(x) = d_n(an+x), \quad \text{and}$$
$$E^{-a}W\hat{d}_n(x) = E^{-a}d_{n-1}(an+x) = d_{n-1}(an+x-a) = \hat{d}_{n-1}(x).$$

By observation (9), the $E^{-a}W$-basic sequence $(\hat{w}_n)$ becomes

$$\hat{w}_n(x) = \frac{x}{x+an}w_n(x+an).$$

The initial conditions will be $\hat{d}_n(-a\ell+b) = 0$ for $n \geq \ell$ if the recursive calculations of $d_n(m)$ at points $(n,m) \in supp(D)$ only refer back to points (i,j) in the allowed region. We use the Binomial Theorem for Sheffer

Sequences (7) again; the solution is

$$\hat{d}_n(x) = \sum_{k=0}^{\ell-1} \hat{d}_k(-a\ell+b)\,\hat{w}_{n-k}(x+a\ell-b) \tag{11}$$

where $D(n,m) = \hat{d}_n(m-an)$. The nonzero initial conditions $\hat{d}_k(-a\ell+b)$ for $k = 0,\ldots,\ell-1$ need to be calculated. The remaining challenge is the basic sequence $(w_n(x))$ as in the first problem above.

We demonstrate the solution method with our example problem using the step set $\mathfrak{S}_b = \{\uparrow\} \cup \mathbf{S}_b$ with a boundary at $n-2$: $a = 1$, $\ell = 2$, and $b = 0$. The basic sequence is $\hat{w}_0(x) = 1$ and for $n > 0$,

$$\begin{aligned}\hat{w}_n(x) &= \frac{x}{x+n} w_n(x+n) \\ &= \sum_{j=0}^{n-1}\sum_{k=0}^{n-j} \binom{-k}{n-j-k}\binom{2n-1+k}{n}\frac{x}{k+n}\binom{3n-j-1+x}{n-j-1}.\end{aligned}$$

The initial conditions $\hat{d}_k(-2) = d_k(k-2) = 0$ for $k > 1$ and from Table 3, we see in bold that $\hat{d}_0(-2) = 1$ and $\hat{d}_1(-2) = 0$. The solution for the path counts, along with the polynomial extension below the boundary, is

$$\begin{aligned}d_n(m) &= \hat{d}_n(m-n) = \hat{w}_n(m-n+2) \\ &= \sum_{j=0}^{n-1}\sum_{k=0}^{n-j} \binom{-k}{n-j-k}\binom{2n-1+k}{n}\frac{m-n+2}{k+n}\binom{2n-j+m+1}{n-j-1}.\end{aligned}$$

Table 3 : $d_n(m)$ for $\mathfrak{S}_b$-paths staying above $n-2$

$m^{\uparrow}$						
5	1	6	35	196	1035	
4	1	5	26	130	589	
3	1	4	18	76	249	
2	1	3	11	33	**0**	
1	1	2	5	**0**	-172	
0	1	1	**0**	-24	-280	
-1	1	**0**	-4	-40	-336	
-2	**1**	-1	-7	-49	-351	
	0	1	2	3	4	5 $\rightarrow n$

4.3 Privileged Access Problems

A **Privileged Access Problem** has a weak boundary at the horizontal axis for n from 0 to $\ell-1$ and a *restricted (half) line* $a(n-\ell)+b$ where a, b, and ℓ are nonnegative integers. We allow paths to step onto the restricted line for $n \geq \ell$ if they arrive there by *privileged access* step vectors from a special set $\mathfrak{P}$.

$$D(n,m) = \sum_{\langle p,q\rangle \in \mathfrak{P}} D(n-p, m-q) \text{ if } n \geq \ell \text{ and } m = a(n-\ell)+b.$$

When designing privileged access we must avoid cycles in the paths. For example, a down step in $\mathfrak{P}$ would be bad if $\uparrow \in \mathfrak{S}$. The path counts are found with the Functional Expansion Theorem (6)

$$\hat{d}_n(x) = \sum_{k=0}^{n} \left\langle L \mid \hat{d}_k \right\rangle \mu(L)^{-1} \hat{w}_{n-k}(x)$$

where $D(n,m) = d_n(m) = \hat{d}_n(m-an)$. The basic sequence $(\hat{w}_n(x))$ is determined by the path recursion from the steps in $\mathfrak{S}$;

$$D(n,m) = \sum_{\langle i,j\rangle \in \mathfrak{S}} D(n-i, m-j) \text{ if } (n,m) \in \operatorname{supp}(D) \smallsetminus \{(n, a(n-\ell)+b)\}.$$

Assuming initial values $\left\langle L \mid \hat{d}_k \right\rangle$ can be found, the challenge then in this problem is to find the inverse operator of the functional L , $\mu(L)^{-1}$. We define L along the boundary line so that

$$\left\langle L \mid \hat{d}_n(x) \right\rangle = 0 \text{ for all } n \geq \ell.$$

We start with the privileged access condition on the line:

$$\hat{d}_n(-a\ell+b) - \sum_{\langle p,q\rangle \in \mathfrak{P}} \hat{d}_{n-p}(a(p-\ell)+b-q) = 0.$$

In functional notation $\left\langle L \mid \hat{d}_n(x) \right\rangle$

$$= \left\langle \operatorname{eval}_{-a\ell+b} \mid \hat{d}_n(x) \right\rangle - \sum_{\langle p,q\rangle \in \mathfrak{P}} \left\langle \operatorname{eval}_{a(p-\ell)+b-q} \mid \hat{d}_{n-p}(x) \right\rangle$$
$$= \left\langle \operatorname{eval}_{-a\ell+b} \mid \hat{d}_n(x) \right\rangle - \sum_{\langle p,q\rangle \in \mathfrak{P}} \left\langle \operatorname{eval}_{-a\ell+b} \operatorname{eval}_{ap-q} \mid \left(E^{-a}W\right)^p \hat{d}_n(x) \right\rangle.$$

Using (4), (5) and Corollary 4,

$$\mu(L) = E^{-a\ell+b}\left(1 - \sum_{\langle p,q\rangle \in \mathfrak{P}} E^{ap-q}\left(E^{-a}W\right)^p\right)$$

where we write $E^{ap-q}\left(E^{-a}W\right)^p$ because it is easier to apply to $\hat{w}_n\left(x\right)$ which is a $E^{-a}W$- Sheffer sequence. The resulting inverse operator of L is

$$\mu\left(L\right)^{-1}=E^{a\ell-b}\left(\frac{1}{1-\sum_{\langle p,q\rangle\in\mathfrak{P}}E^{ap-q}\left(E^{-a}W\right)^p}\right).$$

The possibility of a solution, whether $\mu\left(L\right)^{-1}$ can be applied to the basic sequence, is problem dependent.

We demonstrate this solution method using the standard path step set $\mathfrak{S}=\{\uparrow,\rightarrow\}$ and we allow the paths to go to the boundary $n-2$ for $n\geq 2$ via the **privileged** step set $\mathfrak{P}_b=\mathfrak{S}_b\setminus\{\uparrow\}=\mathbf{S}_b$. Table 4 shows the values on the line in bold. The numbers $d_n(m)$ above the boundary correspond to path counts; numbers below the boundary where $D\left(n,m\right)=0$ continue the polynomial extension. The operator equation for the step set $\mathfrak{S}$ is $\nabla=W$ so the basic sequence is $w_n\left(x\right)=\binom{n-1+x}{n}$ with $\hat{w}_n(x)=\frac{x}{n+x}\binom{2n-1+x}{n}$, the $E^{-1}W$- basic sequence. Using the generating function (2) and the identity $\sum_{n=0}^{\infty}\binom{2n+x}{n}t^n=\frac{2^x}{\sqrt{1-4t}}\left(1+\sqrt{1-4t}\right)^{-x}$ we find

$$\begin{aligned}\mu\left(L\right)^{-1}&=E^{a\ell-b}\left(\frac{1}{1-\sum_{\langle p,q\rangle\in\mathfrak{P}}E^{ap-q}\left(E^{-a}W\right)^p}\right)\\&=E^2\left(\frac{1}{1-\sum_{i\geq1}\frac{1}{1+i}\binom{2i}{i}\left(EW\right)^i}\right)\\&=E^2\left(\sum_{j\geq0}\binom{2j-1}{j}\left(EW\right)^j\right)\\&=\sum_{j\geq0}\binom{2j-1}{j}E^{2j+2}\left(E^{-1}W\right)^j.\end{aligned}$$

The initial conditions for $n\geq 2$ are $\left\langle L\mid\hat{d}_n\left(x\right)\right\rangle=0$. Using

$$\begin{aligned}\left\langle L\mid\hat{d}_n\left(x\right)\right\rangle&=\left\langle \mathrm{eval}_{-2}\mid\hat{d}_n\left(x\right)\right\rangle-\sum_{i\geq1}\frac{1}{1+i}\binom{2i}{i}\left\langle \mathrm{eval}_{-2+2i}\mid\hat{d}_{n-i}\left(x\right)\right\rangle\\&=d_n\left(n-2\right)-\sum_{i\geq1}\frac{1}{1+i}\binom{2i}{i}d_{n-i}\left(n-2+i\right)\end{aligned}$$

and the help of Table 4, $\left\langle L\mid\hat{d}_0\left(x\right)\right\rangle=1$ and $\left\langle L\mid\hat{d}_1\left(x\right)\right\rangle=-1$. In the

Expansion Theorem (6),

$$\begin{aligned}\hat{d}_n(x) &= \sum_{k=0}^{n} \left\langle L \mid \hat{d}_k \right\rangle \mu(L)^{-1} \hat{w}_{n-k}(x) \\ &= \mu(L)^{-1} (\hat{w}_n(x) - \hat{w}_{n-1}(x)) \\ &= \sum_{j\geq 0} \binom{2j-1}{j} E^{2j+2} \left(E^{-1}W\right)^j (\hat{w}_n(x) - \hat{w}_{n-1}(x)).\end{aligned}$$

Applying the operators, $D(n,m) = \hat{d}_n(m-n) =$

$$\sum_{j=0}^{n} \binom{2j-1}{j} (\hat{w}_{n-j}(m-n+2j+2) - \hat{w}_{n-1-j}(m-n+2j+2))$$

where $\hat{w}_n(x) = \frac{x}{n+x}\binom{2n-1+x}{n}$.

Table 4 : $d_n(m)$ for $\{\uparrow, \rightarrow\}$-paths with Privileged step set $\mathbf{S}_b$ to $n-2$

$m^{\uparrow}$						
5	1	6	24	86	316	1257
4	1	5	18	62	230	941
3	1	4	13	44	168	**711**
2	1	3	9	31	**124**	543
1	1	2	6	**22**	93	419
0	1	1	**4**	16	71	326
-1	1	**0**	3	12	55	255
-2	**1**	-1	3	9	43	200
	0	1	2	3	4	5 $\rightarrow n$

5 Results

As was shown in the previous section, it is possible to use the Umbral Calculus tools to obtain explicit solutions for lattice path counting problems with an infinite step set that can not be reinterpreted to a finite, weighted step set. For each infinite step set defined in Subsection 2.2, we show, where found, the necessary components to obtain solutions to similar examples as those demonstrated in Section 4. We list the step set $\mathfrak{S}$, its operator equation $\nabla = \phi(W)$, where W is the delta operator for the $\mathfrak{S}$-path count W-Sheffer sequence $(d_n(x))_{n\geq 0}$, and the basic sequence for $n > 0$ ($w_0(x) = 1$ for all examples). The basic sequence $(w_n(x))$ found with (8) uses the operator equation $\nabla = \phi(W)$ and the basic sequence for the backwards difference operator ∇ which is $\binom{n-1+x}{n}$. The infinite step sets in this paper consist of the up step $\uparrow$ and an infinite subset. The privileged access problems using these sets let the access step sets equal the infinite subset, $\mathfrak{P} = \mathfrak{S} \backslash \{\uparrow\}$. We list the inverse operator of the functional, $\mu(L)^{-1}$, using a generic W which is used here for any generic step set $\mathfrak{S}$ of the problem. All privileged access examples in this paper simply use $\mathfrak{S} = \{\uparrow, \rightarrow\}$ and so $W = \nabla$.

Poisson Set

- $\mathfrak{S}_\lambda = \left\{\uparrow, \langle i, 0\rangle_{\lambda^i/i!} \mid i \in \mathbb{N}_1\right\}$.
- $\nabla = \sum_{i \geq 1} \lambda^i / i! W^i = e^{\lambda W} - 1$.
- $w_n(x) = \sum_{j=1}^{n} \frac{(x)_j \lambda^n}{n!} S(n, j)$ where $S(n, j)$ are the Stirling numbers of the second kind and $(x)_j$ is the rising factorial.

Power Set

- $\mathfrak{S}_p = \left\{\uparrow, \langle \beta^i, \gamma\rangle \mid i \geq 0\right\}, \beta \in \mathbb{N}_1$.
- $E^\gamma \nabla = \sum_{i \geq 0} W^{\beta^i}$.
- $w_n(x)$ is not found. The coefficient $[W^n]\left(\sum_{i \geq 0} W^{\beta^i}\right)^{n-j}$ can be restated as the number $c(n, n-j)$ of compositions of n into $n-j$ parts of the form β^i. See [1] for recent work on the case $\beta = 2$. In that notation $w_n(x) = \sum_{j=0}^{n-1} c(n, n-j)\, q_{n-j}(x)$ where $(q_n(x))$ is the known $E^\gamma \nabla$-basic sequence.
- $\mu(L)^{-1} = E^{a\ell - b}\left(\sum_{j \geq 0}\left(E^{-\gamma} \sum_{i \geq 0} W^{\beta^i}\right)^j\right)$
  $= E^{a\ell-b} E^{-\gamma} \sum_{j \geq 0} \sum_{k \geq 0} c(k, j)\, W^k = E^{a\ell-b} E^{-\gamma} \sum_{k \geq 0} c(k) W^k$
  where $c(k)$ is the number of compositions of k into powers of β.

Central Pathlets

- $\mathfrak{S}_c = \{\uparrow\} \cup \mathbf{C}$ where $\mathbf{C}$ is the set of all central pathlets from $(n-i, m)$ to (n, m) for $i = 1, \ldots, n$.
- $\nabla = \sum_{i \geq 1} \binom{2i}{i} W^i = \frac{1}{\sqrt{1-4W}} - 1$.
- $w_n(x) = 4^n \sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^{j+k} \binom{\frac{-k}{2}}{n} \binom{n-j-1+x}{n-j}$.
- $\mu(L)^{-1} = E^{a\ell-b} \sum_{i \geq 0} \frac{16^i}{3^{i+1}} \Big(2E^{ai}\left(E^{-a}W\right)^i - 8E^{a(i+1)}\left(E^{-a}W\right)^{i+1}$

$$+ \sum_{k \geq 0} \binom{\frac{1}{2}}{k} (-4)^k E^{a(k+i)} \left(E^{-a}W\right)^{k+i}\Big).$$

Dyck Pathlets above

- $\mathfrak{S}_a = \{\uparrow\} \cup \mathbf{S}_a$ where $\mathbf{S}_a$ is the set of all $\{\uparrow, \rightarrow\}$-paths from $(n-i, m-i)$ to (n, m) for $i = 1, \ldots, \min(n, m)$ that stay weakly above a diagonal.

- $\nabla = \frac{1-\sqrt{1-4WE^{-1}}}{2WE^{-1}} - 1 = \frac{2}{1+\sqrt{(1-4WE^{-1})}} - 1$ for problems with boundary line $an - 1$ where a is a positive integer.

- $w_n(x) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \binom{-k}{n-j-k} \binom{2n-1+k}{n} \frac{x}{k+n} \binom{x-j-1}{n-j-1}$.

- for privileged access problems to a boundary $an - 1$, $\mu(L)^{-1} = E^{-b} \sum_{j \geq 0} \binom{2j-1}{j} E^{aj-j} (E^{-a}W)^j$.

Dyck Pathlets below

- $\mathfrak{S}_b = \{\uparrow\} \cup \mathbf{S}_b$ where $\mathbf{S}_b$ is the set of all $\{\downarrow, \rightarrow\}$-paths from $(n-i, m+i)$ to (n, m) for $i = 1, \ldots, n$ that stay weakly below a second diagonal.

- $\nabla = \sum_{i \geq 1} \frac{1}{1+i} \binom{2i}{i} (WE)^i = \frac{1-\sqrt{1-4WE}}{2WE} - 1 = \frac{2}{1+\sqrt{(1-4WE)}} - 1$.

- $w_n(x) = \sum_{j=0}^{n-1} \sum_{k=0}^{n-j} \binom{-k}{n-j-k} \binom{2n-1+k}{n} \frac{x}{k+n} \binom{2n-j-1+x}{n-j-1}$.

- $\mu(L)^{-1} = E^{a\ell-b} \sum_{j \geq 0} \binom{2j-1}{j} E^{aj+j} (E^{-a}W)^j$.

6 Examples

We illustrate a few more examples using the results from above. For each example, we describe the problem, give the explicit solution for $d_n(m)$, and show a table with values for $d_n(m)$ where $d_n(m) = D(n, m)$ above the boundary.

6.1 Poisson Paths staying in First Quadrant

The lattice path can take weighted steps from $\mathfrak{S}_\lambda = \left\{\uparrow, \langle i, 0\rangle_{\frac{\lambda^i}{i!}} \mid i \in \mathbb{N}_1\right\}$ and it stays, naturally, in the first quadrant. The polynomial solution (10) for the total path weight as shown in Table 5 is

$$d_n(m) = w_n(m+1) = \frac{\lambda^n}{n!} \sum_{j=1}^{n} (m+1)_j \, S(n, j)$$

where $S(n, m)$ are the Stirling numbers of the second kind and $\lambda = 1$. The numbers are rounded to two decimal places.

Table 5 : Total Weight for $\mathfrak{S}_\lambda$-paths staying in First Quadrant							
$m^{\uparrow}$							
6	1	7	31.5	113.17	352.63	994.06	
5	1	6	24.0	78.00	222.50	579.30	
4	1	5	17.5	50.83	131.46	313.54	
3	1	4	12.0	30.67	71.00	153.53	
2	1	3	7.5	16.50	33.63	65.03	
1	1	2	4.0	7.33	12.83	21.77	
0	1	1	1.5	2.17	3.13	4.51	
-1	**1**	**0**	**0**	**0**	**0**	**0**	
	0	1	2	3	4	5	$\to n$

6.2 Poisson Paths staying Above a Boundary

We find the total weight of paths from the origin to (n,m) where the path step set is $\mathfrak{S}_\lambda = \left\{\uparrow, \langle i,0\rangle_{\frac{\lambda^i}{i!}} \mid i \in \mathbb{N}_1\right\}$ with $\lambda = 2$, and the paths stay above $n-2$. The $E^{-1}W$-basic sequence is $\hat{w}_0(x) = 1$ and for $n > 0$,

$$\hat{w}_n(x) = \frac{x\lambda^n}{n!}\sum_{j=0}^{n-1}(x+n+1)_j\, S(n,j+1)\,.$$

The initial conditions $\hat{d}_k(-2) = d_k(k-2) = 0$ for $k > 1$ and from Table 6, we see that $\hat{d}_0(-2) = 1$ and $\hat{d}_1(-2) = 0$. The solution (11) for the total path weight when $\lambda = 2$ is

$$\begin{aligned}\hat{d}_n(m-n) &= \hat{w}_n(m-n+2)\\ &= (m-n+2)\sum_{j=0}^{n-1}\frac{2^n(j+m+2)_j}{n!}S(n,j+1)\,.\end{aligned}$$

Table 6 : Total Weight $d_n(m)$ for $\mathfrak{S}_\lambda$ -paths staying above $n-2$							
$m^{\uparrow}$							
5	1	12	90	517.33	2418.00	9088.53	
4	1	10	64	312.00	1186.67	3089.60	
3	1	8	42	162.67	420.67	**0**	
2	1	6	24	61.33	**0**	-1228.27	
1	1	4	10	**0**	-179.33	-1387.20	
0	1	2	**0**	-29.33	-205.33	-1044.80	
-1	1	**0**	-6	-34.67	-150.00	-577.07	
-2	**1**	-2	-8	-24.00	-69.33	-200.00	
	0	1	2	3	4	5	$\rightarrow n$

In terms of Poisson processes, $e^{-\lambda}d_n(m)$ is the probability of n arrivals (in at most $n+m$ hours) and m hours without arrivals, where at the end of every hour there was at most one more hour without arrivals than the total number of arrivals.

6.3 Standard paths with Privileged Central pathlets to the Boundary

We find the total number of standard $\{\uparrow, \rightarrow\}$-paths that stay above line $m = 2(n-1)$ but where there is privileged access to the line for $n \geq 1$ from the access step $\mathfrak{P}_c = \mathfrak{S}_c \backslash \{\uparrow\} = \mathbf{C}$ where $\mathbf{C}$ is the set of all central pathlets. For a point on the boundary line, $(n, 2(n-1))$, a privileged step in $\mathbf{C}$ can come from $(n-i, 2(n-1))$ for $i = 1, \ldots, n$. We use the Functional Expansion Theorem (6), $\hat{d}_n(x) = \sum_{k=0}^{n} \left\langle L \mid \hat{d}_k \right\rangle \mu(L)^{-1} \hat{w}_{n-k}(x)$ where $\hat{w}_n(x) = \frac{x}{2n+x}\binom{3n-1+x}{n}$, the $E^{-2}W$-basic sequence for $\{\uparrow, \rightarrow\}$-paths. The functional is zero on the line except at $n = 0$, $\left\langle L \mid \hat{d}_0(x) \right\rangle = 1$, and so

$$D(n,m) = \sum_{i=0}^{n} \frac{16^i}{3^{i+1}} \Bigg(2\hat{w}_{n-i}(m - 2(n-i-1)) - 8\hat{w}_{n-1-i}(m - 2(n-i-2)) + \sum_{k=0}^{n-i} \binom{\frac{1}{2}}{k} (-4)^k \hat{w}_{n-k-i}(m + 2(k+i-n) + 2) \Bigg).$$

Table 7 : $\{\uparrow, \rightarrow\}$-paths with Privileged Central pathlets to $2(n-1)$

$m\uparrow$					
	1	8	40	178	**826**
5	1	7	32	138	648
4	1	6	25	**106**	510
3	1	5	19	81	404
2	1	4	**14**	62	323
1	1	3	10	48	261
0	1	**2**	7	38	213
-1	1	1	5	31	175
-2	**1**	0	4	26	144
	0	1	2	3	4 $\rightarrow n$

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