Planar Random Walks Inside a Rectangle

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Abstract

We give formulas for the number of certain restricted planar walks from the origin to a given end point in a given number of steps. The restrictions range from half planes over bands and strips to rectangles. The walks must have some axial symmetries; the king walks on a chess board are an example.

1 Introduction

On an empty chess board a king can reach the position of his opponent for the first time after seven moves; there are 393 paths available for achieving this goal. If we allow him nine moves, he can get there in 69106 ways (see the example at the end of this paper). On an infinite board, unrestricted by boundaries, he could make the nine moves in 70515 ways.

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	1	1	1	0	0
0	0	0	1	0	1	0	0

After 1 move

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
0	1	3	6	7	6	3	1
0	3	6	12	12	12	6	3
0	5	9	21	20	21	9	5
0	4	6	15	12	15	6	4

After 3 moves

0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0
5	15	30	45	51	45	30	14
20	55	100	145	160	145	100	50
50	140	250	370	404	370	250	126
80	225	380	565	600	565	380	200
90	260	430	655	685	655	430	230
60	175	280	430	440	430	280	154

After 5 moves

70	160	266	357	393	356	259	133
350	770	1232	1617	1764	1610	1190	623
1155	2535	4032	5292	5760	5265	3885	2037
2590	5635	8820	11515	12460	11445	8470	4480
4480	9765	15183	19873	21420	19740	14560	7728
5880	12810	19698	25788	27636	25599	18858	10080
5810	12705	19439	25543	27272	25347	18599	9975
3640	7965	12110	15925	16940	15798	11578	6237

After 7 moves

The above chess boards show in how many ways any of the 64 positions can be reached after $1, 3, 5$, and 7 moves. There is a way to solve the recursion for this and related problems solely by repeated applications of the reflection principle, exploiting certain symmetries in the number of paths. In this paper we organize the approach to such solutions around a few simple observations. We will show how to count any kind of symmetric planar walk in half planes, quarter planes, bands, half-bands and rectangles, whenever such restriction can be expressed by requiring zeroes on some boundary lines *parallel to an axis of symmetry.* For example, the *diagonal diffusion walk* with steps $\nearrow$, $\searrow$, $\swarrow$, and $\nwarrow$ can be blocked by a diagonal boundary of zeroes because the walk can only reach points where both coordinates are of the same parity.

$\searrow$	$\square$	$\swarrow$	$\square$
$\square$	$\bullet$	$\square$	0
$\nearrow$	$\square$	0	$\square$
$\square$	0	$\square$	

0	0	0	0	0	0	0
0	1	0	2	0	1	0
0	0	0	0	0	0	0
0	2	0	4	0	2	0
0	0	0	0	0	0	0
0	1	0	2	0	0	
0	0	0	0	0		

After 2 moves

1	0	3	0	3	0	1
0	0	0	0	0	0	0
3	0	9	0	9	0	3
0	0	0	0	0	0	0
3	0	9	0	8	0	0
0	0	0	0	0	0	
1	0	3	0	0		

After 3 moves

0 *contribution from the boundary blocks diagonal diffusion*

However, the king walks do not have such parity restriction; a diagonal line of zeroes is not a barrier for them, because some diagonal steps cross right through it, as shown below.

		$\searrow$	0
	$\searrow$	0	$\nwarrow$
$\searrow$	0	$\nwarrow$	
0	$\nwarrow$		

Such problems are very involved; we will discuss some of those in a forthcoming paper. On the other hand there are planar walks with symmetries different from the theme of this paper but easily enumerated by the same ideas. The prince walk (4) is an example.

Diffusion paths (Section 3) and king walks (Section 4) are our main examples. The following table shows which special planar walks are mentioned in this paper.

Name	step vectors	section
Diffusion	$\pm(0,1), \pm(1,0)$	1, 3
Diagonal diffusion	$\pm(1,1), \pm(1,-1)$	1, 3.2
King	$\pm(0,1), \pm(1,0), \pm(1,1), \pm(1,-1)$	1, 4
-	$(0,0), \pm(0,1), \pm(1,0), \pm(1,1), \pm(1,-1)$	3, 4
Page	$\pm(2,1), \pm(1,-2)$	2, 3.5
Prince	$\pm(0,1), \pm(1,0), \pm(1,1)$	2.2

Symmetries among the step vectors are crucial to the enumerative approach we are taking. We start with a detailed study of reflectable points and their four axes of symmetry in Section 2. As expected we find that it is enough to study horizontal, vertical, and diagonal axes of symmetry. We give an example for a similarity transformation into the standard axes in Subsection 3.5.

1.1 Notes and Notation

Planar walks are lattice paths in $\mathbb{Z}^2$ which we view as a subset of the x-y-plane. Because the paths are allowed to intersect themselves, the typical question we ask is for the number of paths $B_k[m,n]$ from the origin to the point (n,m) in k moves. Let $B_k := (B_k[m,n])_{m\in\mathbb{Z},n\in\mathbb{Z}}$ be an infinite array. We call B_k a *board*, and $\{B_k\}_{k\geq 0}$ a sequence of boards, if each $B_k[m,n]$

can be calculated from $B_0, B_1, \ldots, B_{k-1}$ by a **linear** recurrence, and some initial values for each B_j. We will only consider linear recurrences, and will drop the qualifier 'linear' in the sequel. In unrestricted counting the only initial conditions are $B_0[m,n] = 1$ if $m = n = 0$, and zero else. A sequence of boards with such initial conditions is called a *basic* sequence. Our goal is to expand certain restricted boards in terms of basic sequences of boards. The restrictions will be in the form of boundary values along lines. Let u, d, r and l be positive integers. We use the symbol $l\underset{d}{\overset{u}{\square}}r$ to denote the rectangle $-l < x < r$, $-d < y < u$, and ${}_{d}^{l}\diamond_{r}^{u}$ for the rectangle $y - l < x < y + r, -d - x < y < u - x$. Missing indices indicate missing boundaries. For example $D_k[m,n;\underset{d}{\square}r]$ is the number of diffusion paths from $(0,0)$ to (n,m) making k moves and staying strictly inside the shifted quadrant $x < r$, $-d < y$.

There is a notational inconsistency between boards, which are matrices, and $\mathbb{Z}^2$ which we view as a subset of the Euclidean plane. A point $(n,m) \in \mathbb{Z}^2$ falls on 'column' n and 'row' m of $\mathbb{Z}^2$, and the number of walks to (n,m) are therefore reported in the entry $B_k[m,n]$ of the associated board. This switch of indices is confusing, but cannot be avoided when we want to describe the geometry of the enumerative problems in terms of lines, and must add up the counting results in terms of matrices.

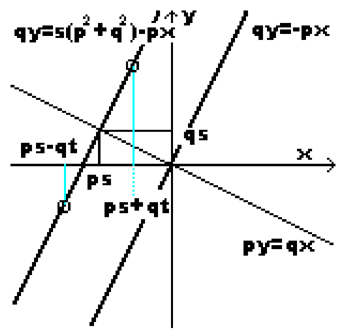
If $\{B_k\}$ is a sequence of board, then the shifted arrays $\{B_k[m-a, n-b]\}$ are boards again, and they follow the same recurrence. Sums and differences of shifted boards are also boards. The diffusion and the King walks follow recursions that are symmetric in the variables m and n. Recursions can have other symmetries; we define those in Section 2. The boards belonging to the corresponding unrestricted planar walks inherit the symmetry from the recursion. It is primarily the symmetry of the boards which allows us to use the simple reflection type tools.

An excellent survey of planar random walks is E. Csáki's recent paper [1]. Diffusion walks inside a band have been counted by S. G. Mohanty [3], and inside a quarter plane by Guy, Krattenthaler, and Sagan [2].

2 Symmetric Boards

In some path counting problems we may encounter boards with symmetries about lines different from the standard coordinate axes or the diagonals. We will show that such boards can be mapped into the standard form with symmetries about the coordinate axes by a similarity transformation. For example, the *page* walk, with steps $\pm(2,1), \pm(1,-2)$ (half of the knight steps), generates a basic sequence of boards which is symmetric about $y = x/2$. The matrix $\frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$ rotates and scales the page walk into the diffusion walk.

For the remainder of this paper let p and q be relative prime integers. We also assume that they are not both negative.



Reflection at $py = qx$

The above diagram shows what we mean by symmetry about $py = qx$: For every s, the lattice points along the perpendicular line $qy = (p^2 + q^2)s - px$ have a mirror image on the opposite side of $py = qx$. If $(ps + qt, qs - pt) \in Z^2$ is such a point, then $(ps - qt, qs + pt)$ is its reflection. Note that s and t need not be integers. For example, if $p = q = 1$, then $(\frac{1}{2}p + \frac{1}{2}q, \frac{1}{2}q - \frac{1}{2}p) = (1, 0)$ has the mirror image $(\frac{1}{2}p - \frac{1}{2}q, \frac{1}{2}q + \frac{1}{2}p) = (0, 1)$.

Definition 1 *A lattice point $(ps + qt, qs - pt) \in Z^2$ is called q/p-*reflectable*. The set of q/p-reflectable points is called the q/p-*grid*. A board U is symmetric about the line $py = qx$, or q/p-*symmetric*, iff*

$$U[qs - pt, ps + qt] = U[qs + pt, ps - qt] \tag{1}$$

for all q/p-reflectable lattice points. A sequence of boards is q/p-symmetric iff every board in the sequence is q/p-symmetric.

The cases $p = 0, q = 1$ and $p = 1, q = 0$ correspond to symmetry about the y-axis and x-axis, respectively. The 0/1-, 1/0-, 1/1-, and $-1/1$-grids are all equal to each other (and to $\mathbb{Z}^2$). This situation is typical, as we will show in the following theorem. A point which is reflectable at a certain line is also reflectable at the line perpendicular to it through the origin, and at the two bisectors.

Theorem 2 *Let p and q be integers such that* $\gcd(p, q) = 1$.

1. *Suppose p and q are of different parity. Any q/p-reflectable point is also $-p/q$-reflectable, $(p+q)/(p-q)$-reflectable, and $(q-p)/(q+p)$-reflectable. The q/p-, $-p/q$-, $(p+q)/(p-q)$-, and $(q - p)/(q + p)$-grid are the same.*
2. *If p and q are both odd, replace $p + q$ by $(p + q)/2$, and $p - q$ by $(p - q)/2$ in 1.*

We will frequently distinguish between pairs of p and q with regard to parity. We will refer to the two cases as "$p + q$ odd" or "$p + q$ even".

Proof. First consider both cases together. Turning the axis of reflection $py = qx$ by 45° counter clockwise gives the line $(p - q)y = (p + q)x$. Further rotations by the same angel give $qy = -px$, $(q + p)y = (q - p)x$, and $py = qx$ again. We only have to show that a point remains reflectable when the axis is turned by 45°. Suppose the point $(u, v) \in \mathbb{Z}^2$ can be q/p-reflected. Then (u, v) is of the form $(ps + qt, qs - pt)$, where

$$s = (qv + up)/\,(p^2 + q^2)$$
$$t = (qu - vp)\,/\,(p^2 + q^2)\,.$$

The q/p-reflection of (u, v) is

$$\begin{aligned}(u', v') &= (ps - qt, qs + pt) \\ &= \left(\frac{2pqv + up^2 - uq^2}{p^2 + q^2}, \frac{q^2 v + 2upq - vp^2}{p^2 + q^2}\right)\end{aligned}$$

which must have integer coordinates. Turning the axis counter clockwise by 45° corresponds to a coordinate transformation where q is replaced by $p+q$, and p by $p-q$ in (u', v') if $p+q$ is odd (and by $(p+q)/2$ and $(p-q)/2$, respectively, if $p+q$ is even). In both cases we obtain the $(p+q)/(p-q)$-reflection of (u, v) as

$$\begin{aligned}&\left(\frac{2(p^2 - q^2)v - 4upq}{2p^2 + 2q^2}, \frac{4vpq + 2u(p^2 - q^2)}{2p^2 + 2q^2}\right) \\ = \; &(-v', u')\end{aligned}$$

which again has integer coordinates. Hence (u, v) is also $(p+q)/(p-q)$-reflectable. ■

2.1 The Rotation

In every grid we see two pairs of perpendicular axes of symmetry, bisecting each other. Only one of them spans the whole grid, as we will show next.

Lemma 3 *Let p and q be integers such that* $\gcd(p, q) = 1$.

1. *If $p+q$ is odd, then the q/p-grid is a vector space over $\mathbb{Z}$ spanned by the basis $\{(p, q), (-q, p)\}$. The vectors $\{(p-q, q+p), (p+q, q-p)\}$ are a basis for a subspace of the q/p-grid.*

2. *If $p+q$ is even, then the q/p-grid is a vector space over $\mathbb{Z}$ spanned by the basis $\{(p+q, q-p)/2, (p-q, q+p)/2\}$. The vectors*

   $\{(p, q), (-q, p)\}$ are a basis for a subspace of the q/p-grid.

Proof. Suppose $p+q$ is odd. All we have to show is that the coefficients s and t in the representation $(qs - pt, ps + qt)$ of a q/p-reflectable point are integers. Then $\{(p, q), (-q, p)\}$ must a basis, and from $(p-q, q+p) = (p, q) + (-q, p)$, $(p+q, q-p) = (p, q) - (-q, p)$ follows that $\{(p-q, q+p), (p+q, q-p)\}$ spans the proper subspace of q/p-grid points with equal parity.

The point $(u, v) = (ps + qt, qs - pt) \in Z^2$ is q/p-reflected into $(ps - qt, qs + pt) = (2ps - u, v + 2pt) \in Z^2$, which shows that $2ps$ and $2qt$ are integers. Suppose q is even and p is odd. If $2s$ is odd then $2ps$ is odd, and from $2u = 2ps + 2qt$ follows $2qt$ is odd, contradicting that q is even. Hence $2s$ is even. If $2t$ is odd then $2v = 2qs - 2pt$ is odd, another contradiction. Hence s and t are both integers. A similar argument shows that s and t are integers when p is even and q is odd.

Now suppose that p and q are both odd. Let $p' = (p+q)/2$ and $q' = (q-p)/2$. We saw in part 2 of Theorem 2 that the q/p-grid and the q'/p'-grid are equal. We can now apply the first part of this lemma because p' and q' are of different parity. The q'/p'-grid contains the span of the vectors $\{(p' - q', q' + p'), (p' + q', q' - p')\} = \{(p, q), (q, -p)\}$. ■

The above lemma makes it easy to represent lines parallel to $py = qx$ in the q/p-grid.

Corollary 4 *If $q+p$ is odd then any line parallel to $py = qx$ in the q/p-grid is of the form $py = qx + t(p^2+q^2)$ where t is an integer. In parametric form,*

$$\{a(p,q) + t(-q,p) \mid a \in \mathbb{Z}\} \tag{2}$$

is the set of all q/p-grid points on that line.

If $q+p$ is even, then any line parallel to $py = qx$ in the q/p-grid is of the form $py = qx + t(p^2+q^2)$ where $2t$ is an integer. In parametric form,

$$\begin{aligned} &\{a(p,q) + t(-q,p) \mid a \in \mathbb{Z}\} \ \text{ if } t \text{ is an integer,} \\ &\{a(p,q) + t(-q,p) \mid 2a \in \mathbb{Z}\} \ \text{ if } 2t \text{ is odd,} \end{aligned} \tag{3}$$

is the set of all q/p-grid points on that line.

Proof. For odd $q+p$ the representation (2) follows directly from Lemma 3. If $q+p$ is even then the grid points are of the form $\frac{w}{2}(p+q, q-p) + \frac{z}{2}(p-q, q+p)$ for some integers w and z. The line parallel to $py = qx$ through this point is

$$\begin{aligned} &\left\{\frac{a+w}{2}(p+q, q-p) + \frac{a+z}{2}(p-q, q+p) \mid a \in \mathbb{Z}\right\} \\ = \ &\left\{\frac{a}{2}(p,q) + \frac{z-w}{2}(-q,p) \mid a \in \mathbb{Z};\ a+z-w \text{ is even}\right\}. \end{aligned}$$

■

Since we know how to find a basis for a q/p-grid, we can make a similarity transformation of this grid into $\mathbb{Z}^2$, preserving the angle between boundaries and the axes of symmetry.

Corollary 5 *Suppose $p+q$ is odd. The the matrix $\frac{1}{p^2+q^2}\begin{pmatrix} p & q \\ -q & p \end{pmatrix}$ maps the q/p-grid into $\mathbb{Z}^2$, carrying the basis $\{(p,q), (-q,p)\}$ into $\{(1,0),(0,1)\}$. If $p+q$ is even, then $\frac{1}{p^2+q^2}\begin{pmatrix} q+p & q-p \\ p-q & q+p \end{pmatrix}$ maps the q/p-grid into $\mathbb{Z}^2$, carrying the basis $\{\frac{1}{2}(q+p, q-p), \frac{1}{2}(p-q, q+p)\}$ into $\{(1,0),(0,1)\}$.*

2.2 Boundaries of Zeroes

All the counting results for symmetric boards in this paper result from the following extremely simple lemma. It is formulated for general q/p-symmetries. We have seen that it is enough to consider symmetries about the coordinate axes and the diagonals; however, the general case presents no extra difficulty.

Lemma 6 *Suppose $py = qx - t(q^2+p^2)$ is a line in the q/p-grid (as in Corollary 4), and that the sequence of boards $\{B_k\}$ is symmetric about $py = qx$. For all $k \geq 0$ and all (n,m) in the q/p-grid define*

$$C_k[m,n] := B_k[m,n] - B_k[m+2pt, n-2qt].$$

The boards C_k follow the same recurrence as B_k and vanish along the line $py = qx - t(q^2+p^2)$.

Proof. The point $(pa+qb, qa-pb)$ lies on the line $py = qx - t(q^2+p^2)$ iff $b = t$ and $a \in \mathbb{Z}$ ($2a \in \mathbb{Z}$ if $2t$, q, and p are odd). By definition, $C_k[qa-pt, pa+qt] = B_k[qa-pt, pa+qt] - B_k[qa+pt, pa-qt]$, which is zero because of q/p-symmetry. ∎

The Lemma shows how to subtract a *correction board* from a given board so that it becomes zero along a line parallel to an axis of symmetry. The proof only verifies that we got the right board; it hides that it is obtained from a very hands-on construction. In physical terms, we place a 'negative charge' symmetric to the origin outside the boundary at $(2qt, -2pt)$. Along the boundary the field of this charge cancels the field originating from a positive charge at the origin. In the following corollary we find a board with zeroes along two lines parallel to an axis of symmetry. We place a negative charge at both sides of the band as shown in the tables below for a diffusion walk bounded by $x = -1$ and $x = 2$. However, for longer paths the negative charges (correction boards) will interfere with each other and need to be corrected by positive charges, etc. The larger k, the more corrections are necessary.

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	-1	0	1	0	0	0	-1	0
-1	0	0	0	1	0	-1	0	-1
0	-1	0	1	0	0	0	-1	0

$k = 1$

0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0
0	0	-1	0	1	0	0	0	-1	0	0
0	-2	0	0	0	2	0	-2	0	-2	0
-1	0	-3	0	3	0	0	0	-4	0	-1
0	-2	0	0	0	2	0	-2	0	-2	0

$k = 2$

0	0	0	-1	0	1	0	0	0	-1	0	0	0
0	0	-5	0	0	0	5	0	-5	0	-5	0	0
0	-10	0	-15	0	15	0	0	0	-25	0	-10	0
-10	0	-40	0	0	0	30	0	-40	0	-50	0	-10
0	-45	0	-50	0	45	0	-5	0	-95	0	-50	0
-24	0	-75	0	-1	0	50	0	-76	0	-99	0	-25
0	-45	0	-50	0	45	0	-5	0	-95	0	-50	0

Further corrections are needed ($k = 5$)

Corollary 7 *Suppose $py = qx - t(q^2+p^2)$ and $py = qx - s(q^2+p^2)$ are lines in the q/p-grid (as in Corollary 4), and that the sequence of boards $\{B_k\}$ is symmetric about $py = qx$. For all $k \geq 0$ and all (n, m) in the q/p-grid define*

$$\begin{aligned} C_k[m,n] \; : &= \sum_{j\geq 0} (B_k[m - 2pj(t-s), n + 2qj(t-s)] \\ &\quad - B_k[m + 2pt + 2pj(t-s), n - 2qt - 2qj(t-s)]) \end{aligned}$$

The boards C_k vanish along the line $py = qx - t(q^2+p^2)$, and are equal to B_k along $py = qx - s(q^2+p^2)$, for all $k \geq 0$, i.e.,

$$\begin{aligned} C_k[qa-pt, pa+qt] &= 0, \text{ and} \\ C_k[qa-ps, pa+qs] &= B_k[qa-ps, pa+qs] \end{aligned}$$

for all $a \in \mathbb{Z}$ ($2a \in \mathbb{Z}$ if $2t, q$, and p are odd). The boards

$$\begin{aligned} A_k[m,n] \; : &= \sum_{j\in\mathbb{Z}} (B_k[m - 2pj(t-s), n + 2qj(t-s)] \\ &\quad - B_k[m + 2pt + 2pj(t-s), n - 2qt - 2qj(t-s)] \end{aligned}$$

vanish along both lines, $py = qx - t(q^2+p^2)$ and $py = qx - s(q^2+p^2)$.

Proof. The boards $\{C_k\}$ and $\{A_k\}$ follow the same recurrence as $\{B_k\}$ because they are sums of shifted boards. We verify the construction by checking the boundary values. In the sum

$$\begin{aligned} C_k[qa-pt, pa+qt] \quad = \quad & \sum_{j\geq 0}(B_k[qa-pt-2pj(t-s), pa+qt+2qj(t-s)] \\ & -B_k[qa+pt+2pj(t-s), pa-qt-2qj(t-s)]) \end{aligned}$$

all terms pairwise cancel by symmetry (applying (1) to $t+2j(t-s)$ instead of t). All but the $j=0$-term cancel out in the expansion

$$\begin{aligned} & C_k[qa-ps, pa+qs] \\ = \; & \sum_{j\geq 0}(B_k[qa-ps-2pj(t-s), pa+qs+2qj(t-s)] \\ & -B_k[qa-ps+2pt+2pj(t-s), pa+qs-2qt-2qj(t-s)]) \\ = \; & \sum_{j\geq 0}(B_k[qa-ps-2pj(t-s), pa+qs+2qj(t-s)] \\ & -B_k[qa+ps+2p(j+1)(t-s), pa-qs-2q(j+1)(t-s)]). \end{aligned}$$

Interchanging s and t gives a board C'_k, say, which vanishes along $py = qx - s(q^2+p^2)$, and equals B along $py = qx - t(q^2+p^2)$. The board A_k is the sum $C_k + C'_k - B_k$, hence $A_k[m,n] =$

$$\begin{aligned} & \sum_{j\geq 0}(B_k[m-2pj(t-s), n+2qj(t-s)] \\ & -B_k[m+2pt+2pj(t-s), n-2qt-2qj(t-s)]) \\ & +\sum_{j\geq 1}(B_k[m+2pj(t-s), n-2qj(t-s)] \\ & -B_k[m+2ps-2p(j-1)(t-s), n-2qs+2q(j-1)(t-s)]). \end{aligned}$$

■

The idea of placing negative charges opposite the boundary applies to even more general symmetries than those required in the above theorems. The following example is not covered by framework of this paper. The *prince* walk $\{P_k\}$ takes the steps $\rightarrow, \uparrow, \leftarrow, \downarrow, \nearrow$, and $\swarrow$. It is 1/1-symmetric, but not 0/1-symmetric.

				1	4	6	4	1
			4	12	16	16	12	4
		6	16	34	48	34	16	6
	4	16	48	60	60	48	16	4
1	12	34	60	90	60	34	12	1
4	16	48	60	60	48	16	4	
6	16	34	48	34	16	6		
4	12	16	16	12	4			
1	4	6	4	1				

The prince board after four steps

Therefore only prince problems involving diagonal boundaries can be enumerated with the help of the above approach. However, $\{P_k\}$ has the "symmetry" $P_k[m,n] = P_k[m-n,-n]$ which is not a line symmetry. If we want to restrict the prince walk by the vertical line $x = r$ we place an "anti-prince" at $(2r, r)$ so that

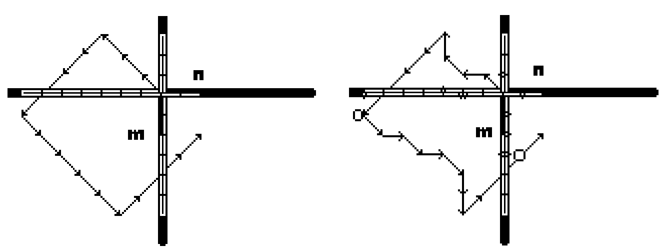
$$P_k[m, n; \square r] = P_k[m, n] - P_k[m - r, n - 2r]. \tag{4}$$

The numbers $P_k[m, n; l\,\square r]$ can also be found (analogously to Corollary 7).

3 Diffusion Walks

An (ordinary) *diffusion walk* is a sequence made from the four steps $\uparrow, \leftarrow, \uparrow$ and $\rightarrow$. Rotating by 45° and scaling the diffusion walk gives the diagonal diffusion walk. There are $\binom{k}{(k+m)/2}\binom{k}{(k+n)/2}$ unrestricted diagonal diffusion walks from $(0,0)$ to (n,m) in k steps; the simplicity of this expression explains the large number of different proofs (usually showing that $D_k[m,n] := \binom{k}{(k+m+n)/2}\binom{k}{(k+m-n)/2}$ ordinary diffusion walks connect the same two points). In view of the enumeration of king moves I want to add my own visualization of the formula. Imagine two independent *linear* walks, the first moving one unit step up or down the y-axis, and the other moving one unit step left or right on the x-axis, k steps each. Vector addition of the two end points of the perpendicular linear walk gives a diagonal diffusion walk, as illustrated in the picture below at the left.

The linear walks are shown in white on the axes



Four steps *Nine steps*

If the vertical path ends at height m and the horizontal path at n, then the diagonal diffusion walk ends at (n, m). This mapping between diagonal diffusion walks and pairs of linear walks is one-to-one; the counting exercise is done after noticing that there are $\binom{k}{(k+m)/2}$ linear walks ending at height m. The idea of representing diffusion walks by independent linear walks goes back to F. Spitzer [4].

Suppose the linear walks can make three kinds of steps, the third being the null step, staying in place. Adding up two independent linear walks with null steps gives a 9-step diffusion walk which makes the 8 different steps of a king and a null step, as in the above figure at the right.

We can now count the King walks by forbidding the planar null steps (see formulas (13) and (14)).

The notation $P_k((a,b)\to(n,m))$ is used in [2] for the set of diffusion paths from (a,b) to (n,m) in k moves. In this notation the boards $\{D_k\}$ have entries

$$D_k[m,n] = |P_k((0,0)\to(n,m))|,$$

and

$$|P_k((a,b)\to(n,m))| = D_k[m-b, n-a].$$

For paths starting at (a,b) we have

$$\begin{aligned} \left| P_k\left((a,b)\to(n,m); l\underset{d}{\overset{u}{\square}}r\right)\right| &= D_k[m-b, n-a; l+a\underset{d+b}{\overset{u-b}{\square}}r-a] \\ \left|P_k\left((a,b)\to(n,m); {}^{l}_{d}\diamond^{u}_{r}\right)\right| &= D_k[m-b, n-a; {}^{l+a-b}_{d+a+b}\diamond^{u-a-b}_{r-a+b}]. \end{aligned}$$

3.1 Weighted Diffusion

A weighted diffusion walk takes the four unit steps $\to$, $\uparrow$, $\leftarrow$, and $\downarrow$, with weights (probabilities) ε, ν, ω, and σ, respectively. The number $\hat{D}_k[m,n]$ of weighted diffusion walks from the origin to (n,m) in k moves can be recursively calculated from $\hat{D}_k[m,n] =$

$$\nu\hat{D}_{k-1}[m-1,n] + \sigma\hat{D}_{k-1}[m+1,n] + \varepsilon\hat{D}_{k-1}[m,n-1] + \omega\hat{D}_{k-1}[m,n+1]$$

for $k = 1, 2, \ldots$ with initial conditions $\hat{D}_0[0,0] = 1$ and $\hat{D}_0[m,n] = 0$ if $(n,m) \neq (0,0)$. The boards $\hat{D}_k$ are zero outside a diamond shaped area, $\hat{D}_k[m,n] = 0$ for $|m \pm n| > k$. They are symmetric about the

- y-axis if $\varepsilon = \omega$ (interchange the East with the West steps),
- x-axis if $\nu = \sigma$,
- diagonal $y = x$ if $\nu = \varepsilon$ and $\sigma = \omega$ (interchange the North with the East and the South with the West steps),
- diagonal $y = -x$ if $\nu = \omega$ and $\sigma = \varepsilon$.

It is easy to show by elementary counting arguments that

$$\hat{D}_k[m,n] = \sum_{i-j=m, a-b=n} \varepsilon^i\omega^j\nu^a\sigma^b \binom{k}{i,j,a,b}$$

which can be written as the single sum $\hat{D}_k[m,n] =$
$\varepsilon^m\nu^n\binom{k}{n+\frac{k-m-n}{2}}\sum_{j\geq 0}(\varepsilon\omega)^j(\nu\sigma)^{(k-m-n)/2-j}\binom{n+\frac{k-m-n}{2}}{j}\binom{m+(k-m-n)/2}{\frac{k-m-n}{2}-j}$

(we interpret the binomial coefficient $\binom{u}{v}$ as zero if u is negative, or if v is not an integer between 0 and u, inc.). Obviously $\hat{D}_k[m,n] = 0$ if $k+m+n$ is odd. If $\varepsilon\omega = \nu\sigma \neq 0$ then the formula simplifies by Vandermonde convolution,

$$\hat{D}_k[m,n \mid \varepsilon\omega = \nu\sigma] = \varepsilon^m\nu^n(\nu\sigma)^{(k-m-n)/2}\binom{k}{\frac{k+n-m}{2}}\binom{k}{\frac{k-m-n}{2}}.$$

If one of the weights equals 0 the planar walk must stay in a half plane.

For simplicity, we will only look at examples of restricted diffusion walks where all weights are 1. However, in all counting problems we state explicitly what kind of symmetries are required, so the results can be easily expanded to appropriately weighted paths.

3.2 One Boundary

We now apply the results of section 2 to ordinary diffusion walks. By symmetry,

$$\begin{aligned} D_k[m,n;\overset{u}{\square}] &= D_k[n,m;\square u] = D_k[n,-m;u\square] = D_k[-m,n;\underset{u}{\square}] \\ D_k[m,n;\diamond_r] &= D_k[-m,-n;{}^r\diamond] = D_k[-n,m;{}_r\diamond] = D_k[n,-m;\diamond^r]. \end{aligned} \tag{5}$$

For the diffusion to stay left of $x = r$ it is necessary and sufficient to respect the initial condition $D_0[m,n] = \delta_{(n,m),(0,0)}$ for all integers m and n, and the boundary condition $D_k[m,r;\square r] = 0$ for all m and k. Using the 1/0-symmetry about the y-axis we get from Lemma 6 (with $p = 0, q = 1, a = m, b = n, t = r$)

$$D_k[m,n;\square r] = D_k[m,n] - D_k[m,n-2r]$$

for all positive integers r. If we cannot apply (5) because of weights, we can find $\hat{D}_k[m,n;\overset{u}{\square}]$ in the case of 0/1-symmetric walks from Lemma 6 with $p = 1, q = 0, a = m, b = n, t = -u$.

Let r be a positive integer. The diffusion above the diagonal $y = x - r$ is obtained from the boundary condition $D_k[m, m+r;\diamond_r] = 0$. The boards D_k are 1/1-symmetric about $y = x$; we can apply Lemma 6 with $p = q = 1$, $m = a - b$, $n = a + b$ and $t = r/2$,

$$D_k[m,n;\diamond_r] = D_k[m,n] - D_k[m+r,n-r]. \tag{6}$$

Diagonal diffusion walks, with steps ↗, ↘, ↙ and ↖, are enumerated in the same way as ordinary diffusion walks as long as the boundary is vertical or horizontal. In the case of diagonal boundaries like $y = x - r$ we have to consider that diagonal diffusion walks reach only lattice points where both coordinates are of the same parity. Therefore we can not require a line of initial values along $(n - r, n)$ if r is odd; we must replace r by $r + 1$ in this case.

Diagonal diffusion above $y = x - 4$

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	3	0	3	0	1	0	0
0	0	0	0	0	0	0	0	0	0	1	0	2	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	9	0	9	0	3	0	0
0	0	0	0	0	0	0	0	0	0	2	0	4	0	2	0	0	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	3	0	9	0	8	0	0	0	-3
0	0	0	0	0	0	0	0	0	0	1	0	2	0	0	0	-2	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-1	0	-1	0	0	0	0	0	0	0	0	0	1	0	3	0	0	0	-8	0	-9
0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	0	-4	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	-3	0	-9	0	-9

$k = 1$ $k = 2$ $k = 3$

3.3 Diffusion Inside a Band

By symmetry,

$$D_k[m,n;\overset{u}{\underset{d}{\square}}] = D_k[n,m;d\square u] \tag{7}$$

$$D_k[m,n;{}^l\diamond_r] = D_k[n,m;{}^r\diamond_l] = D_k[m,-n;{}_r\diamond^l] = D_k[-m,n;{}_l\diamond^r]. \tag{8}$$

With the help of Corollary 7 it is straight forward to find the number of diffusion walks strictly inside the band $l\Box r$, using 1/0-symmetry about the y-axis (let $q = 1, p = 0, a = m, b = n, t = r, s = -l$),

$$D_k[m, n; l\Box r] = \sum_{j\in\mathbb{Z}} \left(D_k[m, n + 2jW] - D_k[m, n - 2r - 2jW]\right)$$

where $W := l+r$ is the band width (for another proof see [3]). Using 1/1-symmetry about the diagonal $y = -x$ we can also get $D_k[m, n; {}^{l}\diamond_{r}]$ from Corollary 7 (let $t = r/2$ and $s = -l/2$), $D_k[m, n; {}^{l}\diamond_{r}] =$

$$\begin{aligned} &\sum_{j\in\mathbb{Z}} \left(D_k[m - jW, n + jW] - D_k[m + r + jW, n - r - jW]\right) \\ = &\binom{k}{\frac{k+m+n}{2}} \sum_{j\in\mathbb{Z}} \left(\binom{k}{\frac{k-m+n}{2} + jW} - \binom{k}{\frac{k-m+n}{2} + jW - r} \right), \end{aligned} \tag{9}$$

where again $W = l + r$.

3.4 Diffusion Inside a Rectangle

If the number of moves k is small compared to the size of the rectangle, only a few correction terms are necessary to get $D_k[m, n; l\overset{u}{\underset{d}{\Box}}r]$. For solving this enumeration problem in general, all we need is Corollary 7 a second time, because the board $D_k[m, n; \overset{u}{\underset{d}{\Box}}]$ is still symmetric about the y-axis.

Theorem 8 *Let u, d, r and l be positive integers, $H := u + d$, the height, and $W := l + r$, the width of the rectangle $-l < x < r, -d < y < u$. The diffusion walks inside this rectangle are enumerated by* $D_k[m, n; l\overset{u}{\underset{d}{\Box}}r] =$

$$\begin{aligned} \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}} \Bigg(&\binom{k}{\frac{k+m+n}{2} + iH + jW} \binom{k}{\frac{k-m+n}{2} - iH + jW} \\ -&\binom{k}{\frac{k+m+n}{2} - iH + jW - u} \binom{k}{\frac{k-m+n}{2} + iH + jW + u} \\ -&\binom{k}{\frac{k+m+n}{2} + iH - jW - r} \binom{k}{\frac{k-m+n}{2} - iH - jW - r} \\ +&\binom{k}{\frac{k+m+n}{2} - iH - jW - r - u} \binom{k}{\frac{k-m+n}{2} + iH - jW - r + u} \Bigg) \end{aligned}$$

Proof. The board $D_k[m, n; \overset{u}{\underset{d}{\Box}}]$ is symmetric about the y-axis. Apply Corollary 7 to $D_k[m, n; \overset{u}{\underset{d}{\Box}}]$ with $s = -l$ and $t = r$, and get

$$D_k[m, n; l\overset{u}{\underset{d}{\Box}}r] = \sum_{j\in\mathbb{Z}} \left(D_k[m, n + 2jW; \overset{u}{\underset{d}{\Box}}] - D_k[m, n - 2jW - 2r; \overset{u}{\underset{d}{\Box}}]\right).$$

We noticed in (7) that $D_k[m, n+2jW; \overset{u}{\underset{d}{\Box}}] = D_k[n+2jW, m; d\Box u]$. Hence $D_k[m, n; l\overset{u}{\underset{d}{\Box}}r] =$

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}}(D_k[n+2jW, m; d\Box u] - D_k[n-2jW-2r, m; d\Box u]) \\
=\ &\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}}(D_k[n+2jW, m+2iH] - D_k[n+2jW, m-2iH-2u] \\
&-D_k[n-2jW-2r, m+2iH] + D_k[n-2jW-2r, m-2iH-2u]).
\end{aligned}$$

■

In the case of weighted diffusion walks we have to make sure that $\hat{D}_k[m, n]$ is symmetric about both axes. This requires that $\varepsilon = \omega$ and $\nu = \sigma$, as we have noted in Section 3.1.

We want to mention two special cases of Theorem 8, which can also be derived directly from Corollary 7.

Diffusion in a horizontal strip: If there is no left barrier, $l = \infty$, then $W = \infty$, and $D_k[m, n; \overset{u}{\underset{d}{\Box}}r]$ equals

$$\begin{aligned}
&\sum_{i\in\mathbb{Z}}\left(\binom{k}{\frac{1}{2}(k+m+n)+iH}\binom{k}{\frac{1}{2}(k-m+n)-iH}\right. \\
&-\binom{k}{\frac{1}{2}(k+m+n)-iH-u}\binom{k}{\frac{1}{2}(k-m+n)+iH+u} \\
&-\binom{k}{\frac{1}{2}(k+m+n)+iH-r}\binom{k}{\frac{1}{2}(k-m+n)-iH-r} \\
&\left.+\binom{k}{\frac{1}{2}(k+m+n)-iH-r-u}\binom{k}{\frac{1}{2}(k-m+n)+iH-r+u}\right).
\end{aligned}$$

Diffusion in a vertical strip can be analogously counted.

Diffusion in a quadrant: If we restrict the diffusion to the quadrant $x < r$ and $y < u$, we only get the term where $i = j = 0$ in Theorem 8. Hence $D_k[m, n; \overset{u}{\Box}r] =$

$$\begin{aligned}
&\binom{k}{\frac{k+m+n}{2}}\binom{k}{\frac{k-m+n}{2}} - \binom{k}{\frac{k+m+n}{2}-u}\binom{k}{\frac{k-m+n}{2}+u} \\
&-\binom{k}{\frac{k+m+n}{2}-r}\binom{k}{\frac{k-m+n}{2}-r} + \binom{k}{\frac{k+m+n}{2}-r-u}\binom{k}{\frac{k-m+n}{2}-r+u}.
\end{aligned}$$

Other quadrants give analogous formulas. For a bijective proof see [2]. The general formula for planar walks in quadrants is given in (12) below.

For diagonal boundaries we get similar results.

Theorem 9 *Let u, d, r and l be positive integers, $H := u + d$, the height, and $W := l + r$ the width of the rectangle $y - l < x < y + r$, $-d - x < y < u - x$. The diffusion walks inside this* *rectangle are enumerated by* $D_k[m,n;\ {}^{l}_{d}\diamond^{u}_{r}] =$

$$\sum_{j\in\mathbb{Z}}\left(\binom{k}{\frac{1}{2}(k+m-n)+jW}-\binom{k}{\frac{1}{2}(k+m-n)+jW+r}\right)$$
$$\times\sum_{i\in\mathbb{Z}}\left(\binom{k}{\frac{1}{2}(k+m+n)+iH}-\binom{k}{\frac{1}{2}(k+m+n)+iH+d}\right).$$

Proof. $D_k[m,n;\ {}_{d}\diamond^{u}]$ is 1/1-symmetric about the diagonal $y=x$ (switch $\rightarrow$ with $\uparrow$, and $\leftarrow$ with $\downarrow$). Using this symmetry we get $D_k[m,n;\ {}^{l}_{d}\diamond^{u}_{r}]$ from Corollary 7 (let $t=r/2, s=-l/2$),
$D_k[m,n;^{l}_{d}\diamond^{u}_{r}] =$
$\sum_{j\in\mathbb{Z}}\left(D_k[m-jW,n+jW;_{d}\diamond^{u}]-D_k[m+r+jW,n-r-jW;_{d}\diamond^{u}]\right)$.

Combine (8) and (9) to expand $D_k[m,n;\ {}^{u}\diamond_{d}]$ as $D_k[m,-n;\ {}_{d}\diamond^{u}]$. Then$D_k[m,n;^{l}_{d}\diamond^{u}_{r}] =$

$$\sum_{j\in\mathbb{Z}}\binom{k}{\frac{k+m-n}{2}-jW}\sum_{i\in\mathbb{Z}}\left(\binom{k}{\frac{k-m-n}{2}+iH}-\binom{k}{\frac{k-m-n}{2}+iH-d}\right)$$
$$-\sum_{j\in\mathbb{Z}}\binom{k}{\frac{k+m-n}{2}+jW+r}\sum_{i\in\mathbb{Z}}\left(\binom{k}{\frac{k-m-n}{2}+iH}-\binom{k}{\frac{k-m-n}{2}+iH-d}\right).$$

■

Diffusion in a diagonal strip: If there is no left diagonal barrier, $l=\infty$, then $W=\infty$, and $D_k[m,n;\ {}_{d}\diamond^{u}_{r}]$ equals

$$\left(\binom{k}{\frac{1}{2}(k+m-n)}-\binom{k}{\frac{1}{2}(k+m-n)+r}\right)$$
$$\times\sum_{i\in\mathbb{Z}}\left(\binom{k}{\frac{1}{2}(k+m+n)+iH}-\binom{k}{\frac{1}{2}(k+m+n)+iH+d}\right).$$

Diffusion in other diagonal strips can be counted analogously.

Diffusion in a diagonal quadrant: If we restrict the diffusion to the rotated quadrant $x<y+r, -d-x<y$, we only get the term where $i=j=0$ in Theorem 8. Hence $D_k[m,n;\ {}_{d}\diamond_{r}]$ equals

$$\left(\binom{k}{\frac{k+m-n}{2}}-\binom{k}{\frac{k+m-n}{2}+r}\right)\left(\binom{k}{\frac{k+m+n}{2}}-\binom{k}{\frac{k+m+n}{2}+d}\right)$$

This is also a special case of (12) below.

3.5 Page Walks

We saw already that the *page* walk, with steps $\pm(2,1),\pm(1,-2)$, can be rotated and scaled by the matrix $\frac{1}{5}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}$ into the diffusion walk. In general, the similarity transformation

$$S=\frac{1}{p^2+q^2}\begin{pmatrix}p & q\\ -q & p\end{pmatrix}$$

maps $(pa+qb, qa-pb)$ into $(a,-b)$ (see Corollary 5).

Our goal in this section is twofold: Showing in an example how such a similarity transformation can be applied, and how the expansion formulas can be used without prior similarity transformation. The basic page boards $\{P_k\}$ are 1/2-symmetric; the walks can only reach points (n,m) of the form $(2a+b, a-2b)$. Hence

$$\begin{aligned} P_k[m,n] &= D_k[(2m-n)/5,(2n+m)/5] \\ &= \binom{k}{(k+(3m+n)/5)/2}\binom{k}{(k+(m-3n)/5)/2} \\ P_k[a-2b,2a+b] &= \binom{k}{(k+a-b)/2}\binom{k}{(k+a+b)/2}. \end{aligned} \tag{10}$$

The number of page boards strictly above $y=(x-5r)/2$ is according to Lemma 6 ($p=2, q=1, t=r$)

$$P_k[m,n]-P_k[m+4r,n-2r] \tag{11}$$

(r must be a positive integer). The parametrization$(pa+qt, qa-pt)$, $a\in\mathbb{Z}$, makes it obvious that the boundary line $py=qx-t(q^2+p^2)$ is mapped onto the horizontal line $(a,-t)$, $a\in\mathbb{Z}$ under the similarity transformation. Therefore $D_k[(2m-n)/5,(2n+m)/5;\underset{r}{\square}]$ is the number of page walks to (n,m) in k moves above $y=(x-5r)/2$. By Lemma 6 (with $p=1, q=0, t=r$) the number of bounded walks $D_k[(2m-n)/5,(2n+m)/5;\underset{r}{\square}]$ equals

$$\begin{aligned} &= D_k[(2m-n)/5,(2n+m)/5]-D_k[(2m-n)/5+2r,(2n+m)/5] \\ &= D_k[(2m-n)/5,(2n+m)/5] \\ &\quad -D_k[(2(m+4r)-(n-r))/5,(2(n-r)+(m+4r))/5] \\ &= P_k[m,n]-P_k[m+4r,n-2r] \end{aligned}$$

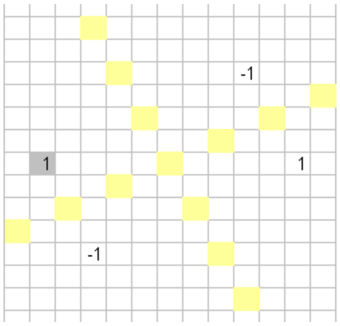
confirming (11).

Page boards are also $-2/1$-symmetric. Again by Lemma 6 (with $q=-2, p=1, t=-u$), the number of page boards strictly left of $y=5u-2x$ is

$$P_k[m,n]-P_k[m-2u,n-4u]$$

(u must be a positive integer). Without enclosing the page walks inside a rectangle first, we can show directly that the number of page moves in the rotated quarter plane strictly above $y=(x-5r)/2$ and left of $y=5u-2x$ equals

$$\begin{aligned} &P_k[m,n]-P_k[m+4r,n-2r]-P_k[m-2u,n-4u] \\ &+P_k[m-2u+4r,n-4u-2r]. \end{aligned}$$



$k = 0$

0	0	0	0	0	0	0	0	-1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	-1	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-1	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	1
0	0	0	0	0	-1	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	-1	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	-1	0	0	0	0	0	0	0	0

$k = 1$

0	0	0	0	0	0	-24	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	-45	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	-40
0	0	0	0	0	-35	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	-70	0	0	0	0
0	35	0	0	0	0	0	0	0	0	0	-49	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-35	0	0	0	0	0
70	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	35	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	35	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	70
0	0	0	0	0	-35	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0
0	-49	0	0	0	0	0	0	0	0	0	35	0
0	0	0	0	-70	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	-35	0	0	0	0	0
-40	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	-45	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	-24	0	0	0	0	0	0

$k = 5$

3 page boards with moves strictly above $y = (x - 5)/2$ and left of $y = 10 - 2x$.

This is a general property: It is easy to verify that the board $W_k[m, n] :=$

$$\begin{aligned} &B_k[m, n] - B_k[m + 2pt, n - 2qt] \\ &-B_k[m - 2qs, n - 2ps] + B_k[m + 2pt - 2qs, n - 2ps - 2qt] \end{aligned} \qquad (12)$$

equals 0 along both, $py = qx - t(p^2 + q^2)$ and $qy = -px + s(p^2 + q^2)$, when B_k is symmetric about $py = qx$ and $qy = -px$.

4 King Walks

We saw in Section 3 that the 9-step diffusion walk can be obtained as the vector sum of two independent perpendicular linear walks with null steps. Sorting by the null steps shows $\sum_{l \geq 0} \binom{k}{l} \binom{k-l}{(k-l+m)/2}$ linear walks with null steps that reach level m in k moves. Therefore,

$$\left(\sum_{l \geq 0} \binom{k}{l} \binom{k - l}{(k - l + m)/2} \right) \left(\sum_{j \geq 0} \binom{k}{j} \binom{k - j}{(k - j + n)/2} \right) \qquad (13)$$

9-step diffusion walks go from $(0,0)$ to (n,m) in k moves. The king moves $K_k[m,n]$ are counted by skipping the planar null steps, which arise when both linear walks make a null step in the same move. Hence we first choose the positions for the l null steps on the vertical axis first, and then j null steps among the remaining $k-l$ moves on the horizontal axis,

$$K_k[m,n] = \sum_{l=0}^{k} \binom{k}{l} \binom{k-l}{(k-l+m)/2} \sum_{j=0}^{k-l} \binom{k-l}{j} \binom{k-j}{(k-j+n)/2}. \tag{14}$$

We mentioned in the introduction that king walks cannot be fenced in by a diagonal line of zeroes. However, vertical or horizontal boundaries work in the same way for king walks as for diffusion walks.

$K_k[m,n;\square 3]$

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
1	1	1	0	0	0	-1	-1	-1
1	0	1	0	0	0	-1	0	-1
1	1	1	0	0	0	-1	-1	-1
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

$k=1$

0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
1	2	3	2	1	0	-1	-2	-3
2	2	4	2	2	0	-2	-2	-4
3	4	8	4	3	0	-3	-4	-8
2	2	4	2	2	0	-2	-2	-4
1	2	3	2	1	0	-1	-2	-3
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

$k=2$

0	0	0	0	0	0	0	0	0
7	6	3	0	-3	-6	-7		
12	12	6	0	-6	-12	-12		
27	27	12	0	-12	-27	-27		
24	27	12	0	-12	-27	-24		
27	27	12	0	-12	-27	-27		
12	12	6	0	-6	-12	-12		
7	6	3	0	-3	-6	-7		
0	0	0	0	0	0	0		

$k=3$

4.1 King Walks Inside a Band or Rectangle

Before we can calculate $K_k[m,n;l\overset{u}{\underset{d}{\square}}r]$, we must derive $K_k[m,n;\overset{u}{\underset{d}{\square}}]$ from Corollary 7 (with $q=0$, $p=1$, $t=-u$, $s=d$), using the 0/1-symmetry of king walks about the x-axis,

$$K_k[m,n;\overset{u}{\underset{d}{\square}}] = \sum_{i\in\mathbb{Z}} \left(K_k[m+2iH,n] - K_k[m-2u-2iH,n]\right)$$

where $H := u+d$ is the height of the rectangle (see (14) for the expansion of K_k). The board $K_k[m,n;\overset{u}{\underset{d}{\square}}]$ is symmetric about the y-axis. Apply Corollary 7 to $K_k[m,n;\overset{u}{\underset{d}{\square}}]$ with $q=1$, $p=0$, $s=-l$ and $t=r$, and get $K_k[m,n;l\overset{u}{\underset{d}{\square}}r] = \sum_{j\in\mathbb{Z}}(K_k[m,n+2jW;\overset{u}{\underset{d}{\square}}] - K_k[m,n-2jW-2r;\overset{u}{\underset{d}{\square}}])$

$$\begin{aligned} = \sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} (&K_k[m+2iH,n+2jW] - K_k[m-2iH-2u,n+2jW] \\ &-K_k[m+2iH,n-2jW-2r] + K_k[m-2iH-2u,n-2jW-2r]) \end{aligned}$$

where u,d,r and l are positive integers, and $W = l+r$. The boards $K_k[m,n;\overset{u}{\underset{d}{\square}}r]$ and $K_k[m,n;\underset{d}{\square}r]$ can be calculated accordingly.

Example 10 *In the Introduction we looked at the king walk on a chess board. The dimensions of this rectangle are given by* $u = 8, d = 1, l = 5$ *and* $r = 4$*. His opponent is in position* $(0, 7)$*. We get* $K_9[7, 0; 5\underset{1}{\overset{8}{\square}}4] = \sum_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}}(K_9[7 + 18i, 18j] - K_9[-18i - 9, 18j] - K_9[7 + 18i, -18j - 8] + K_9[-18i - 9, -18j - 8]) = K_9[7, 0] - 2K_9[-9, 0] - K_9[7, -8] + 2K_9[-9, -8] = 75699 - 2 * 3139 - 333 + 2 * 9 = 69\,106.$

References

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