Lattice Paths with Weighted Left Turns Above a Parallel to the Diagonal

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1. Introduction

A lattice path is a sequence of unit steps in the North $\uparrow$ and East $\rightarrow$ direction, starting at the origin. In a (left) weighted lattice path every left turn $\rightarrow \overset{\uparrow}{\circ}$ gets the multiplicative weight μ.

$$\begin{array}{ccccccccc} & & & & & & & & \uparrow \\ & \circ & \rightarrow & \circ & \rightarrow & \circ & \rightarrow & \mu & \\ & \uparrow & & & & & & & \\ \rightarrow & \mu & & & & & & & \end{array}$$

Lattice Path with Weight μ^2

The total weight $s_n(m)$ of all paths reaching the point (n, m) equals

$$s_n(m) = \sum_{d \geq 0} \binom{m}{d} \binom{n}{d} \mu^d. \tag{1}$$

In Section 2.1 we show why this kind of weighted path is a very "natural" generalization of the ordinary lattice path (where $\mu = 1$).

A ballot path stays strictly above a parallel to the diagonal, $y = x - K$, say, where K is a nonnegative integer. In an ordinary ($\mu = 1$) ballot path we can switch the two types of steps and obtain a path below $y = x + K$. Hence counting ordinary ballot paths above is equivalent to counting paths below corresponding boundaries. However, this is no longer true for weighted paths, because switching the steps moves the weight from left to right turns.

	0	$2\mu + 3\mu^2$	$4\mu + 9\mu^2 + \mu^3$	$6\mu + 18\mu^2 + 4\mu^3$	...	$\leftarrow t_3(u)$
0	2μ	$4\mu + \mu^2$	$6\mu + 3\mu^2$	$8\mu + 6\mu^2$	...	$\leftarrow t_K(u)$
1	$1 + \mu$	$1 + 2\mu$	$1 + 3\mu$	$1 + 4\mu$	...	$\leftarrow t_1(u)$
1	1	1	1	1	...	$\leftarrow t_0(u)$
$u = 0$	$u = 1$	$u = 2$	$u = 3$	$u = 4$	...	degree $v \uparrow$
Total weights of paths with weighted left turns below $y = x + 2$						

The total weight $t_v(u)$ of the ballot paths to (u, v) below $y = x + K$ is easily obtained as

$$t_v(u) = \sum_{l \geq 0} \left(\binom{u}{l} \binom{v}{l} - \binom{u+K}{l} \binom{v-K}{l} \right) \mu^l \tag{2}$$

for all $0 \leq v \leq u + K$. It follows that for $K \leq v \leq u + K$ there are $\binom{u+K}{l}\binom{v-K}{l}$ paths with l left turns, ending at (u, v) and reaching the line $y = x + K$ somewhere. In Corollary 2 we prove (2) via symmetric Sheffer sequences; Theorem 4 shows a different proof for this result, using a variation of 2-rowed plane partitions. Suppose $m > n - K \geq 0$. With the same approach it is shown in Theorem 5 that $\binom{n-K+2}{l+1}\binom{m+K-2}{l-1}$ paths with l left turns end at (n, m) above the line $y = x - K$ after reaching it somewhere.

1	$1+3\mu$	$1+5\mu+3\mu^2$	$\mathbf{1+6}\boldsymbol{\mu}\mathbf{+6}\boldsymbol{\mu}^2\mathbf{+}\boldsymbol{\mu}^3$	$\mathbf{1+6}\boldsymbol{\mu}\mathbf{+6}\boldsymbol{\mu}^2\mathbf{+}\boldsymbol{\mu}^3$	$m=3$
1	$1+2\mu$	$\mathbf{1+3}\boldsymbol{\mu}\mathbf{+}\boldsymbol{\mu}^2$	$\mathbf{1+3}\boldsymbol{\mu}\mathbf{+}\boldsymbol{\mu}^2$		$m=2$
1	$\mathbf{1+}\boldsymbol{\mu}$	$\mathbf{1+}\boldsymbol{\mu}$			$m=1$
1	1				$m=0$
$r_0\uparrow$	$r_1(m)\uparrow$	$r_K(m)\uparrow$	$r_3(m)\uparrow$	$r_4(m)\uparrow$ ...	$\leftarrow n$
Total weights of paths with weighted left turns above $y=x-2$					

In order to find the weight $r_n(m)$ of the ballot paths above the line $y=x-K$ in terms of Sheffer sequences it is no longer admissible to use the initial value $r_n(n-K)=0$ for all $n \geq K$. Instead, we must work with the initial condition $r_n(n-K+1)=r_{n-1}(n-K+1)$ for all $n \geq K$ (bold entries in the table above). We say that the initial values are recursively defined ; the polynomial $r_n(x)$ gets an initial value from its predecessor $r_{n-1}(x)$. In Theorem 3 we show that

$$r_n(m)=\sum_{l\geq 0}\left(\binom{n}{l}\binom{m}{l}-\binom{n-K+2}{l+1}\binom{m+K-2}{l-1}\right)\mu^l$$

as a special case ($k=1$) of a result for skew-symmetric Sheffer sequences. More general results on Sheffer sequences with recursively defined initial values will be shown in a forthcoming paper.

2. Sheffer Sequences

For the following we need a few definitions from the Umbral Calculus [6] . A Sheffer sequence $\{p_n(x)\}_{n\geq 0}$ is a sequence of polynomials such that $\deg p_n = n$ and

$$\sum_{n\geq 0} p_n(x)t^n = p(t)e^{x\beta(t)},$$

where $p(t)=\sum_{n\geq 0} p_n(0)t^n$ and $\beta(t)$ are formal power series of order 0 and 1, respectively. Let $B(t)$ be the formal inverse of $\beta(t)$, and D be the derivative operator on the algebra of polynomials. The linear operator $B(D)$ is called the delta operator for $\{p_n\}$; for all $n \geq 0$

$$Bp_n = p_{n-1}.$$

$\left\{\binom{n+x}{n}\right\}_{n\geq 0}$ is an example for a Sheffer sequence with associated delta operator $\nabla = 1-E^{-1}$, the backwards difference operator, where E^a denotes the translation operator $E^a f(x) = f(x+a)$. The basic sequence associated with the delta operator B is the Sheffer sequence $\{b_n\}$ with initial values $b_n(0)=\delta_{0,n}$.

2.1 Symmetry

A Sheffer sequence $\{p_n(x)\}$ is symmetric if

$$p_n(m) = p_m(n)$$

for all nonnegative integers m and n. Obviously, $\binom{n+m}{n}$ is symmetric, which is the reason for so many elegant results in ordinary lattice path counting.

The difference of two different Sheffer sequences for the same delta operator is again a Sheffer sequence (for the same operator). If $\{p_n\}$ is symmetric then

$$t_n(x) := p_n(x) - p_{n-K}(x+K) \tag{3}$$

is a Sheffer sequence which has initial values $t_n(n-K)=0$ for all $n \geq K$. The well-known solution for the ordinary ballot problem can be obtained this way. But are there any other symmetric Sheffer sequences besides $\left\{\binom{n+x}{n}\right\}$? In [5] we have shown that except for a scaling factor there is only one parameter that describes the whole class of symmetric Sheffer sequences:

Theorem 1 All symmetric Sheffer sequences are of the form $\{\alpha s_n(x)\}_{n\geq 0}$ where α is a nonzero scaling factor, and

$$s_n(x) = \sum_{l=0}^{n} \binom{n}{l}\binom{x}{l}\mu^l$$

($\mu \neq 0$). The corresponding delta operator Ω has the expansion

$$\Omega = \frac{\Delta}{\mu + \Delta} \tag{4}$$

in terms of the forward difference operator $\Delta = E^1 - 1$.

In other words, the Sheffer sequence $\{s_n\}$ has the same symmetry as $\left\{\binom{n+x}{n}\right\}$, which is the reason for many elegant results on the enumeration of lattice paths with weighted left turns.

All Sheffer sequences for Ω satisfy the recurrence relation

$$s_n(x) = s_n(x-1) + s_{n-1}(x) + (\mu - 1)s_{n-1}(x-1) \tag{5}$$

and the identity

$$s_n(x) - s_n(x-1) = \mu \sum_{i=0}^{n-1} s_i(x-1), \tag{6}$$

because (4) has the solution

$$\Delta = \mu\Omega/(1-\Omega). \tag{7}$$

It is easy to verify that lattice paths with weighted left turns are enumerated by the recurrence (5), with initial values $s_n(0) = 1$ for all $n = 0, 1, \ldots$ if no other restrictions are present. Hence the total weight (1) is obtained.

Corollary 2 The Ω-Sheffer sequence

$$t_n(x) := s_n(x) - s_{n-K}(x+K)$$

has the initial values $t_n(0) = 1$ for all $n = 0, \ldots, K-1$, and $t_n(n-K) = 0$ for all $n \leq K$.

Proof. Follows from (3). ■

2.2 Skew-Symmetric Sheffer Polynomials

The generating functions

$$(1-t)^{-k}\left(1 + \frac{\mu t}{1-t}\right)^x, \quad k \in \mathbb{Z}$$

[5] belong to an important class of Ω-Sheffer sequences which will be called skew-symmetric in the following. They can be expanded as

$$s_n^{(k)}(x) = \sum_{l=0}^{n} \binom{x}{l}\binom{n+k-1}{n-l}\mu^l. \tag{8}$$

The case $k = 1$ has been discussed in the previous section; $k = 0$ gives the basic sequence for Ω. It is easy to check that skew-symmetric Sheffer sequences have the property

$$s_n^{(k)}(m-k+1) = \mu^{1-k} s_m^{(2-k)}(n+k-1) \tag{9}$$

for all integers k, and for all non-negative integers n and m.

From the expansion (8) we can derive another important property of skew-symmetric Sheffer sequences,

$$\Delta s_n^{(k)}(x) = \mu s_{n-1}^{(k+1)}(x). \tag{10}$$

3. Recursive Initial Conditions

Suppose $\{t_n(x)\}$ and $\{r_n(x)\}$ are Sheffer sequences for the same delta operator. Assume that $\{t_n(x)\}$ and the basic sequence $\{b_n(x)\}$ are known, and that $r_n(x) = t_n(x)$ for all n below some given positive integer K. For $n \geq K$ we recursively define the initial values

$$r_n(n-K+\alpha) = r_{n-1}(n-K+\alpha), \tag{11}$$

where α is some given shift. This recursive initial value problem can be solved (see [4]), in the sense that $r_n(x)$ can be expanded in terms of $t_i(x)$ and $b_i(x)$. In this paper we restrict our attention to skew-symmetric Sheffer sequences where a special approach leads to an elegant solution.

If we write

$$r_n(x) = t_n(x) + q_{n-K}(x-\alpha)$$

then $\{q_n(x)\}$ is a Sheffer sequence for the same delta operator, satisfying the condition

$$q_n(n) - q_{n-1}(n) = t_{n+K-1}(n+\alpha) - t_{n+K}(n+\alpha) \tag{12}$$

for all $n \geq 0$.

3.1 Skew-Symmetry

Let $\left\{s_n^{(k)}(x)\right\}_{n=0,1,\ldots}$ be a skew-symmetric Sheffer sequence as defined in (8).

Theorem 3 If $r_n^{(k)}(x) := s_n^{(k)}(x)$ for all $n = 0, \ldots, K-1$, and $r_n^{(k)}(n-K+k-2) = r_{n-1}^{(k)}(n-K+k-2)$ for all $n \geq K$, then

$$r_n^{(k)}(x) = \sum_{l \geq 0} \left(\binom{x}{l}\binom{n+k-1}{l+k-1} - \binom{x+K+2k-4}{l+k-2}\binom{n-K-k+3}{l+1} \right) \mu^l.$$

Proof. Let $t_n(x) = s_n^{(k)}(x)$ and $\alpha = 2-k$ in equation (12),

$$\begin{aligned} q_n(n) - q_{n-1}(n) &= s_{n+K-1}^{(k)}(n-k+2) - s_{n+K}^{(k)}(n-k+2) \\ &= \mu^{1-k}\left(s_{n+1}^{(2-k)}(n+K+k-2) - s_{n+1}^{(2-k)}(n+K+k-1)\right), \end{aligned}$$

because of (9).

Hence, the n-th degree Ω-Sheffer polynomials $q_n(x) - q_{n-1}(x)$ and

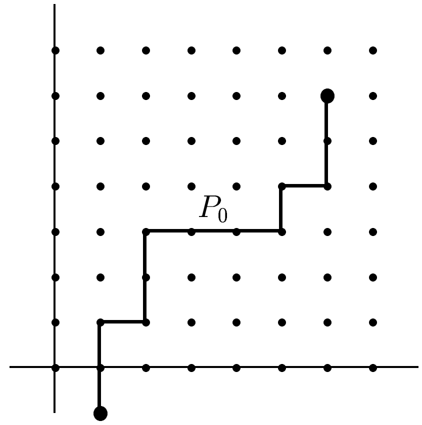
$$\mu^{1-k}\left(s_{n+1}^{(2-k)}(x+K+k-2) - s_{n+1}^{(2-k)}(x+K+k-1)\right)$$

all have one value in common, and therefore must be identical (this is a consequence of the Binomial Theorem for Sheffer sequences [6]). In terms of Δ and Ω,

$$q_n(x) - q_{n-1}(x) = (1-\Omega)\, q_n(x) = \Omega\,(1-\Omega)\, q_{n+1}(x) = -\mu^{1-k}\Delta s_{n+1}^{(2-k)}(x+K+k-2).$$

Applying (7) and (10) shows that

$$\begin{aligned} q_n(x) &= \Omega^2 q_{n+2}(x) = -\frac{\mu^{1-k}\Delta\Omega}{(1-\Omega)} s_{n+2}^{(2-k)}(x+K+k-2) = -\mu^{-k}\Delta^2 s_{n+2}^{(2-k)}(x+K+k-2) \\ &= -\mu^{2-k} s_n^{(4-k)}(x+K+k-2). \end{aligned}$$



Now we can expand $r_n^{(k)}(x)$.

$$\begin{aligned} r_n^{(k)}(x) &= s_n^{(k)}(x) + q_{n-K}(+k-2) = s_n^{(k)}(x) - \mu^{2-k} s_{n-K}^{(4-k)}(x+K+2k-4) \\ &= \sum_{l\geq 0}\left(\binom{x}{l}\binom{n+k-1}{n-l} - \mu^{2-k}\binom{x+K+2k-4}{l}\binom{n-K-k+3}{n-K-l}\right)\mu^l \\ &= \sum_{l\geq 0}\left(\binom{x}{l}\binom{n+k-1}{l+k-1} - \binom{x+K+2k-4}{l+k-2}\binom{n-K-k+3}{l+1}\right)\mu^l. \end{aligned}$$

■

At the end of the next section we provide a combinatorial interpretation of $r_n^{(k)}(x)$ in terms of two-rowed arrays.

4. Combinatorics

In this section we describe a combinatorial approach to the enumeration of weighted lattice paths. Basically, it can be extracted from [3] , or, equivalently, from the restriction to two paths of the Gessel˘Viennot theory of nonintersecting lattice paths [1] , [2] , [7, Sec. 1] .

Our approach to the enumeration of lattice paths with respect to their number of turns is by encoding lattice paths in terms of two-rowed arrays. Let (p_1,q_1), (p_2,q_2), ..., (p_l,q_l) be the left turns of a path P. Then the two-rowed array representation of P is defined by the two-rowed array

$$\begin{array}{cccc} p_1 & p_2 & \ldots & p_l \\ q_1 & q_2 & \ldots & q_l, \end{array} \tag{13}$$

which consists of two strictly increasing sequences. If P runs from (a,b) to (c,d) then $a+1 \leq p_1 < p_2 < \ldots < p_l \leq c$ and $b \leq q_1 < q_2 < \ldots < q_l \leq d-1$ are satisfied. If we wish to make this fact transparent, we write

$$\begin{array}{rccccl} a+1 \leq & p_1 & p_2 & \ldots & p_l & \leq c \\ b \leq & q_1 & q_2 & \ldots & q_l & \leq d-1. \end{array} \tag{14}$$

For a given starting point and a given final point, by definition the empty array is the representation for the only path that has no left turn. For the path in the figure above we obtain the left turn representation

$$\begin{array}{ccc} 2 & 5 & 6 \\ 1 & 3 & 4 \end{array}$$

or with bounds included,

$$\begin{array}{rccccl} 2 \leq & 2 & 5 & 6 & & \leq 6 \\ -1 \leq & 1 & 3 & 4 & & \leq 5 \end{array}.$$

Two-rowed arrays with rows of unequal length will also be considered. The arrays have the property that the rows are strictly increasing. So by convention, whenever we speak of two-rowed arrays we mean two-rowed arrays with strictly increasing rows. For these arrays we will use a notation of the kind (14) as well.

Theorem 4 Let (u, v) be located below the line $y = x + K$. There is an explicit bijection between lattice paths from $(0, 0)$ to (u, v) with l left turns, reaching the line $y = x + K$ somewhere, and two-rowed arrays of the form

$$\begin{array}{rcccl} -K+1 \leq & \bar{p}_1 & \dots & \bar{p}_l & \leq u \\ K \leq & \bar{q}_1 & \dots & \bar{q}_l & \leq v-1. \end{array} \tag{15}$$

Hence, their cardinality equals $\binom{u+K}{l}\binom{v-K}{l}$.

Proof. Consider a path P from $(0, 0)$ to (u, v) with l left turns, reaching the line $y = x + K$ somewhere. In terms of its two-rowed array representation,

$$\begin{array}{rccccl} 1 \leq & p_1 & p_2 & \dots & p_l & \leq u \\ 0 \leq & q_1 & q_2 & \dots & q_l & \leq v-1, \end{array} \tag{16}$$

it means that there is an index I, $0 \leq I \leq l$, such that $q_{I+1} \geq p_I + K$ (by convention, $p_0 := 0$ and $q_{l+1} := v$). Without loss of generality assume that I is the largest such integer. Map the two-rowed array of P to

$$\begin{array}{cccccc} q_1-K+1 & \dots & q_I-K+1 & p_{I+1} & \dots & p_l \\ p_1+K-1 & \dots & p_I+K-1 & q_{I+1} & \dots & q_l. \end{array} \tag{17}$$

Note that both rows are strictly increasing because of $q_I - K + 1 < q_{I+1} - K + 1 < q_{I+2} - K + 1 \leq p_{I+1}$ and $p_I + K - 1 < q_{I+1}$. It is not difficult to see that (17) is of type (15).

The inverse of this map is defined in the same way. We leave it to the reader to check the details. ■

Theorem 5 Let (n, m) be located above the line $y = x - K$. There is an explicit bijection between lattice paths from $(0, 0)$ to (n, m) with l left turns, reaching the line $y = x - K$ somewhere, and two-rowed arrays of the form

$$\begin{array}{rccccccl} K-1 \leq & \bar{p}_0 & \bar{p}_1 & \bar{p}_2 & \dots & \bar{p}_l & \leq n \\ -K+2 \leq & & & \bar{q}_2 & \dots & \bar{q}_l & \leq m-1. \end{array} \tag{18}$$

Hence, their cardinality equals $\binom{n-K+2}{l+1}\binom{m+K-2}{l-1}$.

Proof. Consider a path P from $(0, 0)$ to (n, m) with l left turns, reaching the line $y = x - K$ somewhere. In terms of its two-rowed array representation (16) this means that there is an index I, $1 \leq I \leq l$ such that $q_I \leq p_I - K$. Without loss of generality assume that I is the largest such integer. Map the two-rowed array of P to

$$\begin{array}{cccccccc} q_1+K-1 & q_2+K-1 & q_3+K-1 & \dots & q_I+K-1 & p_I & p_{I+1} & \dots & p_l \\ & & p_1-K+1 & \dots & \dots & p_{I-1}-K+1 & q_{I+1} & \dots & q_l. \end{array} \tag{19}$$

Note that both rows are strictly increasing because of $q_I + K - 1 < p_I$ and $p_{I-1} - K + 1 < p_I - K + 1 < p_{I+1} - K + 1 \leq q_{I+1}$. Again, it is not difficult to see that (19) is of type (18).

The inverse of this map is defined in the same way. We leave it to the reader to check the details. ■

We remark that in the same way it can be seen that for $k \leq 1$ and $K+2k-3 \geq 0$

$$\binom{x}{l}\binom{n+k-1}{l+k-1} - \binom{x+K+2k-4}{l+k-2}\binom{n-K-k+3}{l+1}$$

counts two-rowed arrays of the form

$$\begin{array}{cccccccc} 1 & \leq & & & p_1 & \dots & p_{l+k-1} & \leq n+k-1 \\ 0 & \leq & q_k & \dots & q_1 & \dots & q_{l+k-1} & \leq x-1 \end{array} \qquad (20)$$

with the property that $q_i > p_i - K - 2k + 2$ for all i. Without the restrictions $k \leq 1$ and $K+2k-3 \geq 0$, one has a similar but more involved combinatorial interpretation.

References

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