# The Distribution of the Size of the Intersection of a $k$-Tuple of Intervals

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**Abstract**

Let $(I_1, I_2, ..., I_k)$ be a random $k$-tuple of subintervals of the discrete interval $[1, n]$, and $L_n$ the random variable that measures the size of their insersection. We derive the exact and asymptotic distribution of $L_n$ under the assumption of equally likely drawn $k$-tuples. The enumeration of such $k$-tuples and refinements of the given statistic lead to interesting relations to other topics, like octahedral numbers and bipartite graphs.

## 1 Introduction

How do we decide whether a sequence of subintervals from $[1, n] = \{1, \dots, n\}$ is "random", i.e., independently and equally likely drawn from all $\binom{n+1}{2}^k$ subintervals, if all we get to see is the size of their intersection? This paper was motivated by investigating the distribution of the intersection size of $k$ subintervals of $[1, n]$. For example, if 5 subintervals from $[1, 10]$ intersect in 4 or more points, we can be 99.5% certain that they are not randomly generated. Alternatively, suppose we can draw the subintervals one after the other. If $n$ is larger than 4, and the first 10 intervals still have a nonempty intersection, we can be more than 99% sure that the intervals are not random. We will see in Section 6 that the probability of drawing $k$ intersecting intervals approaches $2^k/\binom{2k}{k} \approx 2^{-k}\sqrt{k\pi}$ for $n \to \infty$.

Let $\Lambda^k_{n;l} := \{(I_1, I_2, ..., I_k) \mid I_j$ is a subinterval of $[1, n]$ for all $j \in [1, k]$, and $\left|\bigcap_{j=1}^k I_j\right| = l\}$. The enumeration begins with the basic observation in Section 2 that $\Lambda^k_{n;l} = \Lambda^k_{n+1-l;1}$ for $l > 0$. Whenever feasible, we will therefore write $\Lambda^k_j$ instead of $\Lambda^k_{j;1}$. In this notation

$$\Pr\left(\text{The size of the intersection of 5 subintervals of } [1,10] \text{ is 4 or larger}\right)$$
$$= \binom{11}{2}^{-5} \sum_{l=4}^{10} \left|\Lambda^5_{11-l,1}\right| = 0.00474$$

using one of the many formulas we will derive for $\left|\Lambda^k_{n,1}\right|$.

A Short Table of $\left|\Lambda_n^k\right|$

| $k\downarrow$ | $n=1$ | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| 2 | 1 | 6 | 19 | 44 | 85 | 146 | 231 |
| 3 | 1 | 14 | 87 | 340 | 1001 | 2442 | 5215 |
| 4 | 1 | 30 | 355 | 2300 | 10 213 | 35 162 | 100 935 |
| 5 | 1 | 62 | 1383 | 14644 | 97145 | 469 146 | 1803 007 |

In Section 6.1 we determine the expectation of $L_n$, well approximated by $2^k\left(n+1\right)^{k+1}k!k!/\left(n^k\left(2k+1\right)!\right)$, and the variance.

In terms of *subsets* instead of subintervals a related problem is discussed in Stanley's *Enumerative Combinatorics I* [5, Example 1.1.16]; the additional structure gained from intervals makes $\Lambda_n^k$ a very interesting object to study, with surprisingly many aspects and refinements.

Experimenting with small values of $k$ was fruitful; we found by ad hoc arguments and algebra that

$$\begin{aligned}
|\Lambda_n^2| &= \tbinom{n}{1}+4\tbinom{n}{2}+4\tbinom{n}{3}\\
|\Lambda_n^3| &= \tbinom{n}{1}+12\tbinom{n}{2}+48\tbinom{n}{3}+72\tbinom{n}{4}+36\tbinom{n}{5}\\
|\Lambda_n^4| &= \tbinom{n}{1}+28\tbinom{n}{2}+268\tbinom{n}{3}+1056\tbinom{n}{4}+1968\tbinom{n}{5}+1728\tbinom{n}{6}+576\tbinom{n}{7}
\end{aligned}$$

A combinatorial interpretation of the coefficients (called $c\left(p,k\right)$) of $\binom{n}{p}$ in the expansion of $\left|\Lambda_n^k\right|$ will be given in Section 4.1. We found the (octahedral) numbers $|\Lambda_n^2|$ especially noteworthy; they are discussed in Section 3.

Expanding $\left|\Lambda_n^k\right|$ in powers of $n$ shows another interesting feature:

$$\begin{aligned}
&|\Lambda_n^2| = \tfrac{1}{3}n+\tfrac{2}{3}n^3, \qquad |\Lambda_n^3| = \tfrac{1}{5}n+\tfrac{1}{2}n^3+\tfrac{3}{10}n^5,\\
&|\Lambda_n^4| = \tfrac{4}{35}n^7+\tfrac{4}{15}n^3+\tfrac{2}{5}n^5+\tfrac{23}{105}n,\ |\Lambda_n^5| = \tfrac{3}{7}n-\tfrac{5}{126}n^3+\tfrac{1}{3}n^5+\tfrac{5}{21}n^7+\tfrac{5}{126}n^9
\end{aligned}$$

Only odd powers of $n$ occur in those expansion; however, the negative coefficient in $|\Lambda_n^5|$ discourages a search for combinatorial significance. It turns out that the Bernoulli numbers $B_k$ are to blame (Section 5); on the other hand, they give us a rather precise approximation of our numbers, $\left|\Lambda_n^k\right|\approx\frac{\left((n+1)^{2k+1}-2n^{2k+1}+(n-1)^{2k+1}\right)k!k!}{(2k+1)!}-\frac{2\left((n+1)^k-2n^k+(n-1)^k\right)B_{k+1}}{k+1}$ for odd $k$, and a similar approximation for even $k$ (Corollary 9).The numerical experiments hint at another expansion again in odd degrees but of a different basis,

$$\begin{aligned}
&|\Lambda_n^2| = \tbinom{n}{1}+4\tbinom{n+1}{3}, \qquad |\Lambda_n^3| = \tbinom{n}{1}+12\tbinom{n+1}{3}+36\tbinom{n+2}{5}\\
&|\Lambda_n^4| = \tbinom{n}{1}+28\tbinom{n+1}{3}+240\tbinom{n+2}{5}+576\tbinom{n+3}{7}.
\end{aligned}$$

Those coefficients are indeed positive integers, and they are derived in Section 5.1. The most detailed refinement of $\Lambda_n^k$ that we consider is the number of all $k$-tuples that intersect in the single number $p$, consist of $h$ different subintervals, and have $u$ left endpoints and $v$ right endpoints. The number of such $k$-tuple of subintervals equals $\binom{p-1}{u-1}\binom{n-p}{v-1}B(u,v,h)S(k,h)h!$, (Section 4.3), where $S\left(k,h\right)$ stands for the Stirling numbers of the second kind, and $B(u,v,h)$ is the number of ways to select $h$ elements from an $u\times v$ matrix such that at least one element is chosen from each row and each column. At the same time, the numbers $B(u, v, h)$ are the the connection coefficients in the product formula (5)

$$\binom{mn}{h} = \sum_{i=0}^{m} \sum_{j=0}^{n} \binom{m}{i} \binom{n}{j} B(i, j, h)$$

bringing a combinatorial interpretation to a rather bland application of linear algebra (Remark 6). There is a closely related second interpretation of this formula in terms of bipartite graphs (Remark 5).

Deriving the size of the intersection is a classical topic in computer science; an algorithm was presented by McCreight [3]. For most of our expansions we need properties of Stirling numbers. Jacobo Stirling published the first table of "his" numbers in 1730, and a systematic treatment of Stirling numbers appeared in Jordan's *Calculus of Differences* [2] 200 years later. However, for convenience and accessibility we will refer to R. Stanley's text book, *Enumerative Combinatorics I*, whenever possible.

## 2 Two Basic Observations

The following two simple lemmas are the basis for our combinatorial approaches.

**Lemma 1** $\Lambda^k_{n;l} = \Lambda^k_{n+1-l;1}$ *for* $l > 0$

**Proof.** It suffices to show that $\Lambda^k_{n;l} = \Lambda^k_{n-1;l-1}$ for $l > 1$. Suppose $(I_1, I_2, ..., I_k)$ is a $k$-tupel of subintervals of $[1, n]$, and $\left|\bigcap_{j=1}^k I_j\right| = l$. The intersection is again an interval, and let $r$ be the right endpoint of this interval. Now remove $r$ from each of the $k$ intervals, and decrease all elements larger than $r$ by 1. This way all $k$ intervals are mapped into subintervals of $[1, n-1]$, and their intersection has size $l - 1$. Vice versa, we can any $k$-tuple from $\Lambda^k_{n-1;l-1}$ to a $k$-tuple from $\Lambda^k_{n;l}$ by increasing all numbers larger than the right endpoint of the $l-1$-element intersection interval by 1, and enlarging the intersection interval by 1 at the right end. ■

Sequences of subintervals from $[1, n]$ are nothing but sequences of ordered pairs of endpoints. The following lemma translates the intersection cardinality into a condition on the endpoints.

**Lemma 2** *Let* $I_j = [l_j, r_j]$, $j = 1, 2, ..., k$ *be intervals such that* $I_j \subseteq [1, n]$*. Then*

$$x \in \bigcap_{j=1}^{k} I_j \Leftrightarrow \max_{1 \leq i \leq k} \{l_j\} \leq x \leq \min_{1 \leq j \leq k} \{r_j\}$$

*Furthermore,*

$$\left|\bigcap_{j=1}^{k} I_j\right| = l \Leftrightarrow \min_{1 \leq j \leq k} \{r_j\} - \max_{1 \leq i \leq k} \{l_j\} = l - 1$$

**Proof.** The lemma follows because

$$x \in \bigcap_{j=1}^{k} I_j \Leftrightarrow l_j \leq x \leq r_j, \quad j = 1, ..., k \Leftrightarrow \max_{1 \leq i \leq k} \{l_i\} \leq x \leq \min_{1 \leq j \leq k} \{r_j\}$$

■

# 3 Octahedral Numbers

Let $\Lambda_n^2$ be the set of pairs of discrete subintervals of $[1, n]$ that intersect in one point. If $I = [a, b]$ is the first and $J = [c, d]$ the second interval, then we have following situations for which $|I \cap J| = 1$,

$$a = b = c = d$$
$$a = b = c < d, \ c = d = a < b, a < b = c = d, c < d = a = b$$
$$a < b = c < d, c < d = a < b, a < c = d < b, c < a = b < d$$

Considering the number of different points (1,2 or 3) in the above cases it is easy to see that

$$\left|\Lambda_n^2\right| = \binom{n}{1} + 4\binom{n}{2} + 4\binom{n}{3}$$

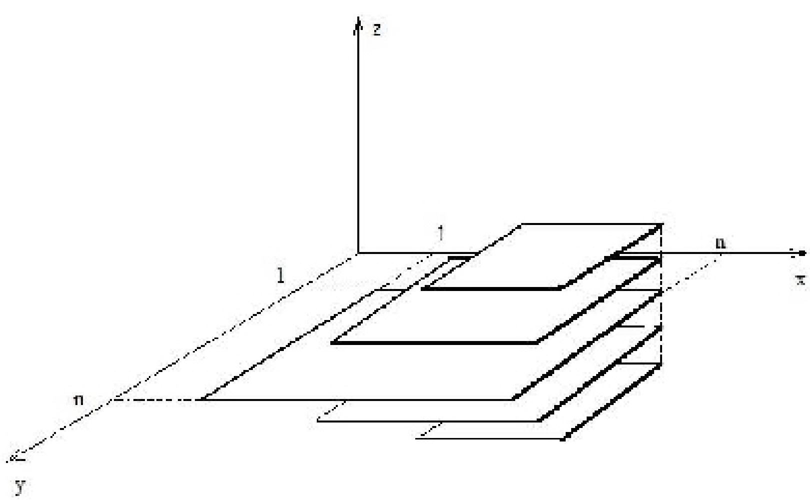
If we rearrange the expression on the right hand side, we get the well known octahedral numbers

$$1 + 2^2 + \ldots + (n-1)^2 + n^2 + (n-1)^2 + \ldots + 2^2 + 1$$

A bijection between $\Lambda_n^2$ and the discrete octahedron with an $n \times n$ center square can be obtained as follows. Let $x \wedge y := \min\{x, y\}$. In the Euclidean 3-space $\mathbb{Z}^3$ we represent the octahedron as a union of squares (layers)

$$A_l = \{(x, y, l) \mid |l| < (x \wedge y) \leq n, \quad x, y \in \mathbb{N}\}$$

for $l \in [-n+1, n-1]$.




*A Representation of the Discrete Octahedron*

Each layer $A_l$ contains $(n-|l|)^2$ points with integral coordinates. Therefore,

$$\sum_{l=-n+1}^{n+1} |A_l| = 1 + 2^2 + \ldots + (n-1)^2 + n^2 + (n-1)^2 + \ldots + 2^2 + 1$$

Consider the mapping $\Theta : \bigcup_{l=-n+1}^{n-1} A_l \to \Lambda_n^2$ defined as

$$\Theta(x,y,l) = \begin{cases} ([x \wedge y, x], [-l, y]) & \text{if } l < 0 \\ ([x \wedge y, x], [x \wedge y, y]) & \text{if } l = 0 \\ ([l, y], [x \wedge y, x]) & \text{if } l > 0 \end{cases}$$

Obviously

$$\Theta(x,y,l) = \Theta(x_1,y_1,l) \Rightarrow (x,y) = (x_1,y_1)$$

for all $l \in [-n+1, n-1]$. Hence $\Theta$ is injective. Consequently, define $\Theta^{-1} : \Lambda_n^2 \to \bigcup_{l=-n+1}^{n-1} A_l$ by

$$\Theta^{-1}([l_1,r_1],[l_2,r_2]) = \begin{cases} (r_1, r_2, -l_2) & \text{if } l_2 < l_1 \\ (r_1, r_2, 0) & \text{if } l_2 = l_1 \\ (r_2, r_1, l_1) & \text{if } l_1 < l_2 \end{cases}$$

Again, it is easy to check that $\Theta^{-1}$ is injective. We show that $\Theta$ is a bijection by proving that $\Theta^{-1}$ is indeed its inverse. Let $(x,y,l) \in A_l$, thus $|l| < (x \wedge y) \leq n$.
If $l > 0$, then $\Theta^{-1}(\Theta((x,y,l)) = \Theta^{-1}([l,y],[x \wedge y, x]) = (x,y,l)$.
If $l < 0$ then $\Theta^{-1}(\Theta(x,y,l)) = \Theta^{-1}([x \wedge y, x], [|l|, y]) = (x, y, -|l|)$.
If $l = 0$ then $\Theta^{-1}(\Theta(x,y,0)) = \Theta^{-1}([x \wedge y, x], [x \wedge y, y]) = (x,y,0)$.

# 4 The General Case

From Lemma 2 follows that $(I_1, I_2, ..., I_k) \in \Lambda_n^k$ iff

$$\max_{1 \leq i \leq k} \{l_j\} = \min_{1 \leq j \leq k} \{r_j\}$$

where $l_j$ are the left endpoints and $r_j$ are the right endpoints of the intervals $I_j = [l_j, r_j]$ in this $k$-tuple. Suppose the intervals intersect in $p \in [1,n]$. The number of ways the left endpoint can be chosen equals the number $p^k - p^{k-1}$ of mappings from $[1,k]$ to $[1,p]$ that contain $p$ as an image. Interpreting the right endpoints in the same way shows that

$$\left|\Lambda_n^k\right| = \sum_{p=1}^{n} \left(p^k - (p-1)^k\right)\left((n+1-p)^k - (n-p)^k\right) \tag{1}$$

This is probably the most "basic" answer to our problem. It hides the polynomial character of the numbers $\left|\Lambda_n^k\right|$, which will become more apparent in the following refinements of the problem.

Note that

$$\sum_{l=1}^{n} \left|\Lambda_l^k\right| = \sum_{p=1}^{n} \left(p^k - (p-1)^k\right) \sum_{l=p}^{n} \left((l+1-p)^k - (l-p)^k\right) \quad (2)$$
$$= \sum_{p=1}^{n} \left(p^k - (p-1)^k\right) (n+1-p)^k$$

This sum plays a role in determining the probability of selecting $k$ nonintersecting subintervals (Section 6)

## 4.1 Endpoint Sets

Consider the mapping $\Phi : \Lambda_n^k \longrightarrow [1, n]$ defined as

$$\Phi([l_1, r_1], [l_2, r_2], ..., [l_k, r_k]) = \bigcup_{i=1}^{k} \{l_i, r_i\}.$$

The set on the right-hand side in the argument of the mapping $\Phi$ we will call the *endpoint* set of the $k$-tuple. We know from Lemma 2 that

$$1 \leq \left|\bigcup_{i=1}^{k} \{l_i, r_i\}\right| \leq 2k - 1$$

and we can divide set $\Lambda_n^k$ into equivalence classes regarding different endpoint sets. We are interested in the sizes of those equivalence classes, i.e., we want to find $|\Phi^{-1}\{i_1, i_2, ..., i_m\}|$ where $\{i_1, i_2, ..., i_m\} \subseteq [1, n]$ for some fixed $m$, such that $1 \leq m \leq 2k - 1$. Without loss of generality we can assume that $\{i_1, i_2, ..., i_m\} = [1, m]$. Therefore, we want the number of all $k$-tuples from $\Lambda_n^k$ such that

$$\Phi([l_1, r_1], [l_2, r_2], ..., [l_k, r_k]) = [1, m].$$

Denote this number as $c(m, k) = |\Phi^{-1} [1, m]|$. For finding this number it is helpful to notice that

- every number from $[1, m]$ has to occur in the corresponding $k$-tuple
  $([l_1, r_1], [l_2, r_2], ..., [l_k, r_k]) \in [1, m]$ at least once.

- since $k$-tuples from $\Phi^{-1} [1, m]$ intersect in one point, that intersection point must be in the set $[1, m]$.

Suppose that $\bigcap_{j=1}^{k} [l_j, r_j] = \{p\}$, where $p \in [1, m]$. From $\max_{1 \leq i \leq k}\{l_j\} = p = \min_{1 \leq j \leq k}\{r_j\}$ (Lemma 2) follows $\{l_1, l_2, ..., l_k\} = [1, p]$ and $\{r_1, r_2, ..., r_k\} = [p,\ m]$. Denote the number of occurrences of $i \in [1,\ p]$ among the left end $k$-tuple $(l_1, l_2, ..., l_k)$ as $t_i$. Therefore, $t_1 + t_2 + ... + t_p = k$ where $t_i \geq 1$ for all $i \in [1,\ p]$. For every such composition $t_1 + t_2 + ... + t_p = k$ we have $\binom{k}{t_1, t_2, ..., t_p}$ different orders of left endpoints. Similarly, we have $\binom{k}{w_p, w_{p+1}, ..., w_m}$ different orders of right endpoints for the composition $w_p + w_{p+1} + ... + w_m = k$, where $w_i$ represents the number of occurrences of $i \in [p, m]$ among the right endpoint $k$-tuple $(r_1, r_2, ..., r_k)$. Thus

$$c(m,k) = \sum_{p=1}^{m} \sum_{t_u \geq 1,\, t_1+t_2+...+t_p=k} \binom{k}{t_1, t_2, ..., t_p} \times \sum_{w_v \geq 1,\, w_p+w_{p+1}+...+w_m=k} \binom{k}{w_p, w_{p+1}, ..., w_m}$$

It is well known [2, § 60] that

$$\sum_{t_u \geq 1,\, t_1+t_2+...+t_p=k} \binom{k}{t_1, t_2, ..., t_p} = p!S\left(k,p\right) = \sum_{j=0}^{p} \binom{p}{j} \left(-1\right)^{p-j} j^k,$$

the Stirling number of the second kind. Indeed, this may be seen as a direct consequence of the exponential generating function of the Stirling numbers of the second kind (as in Jordan's book), or as an application of the multinomial formula and the exclusion-inclusion principle. Hence

$$c(m,k) = \sum_{p=1}^{m} p!S(k,p)(m+1-p)!S(k,m+1-p) \tag{3}$$

Now we can easily evaluate the number of all $k$-tuples from the set $\Lambda_n^k$ that intersect in $m$ as $\binom{n}{m}c(\ m,k)$. As we noticed earlier, $1 \leq m \leq 2k-1$, hence

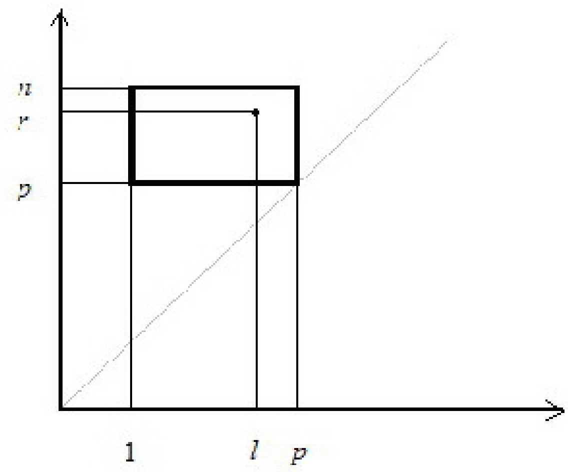
$$\left|\Lambda_n^k\right| = \sum_{m=1}^{2k-1} \binom{n}{m} c(m,k) = \sum_{m=1}^{2k-1} \binom{n}{m} \sum_{p=1}^{m} p!S(k,p)(m+1-p)!S(k,m+1-p) \tag{4}$$

This formula for $\left|\Lambda_n^k\right|$ makes it obvious that $\left|\Lambda_n^k\right|$ can be extended to a polynomial $\lambda_{2k-1}\left(n\right)$ of degree $2k-1$ in $n$.

**Remark 3** *Formula (3) motivates us to interpret the total number of different orders of left endpoints of $k$-tuples from $\Phi^{-1}\left[1,m\right]$ intersecting in $\{p\}$, as the number of surjective mappings from the set $\{l_1, l_2, ..., l_k\}$ to the set $[1,p]$. That number is equal to $p!S(k,p)$ (see [5, 1.4]), and similarly for right endpoints the number of surjections from $\{r_1, r_2, ..., r_k\}$ to $[p,m]$ equals $(m+1-p)!S(k,m+1-p)$.*

## 4.2 Geometric Interpretation

We will map intervals to points via the mapping $F : \{[i,j] \mid [i,j] \subseteq [1,n]\} \to \{(i,j) \mid 1 \leq i \leq j \leq n\}$ defined as $F([i,j]) = (i,j)$. Clearly, $F$ is a bijection between the set of all subintervals of $[1,n]$ and a discrete right triangle in the coordinate plane. This bijection provides a geometric interpretation of our problem. As we pointed out earlier, every $k$-tuple from the set $\Lambda_n^k$ must have an intersection in $1, 2, ...$ or $n$. Suppose the $k$-tuple $([l_1, r_1], [l_2, r_2], ..., [l_k, r_k])$ intersects in $\{p\}$. Then we know that $\max_{1 \leq i \leq k} l_j = p = \min_{1 \leq j \leq k} r_j$. This means that every interval $[l_i, r_i]$ from the observed $k$-tuple is mapped by $F$ to the point $(l_i, r_i)$ in the rectangle $R_p = \{(i,j) \mid 1 \leq i \leq p \leq j \leq n\}$.




*Mapping Subintervals to Points*

Conversely, every choice of $k$ points (with multiplicities) from the rectangle $R_p$ containing at least one point with first coordinate equal to $p$, and one point with second coordinate equal to $p$, will give us a $k$-tuple that intersects in $\{p\}$. Notice that the number of different points which we take from $R_p$ in this way equals the number of different intervals in the $k$-tuple that intersect in $p$. This leads us to a new way of expressing the cardinality of the set $\Lambda_n^k$. In the following subsection we will obtain the cardinality of $\Lambda_n^k$ by summing up the number of all $k$-tuples which consist of $h$ different intervals, $h = 1, 2, \ldots, k$.

### 4.2.1 Number of all $k$-tuples consisting of $h$ different intervals

Consider the mapping $\Gamma : \Lambda_n^k \to \mathbb{N}$ defined as

$$\Gamma(I_1, I_2, ..., I_k) = |\{I_1, I_2, ..., I_k\}|$$

This mapping brakes $\Lambda_n^k$ into equivalence classes such that $\Lambda_n^k = \bigcup_{h=1}^k \Gamma^{-1}(h)$, which implies $\left|\Lambda_n^k\right| = \sum_{h=1}^k |\Gamma^{-1}(h)|$. Let us investigate the size of $|\Gamma^{-1}(h)|$. As we already know, the intersection of every tuple from $\Gamma^{-1}(h)$ must be an element of $[1, n]$. Suppose the $k$-tuple intersects in some number $p$. We count the number of all $k$-tuples which intersect in $p$ and consists of exactly $h$ different intervals. Going back to the geometric interpretation, denote by $T(p, h)$ the number of ways of choosing $h$ different points from the rectangle $R_p$ such that at least one is chosen with $x$-coordinate $p$, and at least one with $y$-coordinate $p$. It is not difficult to see, by the inclusion-exclusion principle, that:

$$T(p, h) = \binom{p(n-p+1)}{h} - \binom{(p-1)(n-p+1)}{h} - \binom{p(n-p)}{h} + \binom{(p-1)(n-p)}{h}$$

From $h$ different intervals we can make $h!S(k, h)$ different $k$-tuples. Therefore, the number of $k$-tuples from the set $\Gamma^{-1}(h)$ which intersect in $p$ is $T(p, h)h!S(k, h)$. It follows that

$$|\Gamma^{-1}(h)| = \sum_{p=1}^{n} T(p, h)h!S(k, h)$$

and

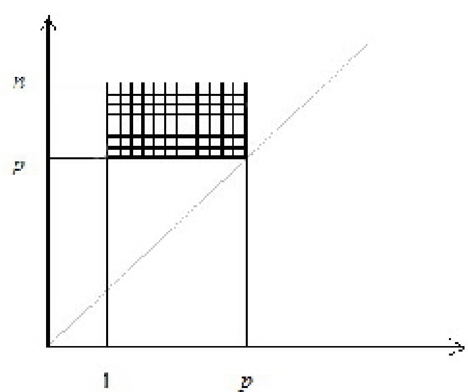
$$|\Lambda_n^k| = \sum_{h=1}^{k} |\Gamma^{-1}(h)| = \sum_{h=1}^{k}\sum_{p=1}^{n} T(p,h)h!S(k,h) =$$

$$= \sum_{h=1}^{k}\sum_{p=1}^{n}\left(\binom{p(n-p+1)}{h} - \binom{(p-1)(n-p+1)}{h}\right.$$

$$\left. -\binom{p(n-p)}{h} + \binom{(p-1)(n-p)}{h}\right) h!S(k,h)$$

## 4.3 The Number of $k$-Tuples from $\Gamma^{-1}(h)$ Determined by the Number of Left and Right Endpoints

All refinements of $|\Lambda_n^k|$ so far seem to indicate that Stirling numbers of the second kind are an unavoidable component. However, if we want to calculate the number of all $k$-tuples from $\Gamma^{-1}(h)$ such that $|\{l_1, l_2, ..., l_k\}| = u$ and $|\{r_1, r_2, ..., r_k\}| = v$ then the Stirling numbers of the *first* kind will appear in this refinement of the formula for $|\Lambda_n^k|$. This leads us to a new approach in calculating $|\Lambda_n^k|$. We use the standard method for determining the number of $k$-tuples which satisfy some conditions, by dividing them into disjoint subsets regarding the intersection point. Therefore, for the purpose of finding all $k$-tuples from $\Gamma^{-1}(h)$ with a fixed number of left endpoints $u$ and a fixed number of right endpoints $v$, we first observe just those that intersect in a given number $p$. In our geometric interpretation this means that we are choosing $h$ different points from $R_p$ such that at least one is chosen from column $p$ (i.e., with $x$-coordinate $p$), and at least one from row $p$ (with $y$-coordinate $p$). In addition, the number of selected columns (number of left endpoints) from $R_p$ has to be exactly $u$, and the number of rows (the number of right endpoints) from $R_p$ has to be exactly $v$.




Selecting $h$ Points from $R_p$

Since we must always choose column $p$ and row $p$, we conclude that we have to choose another $u-1$ columns and $v-1$ rows. In how many different ways can this be done? The answer to this question is related to the answer of the next problem.

**The disappearing matrix** Consider a matrix of dimension $i \times j$. We select $h$ elements from the matrix and erase every row and column that contain any of the $h$ elements. In how many different ways can we select $h$ elements such that the whole matrix must be erased? Denote this number of ways by $B(i,j,h)$. Notice that $B(i,j,h) = B(j,i,h)$.

We call any two sets of $h$ matrix elements from an $m \times n$ matrix *equivalent*, if they erase the same submatrix. If the erased submatrix has $i$ rows and $j$ columns, then the size of the corresponding equivalence class is $B(i,j,h)$. An $m \times n$-matrix has $\binom{m}{i}\binom{n}{j}$ submatrices of size $i \times j$. Hence our partitioning of the $\binom{mn}{h}$ subsets of $h$ matrix elements into equivalence classes shows that

$$\binom{mn}{h} = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}B(i,j,h) \tag{5}$$

**Remark 4** *Let $p_h(x) := \binom{x}{h}$, a polynomial of degree $h$ in $x$. The sequence $(p_h)_{h\geq 0}$ is an example of an important class, the polynomials of binomial type, for which the "binomial theorem" holds,*

$$p_h(m+n) = \sum_{i=0}^{h} p_i(m)\, p_{h-i}(n)$$

*Of course, $x^h/h!$ is the most prominent member of this family. However, (5) is a "product formula", and such formulas are usually more difficult to obtain, with one notable exception, $x^h$. We will give an algebraic proof of (5) in Remark 6 below that will uncover the actual multiplication step $(mn)^h = m^h n^h$ in this formula.*

**Remark 5** *From the definition of $B(i,j,h)$ it is easy to see that $B(i,j,h)$ is also the number of bipartite labelled simple graphs $G([i],[j];[h])$* without isolated vertices*, vertex sets of cardinality $i$ and $j$, respectively, and edge set of size $h$. The product formula (5) gives the number of graphs $G([n],[m];[h])$ (isolated vertices allowed). Summing over the number of edges results in the well known formula $2^{mn}$ for the number of bipartite labelled simple graphs with vertex sets of cardinality $m$ and $n$, respectively. See [4] for more general results on the enumeration of $k$-colored labelled graphs.*

Let us suppose for a moment that we have an explicit form of the numbers $B(u,v,h)$. Now the problem of finding the number of all $k$-tuples from $\Gamma^{-1}(h)$ with $u$ left endpoints and $v$ right endpoints becomes much easier. Let us consider those $k$-tuples that intersect in the number $p$. As we explained before, it is enough to take $u-1$ columns and $v-1$ rows from $R_p$. Therefore, we select some matrix of dimension $u \times v$, and $h$ points which we take in order to "erase" the whole matrix. It follows that

$$\binom{p-1}{u-1}\binom{n-p}{v-1}B(u,v,h)$$

is the number of ways of choosing $h$ points from the rectangle $R_p$ such that they take exactly $u$ columns and $v$ rows including row $p$ and column $p$. As we saw before, $h$ different intervals can be copied to $k$ places in $S(k,h)h!$ different ways, which is simply the number of surjective mappings from a $k$ element set to an $h$ element set. Finally,

$$\binom{p-1}{u-1}\binom{n-p}{v-1}B(u,v,h)S(k,h)h!$$

is the number of all $k$-tuples that intersect in $p$, consist of $h$ different subintervals, have $u$ left endpoints and $v$ right endpoints. From this very detailed statistic on the $k$-tuples we can obtain many results. For example, if we sum over the intersection point $p$, we get the number

$$\sum_{p=1}^{n} \binom{p-1}{u-1} \binom{n-p}{v-1} B(u,v,h) S(k,h) h! = \binom{n}{v+u-1} B(u,v,h) S(k,h) h! \tag{6}$$

of all $k$-tuples from $\Lambda_n^k$ that belong to $\Gamma^{-1}(h)$ (consist of $h$ different intervals) having $u$ left endpoints and $v$ right endpoints.

If we sum (6) over the number $u$ of left endpoints, we get the number

$$\sum_{u=1}^{k} \binom{n}{v+u-1} B(u,v,h) S(k,h) h!,$$

of all $k$-tuples which belong to $\Gamma^{-1}(h)$, and have exactly $v$ right endpoints.

If we sum (6) over the number of left and right endpoints, we obtain the number of all $k$-tuples from $\Lambda_n^k$ which belong to $\Gamma^{-1}(h)$,

$$|\Gamma^{-1}(h)| = \sum_{v=1}^{k} \sum_{u=1}^{k} \binom{n}{v+u-1} B(u,v,h) S(k,h) h!.$$

If we keep the number of endpoints fixed, and sum (6) over the number $h$ of different intervals, then we obtain the number

$$\sum_{h=1}^{k} \binom{n}{v+u-1} B(u,v,h) S(k,h) h! \tag{7}$$

of all $k$-tuples from $\Lambda_n^k$ that have exactly $u$ left endpoints and $v$ right endpoints.

Finally, we find $|\Lambda_n^k|$,

$$\left|\Lambda_n^k\right| = \sum_{h=1}^{k} \sum_{v=1}^{k} \sum_{u=1}^{k} \binom{n}{v+u-1} B(u,v,h) S(k,h) h! \tag{8}$$

**Evaluating** $B(u,v,h)$ As we already noticed in (6), the number of all $k$-tuples from $\Lambda_n^k$ that have exactly $u$ left endpoints and $v$ right endpoints equals

$$\binom{n}{u+v-1} \sum_{h=1}^{k} B(u,v,h) S(k,h) h!$$

On the other hand, the number of all $k$-tuples from $\Lambda_n^k$ that have exactly $u$ different left endpoints and $v$ different right endpoints can be evaluated as follows. Every $k$-tuple with $u$

left endpoints and $v$ right endpoints takes exactly $u+v-1$ numbers from $[1,n]$ (remember that left and right endpoints must overlap in one point). Therefore, we can choose $\binom{n}{u+v-1}$ different subsets of $[1,n]$ to construct $k$-tuples of this type. The left endpoints take the first $u$ places; this can be done in $u!S(k,u)$ different ways. Similarly, there are $v!S(k,v)$ ways to arrange the right endpoints. Therefore, the number of all $k$-tuples from $\Lambda_n^k$ that have exactly $u$ left endpoints and $v$ right endpoints is

$$\binom{n}{u+v-1} u!S(k,u)v!S(k,v).$$

Comparing the last two formulas, we see that

$$\sum_{h=1}^{k} B(u,v,h)S(k,h)h! = u!S(k,u)v!S(k,v)$$

The well known inversion formula for Stirling numbers [5, Prop. 1.4.1] shows that

$$B(u,v,h) = \frac{u!v!}{h!}\sum_{r=0}^{h} s(h,r)S(r,u)S(r,v)$$

where $s(h,i)$ stands for the (alternating) Stirling numbers of the first kind.

**Remark 6** *We can now verify the product formula (5).*

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}B(i,j,h) = \sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}\frac{i!j!}{h!}\sum_{r=0}^{h} s(h,r)S(r,i)S(r,j)$$
$$= m!n!\sum_{r=0}^{h}\frac{s(h,r)}{h!}\sum_{i=0}^{m}\frac{S(r,i)}{(m-i)!}\sum_{j=0}^{n}\frac{S(r,j)}{(n-j)!}$$

*From* $\sum_{i=0}^{m}\frac{S(r,i)}{(m-i)!} = \frac{m^r}{m!}$ *and* $\sum_{r=0}^{h}\frac{s(h,r)}{h!}x^r = \binom{x}{h}$ *(see [5, 1.4]) follows*

$$\sum_{i=0}^{m}\sum_{j=0}^{n}\binom{m}{i}\binom{n}{j}B(i,j,h) = \sum_{r=0}^{h}\frac{s(h,r)}{h!}(mn)^r = \binom{mn}{h}$$

*Note that this type of product formula for a polynomial sequence* $(p_n(x))_{n\geq 0}$*, say, is achieved by straightforward basis transformation* $p_n(x) \mapsto x^n$ *and its inverse,* $x^n \mapsto p_n(x)$*, in the vector space of polynomials. Such a product formula will surface whenever an inverse pair of transformations is explicitly known.*

We also check our formula for $B(u, v, h)$ by calculating the cardinality of $\Lambda_n^k$ according to (8), using the identity $\delta_{i,k} = \sum_{h=i}^{k} S(k,h)\, s(h,i)$ ([5, 1.4.1])

$$\begin{aligned}\left|\Lambda_n^k\right| &= \sum_{h=1}^{k} S(k,h) \sum_{v=1}^{k}\sum_{u=1}^{k} \binom{n}{v+u-1} u!v! \sum_{i=0}^{h} s(h,i)S(i,u)S(i,v) \\ &= \sum_{v=1}^{k}\sum_{u=1}^{k} \binom{n}{v+u-1} u!v! S(k,u)S(k,v) \\ &= \sum_{p=1}^{2k-1}\sum_{s=1}^{k} \binom{n}{p} s!\,(p+1-s)!S(k,s)S(k,p+1-s)\end{aligned}$$

in agreement with (4).

# 5 The Polynomials

We saw in (4) that the numbers $\left|\Lambda_n^k\right|$ can be extended from their support to polynomials $\lambda_{2k-1}(x)$, say, in $x \in \mathbb{R}$ of degree $2k-1$,

$$\lambda_{2k-1}(x) = \sum_{m=1}^{2k-1} \binom{x}{m} \sum_{p=1}^{k} p!S(k,p)\,(p+1-p)!S(k,m+1-p)$$

By (1) $\lambda_{2k-1}(n) = \sum_{p=0}^{n} \left(p^k - p^{k-1}\right)\left((n+1-p)^k - (n+1-p)^{k-1}\right)$ for all positive integers $n$. We want to have a closer look at this aspect of the polynomials.

**Lemma 7** *The functions $\beta_{2k+1}(n) := \sum_{j=1}^{n} j^k (n-j)^k$, $n, k \in \mathbb{N}_0$, can be extended to polynomials $\beta_{2k+1} \in \mathbb{R}[x]$ of degree $2k+1$,*

$$\beta_{2k+1}(x) = \frac{x^{2k+1}k!k!}{(2k+1)!} + (-1)^k \sum_{j=0}^{(k-1)/2} \frac{B_{2(k-j)}}{k-j} \binom{k}{2j+1} x^{2j+1}$$

*where $B(n) = \sum_{j=0}^{n}\sum_{i=0}^{j} \frac{(-1)^i}{j+1}\binom{j}{i} i^n$ is the $n$-th Bernoulli number.*

**Proof.** The functions $f_{k+1}(n) := \sum_{j=1}^{n} j^k$ can be extended to polynomials $f \in \mathbb{R}[x]$ of degree $k+1$ because $\nabla f_{k+1}(n) = f_{k+1}(n) - f_{k+1}(n-1) = n^k$ for all $n \in \mathbb{N}_1$ can be extended to the polynomial $x^k$. It is well known [2, § 83] that

$$f_{k+1}(n) = \sum_{j=0}^{k} \binom{k+1}{j} B_j \frac{(n+1)^{k+1-j}}{k+1}$$

Hence $\beta_{2k+1}(n+1)$

$$
\begin{aligned}
&= \sum_{j=0}^{n} j^k (n+1-j)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (n+1)^i \sum_{j=1}^{n} j^{2k-i} \\
&= \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} (n+1)^i \sum_{j=0}^{2k-i} \binom{2k-i+1}{j} \frac{B_j (n+1)^{2k-i+1-j}}{2k-i+1} \\
&= \sum_{j=0}^{2k} B_j (n+1)^{2k+1-j} \sum_{i=0}^{2k-j} \binom{k}{i} \binom{2k-i+1}{j} \frac{(-1)^{k-i}}{2k-i+1}
\end{aligned}
$$

■

Note that for $j=0$ we obtain the term $B_0 (n+1)^{2k+1} \sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{k-i}}{2k-i+1}$, where

$$
\begin{aligned}
\sum_{i=0}^{k} \binom{k}{i} \frac{(-1)^{k-i}}{2k-i+1} &= \sum_{i=0}^{\infty} \binom{k}{i} \left. \frac{(-1)^i x^{k+1+i}}{k+1+i} \right|_{x=1} = \int_0^1 x^k (1-x)^k \, dx \\
&= \operatorname{Beta}(k,k) = \frac{k!k!}{(2k+1)!}
\end{aligned}
$$

hence $\beta_{2k+1}(n+1)$

$$
= \frac{(n+1)^{2k+1} k!k!}{(2k+1)!} + \sum_{j=1}^{2k} B_j (n+1)^{2k+1-j} \sum_{i=0}^{2k-j} \binom{k}{i} \binom{2k-i+1}{j} \frac{(-1)^{k-i}}{2k-i+1}.
$$

For $j>0$ holds

$$
\begin{aligned}
&\sum_{i=0}^{2k-j} \binom{k}{i} \binom{2k-i+1}{j} \frac{(-1)^{k-i}}{2k-i+1} \\
&= \frac{1}{j} \sum_{i=0}^{2k-j} \binom{k}{i} \left( \binom{2k-i+1}{j} - \binom{2k-i}{j} \right) (-1)^{k-i} \\
&= \frac{(-1)^{k-j-1}}{j} \left( \binom{k-j}{2k+1-j} - \binom{k}{2k-j+1} \right).
\end{aligned}
$$

Hence $\beta_{2k+1}(x)$

$$
\begin{aligned}
&= \frac{x^{2k+1} k!k!}{(2k+1)!} + \sum_{j=1}^{2k} B_j x^{2k+1-j} \frac{(-1)^{k-j-1}}{j} \left( \binom{k-j}{2k+1-j} - \binom{k}{2k+1-j} \right) \\
&= \frac{x^{2k+1} k!k!}{(2k+1)!} + \sum_{j=1}^{k} B_{2k+1-j} x^j \frac{(-1)^{k-j}}{2k+1-j} \left( \binom{k}{j} (-1)^j - \binom{k}{j} \right)
\end{aligned}
$$

Starting at $j=3$, the Bernoulli numbers $B_j$ are 0 for odd $j$. Therefore, $\beta_{2k+1}(x)$

$$
\begin{aligned}
&= \frac{x^{2k+1}k!k!}{(2k+1)!} + \sum_{j=0}^{(k-1)/2} B_{2(k-j)} x^{2j+1} \frac{(-1)^{k+1}}{2(k-j)} \left( \binom{2j-k}{2j+1} - \binom{k}{2j+1} \right) \\
&= \frac{x^{2k+1}k!k!}{(2k+1)!} + (-1)^k \sum_{j=0}^{(k-1)/2} \frac{B_{2(k-j)}}{k-j} \binom{k}{2j+1} x^{2j+1}.
\end{aligned}
$$

We call the polynomials $\beta_{2k+1}(x)$ *beta polynomials*, because

$$\lim_{n\to\infty} \beta_{2k+1}(n)\, n^{-2k-1} = \frac{k!k!}{(2k+1)!}.$$

Lemma 7 allows us to improve on this result.

**Corollary 8** *If $k$ is odd, then $\beta_{2k+1}(x)/x^k = \frac{x^{k+1}k!k!}{(2k+1)!} - \frac{2B_{k+1}}{k+1} + o\left(x^k\right)$. If $k$ is even, then $\beta_{2k+1}(x)/x^{k-1} = \frac{x^{k+2}k!k!}{(2k+1)!} + \frac{2kB_{k+2}}{k+2} + o\left(x^{k-1}\right)$.*

The second order backwards difference connects the beta polynomials to the polynomials $\lambda_{2k-1}(x)$, because

$$
\begin{aligned}
\nabla^2 \beta_{2k+1}(n+1) &= \beta_{2k+1}(n+1) - 2\beta_{2k+1}(n) + \beta_{2k+1}(n-1) \\
&= \sum_{s=0}^{n} \left(s^k - s^{k-1}\right)\left((n+1-s)^k - (n+1-s)^{k-1}\right) \qquad (9) \\
&= \lambda_{2k-1}(n)
\end{aligned}
$$

for all positive integers $n$, and hence

$$\nabla^2 \beta_{2k+1}(x+1) = \lambda_{2k-1}(x)$$

for all $x \in \mathbb{R}$. Therefore we obtain the following asymptotics for $\left|\Lambda_n^k\right|$.

**Corollary 9** *For odd $k$*

$\left|\Lambda_n^k\right| \approx \frac{\left((n+1)^{2k+1} - 2n^{2k+1} + (n-1)^{2k+1}\right)k!k!}{(2k+1)!} - \frac{2\left((n+1)^k - 2n^k + (n-1)^k\right)B_{k+1}}{k+1}$.

*For even $k$,*

$\left|\Lambda_n^k\right| \approx \frac{\left((n+1)^{2k+1} - 2n^{2k+1} + (n-1)^{2k+1}\right)k!k!}{(2k+1)!} + \frac{2k\left((n+1)^{k-1} - 2n^{k-1} + (n-1)^{k-1}\right)B_{k+2}}{k+2}$.

*The number of nonintersecting $k$-tuples of intervals is approximately*

$\binom{n+1}{2}^k - \frac{\left((n+1)^{2k+1} - n^{2k+1}\right)k!k!}{(2k+1)!} + \frac{2\left((n+1)^k - n^k\right)B_{k+1}}{k+1}$ *for odd $k$, and*

$\binom{n+1}{2}^k - \frac{\left((n+1)^{2k+1} - n^{2k+1}\right)k!k!}{(2k+1)!} - \frac{2\left((n+1)^{k-1} - n^{k-1}\right)B_{k+2}}{k+2}$ *for even $k$.*

**Proof.** If $k$ is odd, then $\beta_{2k+1}(x) \approx \frac{x^{2k+1}k!k!}{(2k+1)!} - \frac{2x^k B_{k+1}}{k+1}$ according to the previous Corollary, hence

$\left|\Lambda_n^k\right| = \lambda_{2k-1}(n) \approx$

$\frac{(n+1)^{2k+1}k!k!}{(2k+1)!} - \frac{2(n+1)^k B_{k+1}}{k+1} - 2\left(\frac{n^{2k+1}k!k!}{(2k+1)!} - \frac{2n^k B_{k+1}}{k+1}\right) + \frac{(n-1)^{2k+1}k!k!}{(2k+1)!} - \frac{2(n-1)^k B_{k+1}}{k+1}$
$= \frac{\left((n+1)^{2k+1}-2n^{2k+1}+(n-1)^{2k+1}\right)k!k!}{(2k+1)!} - \frac{2\left((n+1)^k-2n^k+(n-1)^k\right)B_{k+1}}{k+1}$

Recall that the number of $k$-tuples of subintervals from $[1,n]$ intersecting in $l \geq 1$ points equals the number of $k$-tuples chosen from $[1, n+1-l]$ intersecting in 1 point. Hence the number of nonintersecting $k$-tuples equals $\binom{n+1}{2}^k - \sum_{l=1}^{n} \left|\Lambda_l^k\right|$

$$\begin{aligned}
&= \binom{n+1}{2}^k - \sum_{s=1}^{n} s^k (n+1-s)^k + \sum_{s=1}^{n} (s-1)^k (n+1-s)^k \\
&= \binom{n+1}{2}^k - \beta_{2k+1}(n+1) + \beta_{2k+1}(n) \\
&\approx \binom{n+1}{2}^k + \frac{\left(n^{2k+1} - (n+1)^{2k+1}\right) k!k!}{(2k+1)!} - \frac{2\left(n^k - (n+1)^k\right) B_{k+1}}{k+1}
\end{aligned}$$

For even $k$ the result follows in the same way. ■

It is no surprise that the above approximation to $\left|\Lambda_n^k\right|$ works well, even for small $n$. The following small table shows some relative approximation errors $\left(\left|\Lambda_n^k\right| - \text{approximation}\right) / \left|\Lambda_n^k\right|$ for $k = 5, 6$, and $n$ between 3 and 6.

| | $n = 3$ | 4 | 5 | 6 |
|---|---|---|---|---|
| $k = 5$ | $-2.2 \cdot 10^{-3}$ | $-5.8 \cdot 10^{-4}$ | $-1.8 \cdot 10^{-4}$ | $-6.8 \cdot 10^{-5}$ |
| $k = 6$ | $-9.7 \cdot 10^{-3}$ | $8.0 \cdot 10^{-5}$ | $1.0 \cdot 10^{-5}$ | $1.8 \cdot 10^{-6}$ |

## 5.1 Expansion in Odd Degrees

The beta polynomials are sums of odd powers, and therefore $\lambda_{2k-1}(x) = \nabla^2 \beta_{2k+1}(x+1)$ is a polynomial that contains only odd powers of $x$. The experimental results we mentioned in the introduction let us conjecture that $\left|\Lambda_n^k\right|$ can be expanded in terms of binomial coefficients of odd degrees,

$$\left|\Lambda_n^k\right| = \sum_{j=1}^{k} \binom{n+j-1}{2j-1} d(j,k)$$

for some positive integers $d(j,k)$. We will now determine those coefficients in terms of the numbers $c(m,k)$ we discussed earlier (see (**??**)). First we need one more property of $c(m,k)$.

**Lemma 10** *For all positive integers $m$ holds*

$$c(m,k) = \sum_{i=m}^{2k-1} (-1)^{i-1} c(i,k) \binom{i-1}{m-1}$$

**Proof.** We noted that the polynomials $\lambda_{2k-1}(x)$ are odd, $\lambda_{2k-1}(x) = -\lambda_{2k-1}(-x)$. By expansion (4) $\lambda_{2k-1}(n)$

$$
\begin{aligned}
&= \sum_{m=1}^{2k-1} \binom{n}{m} c(m,k) = -\lambda_{2k-1}(-n) = -\sum_{m=1}^{2k-1} \binom{-n}{m} c(m,k) \\
&= \sum_{i=1}^{2k-1} (-1)^{i-1} c(i,k) \binom{n+i-1}{i} \\
&= \sum_{i=1}^{2k-1} (-1)^{i-1} c(i,k) \sum_{m=1}^{2k-1} \binom{n}{m} \binom{i-1}{i-m} \\
&= \sum_{m=1}^{2k-1} \binom{n}{m} \sum_{i=m}^{2k-1} (-1)^{i-1} c(i,k) \binom{i-1}{m-1}
\end{aligned}
$$

Comparing coefficients of the polynomial basis $\binom{n}{m}$ shows that

$$
c(m,k) = \sum_{i=m}^{2k-1} (-1)^{i-1} c(i,k) \binom{i-1}{m-1}.
$$

■

**Lemma 11** *Let* $d(j,k) := \sum_{i=1}^{2k-1} (-1)^{i-1} \frac{i-1}{i-j} \binom{i-j}{j-1} c(i,k)$ *for all* $j = 1, \ldots, k$. *Then*

$$
\left|\Lambda_n^k\right| = \sum_{j=1}^{k} \binom{n+j-1}{2j-1} d(j,k)
$$

*Note that* $\frac{i-1}{i-j}\binom{i-j}{j-1}$ *must be interpreted as* 1 *if* $j = 1$, *and as* $\frac{i-1}{j-1}\binom{i-j-1}{j-2}$ *if* $i = j > 1$.

**Proof.** Because of

$$
\begin{aligned}
\sum_{j\geq 1} \binom{n+j-1}{2j-1} d(j,k) &= \sum_{j\geq 1} \sum_{m=0}^{2j-1} \binom{n}{m} \binom{j-1}{2j-1-m} d(j,k) \\
&= \sum_{m=0}^{2k-1} \binom{n}{m} \sum_{j=1}^{m} \binom{j-1}{m-j} d(j,k)
\end{aligned}
$$

and $\left|\Lambda_n^k\right| = \sum_{m=1}^{2k-1} \binom{n}{m} c(m,k)$ it suffices to show that

$$
\begin{aligned}
c(m,k) &= \sum_{j=1}^{m} \binom{j-1}{m-j} d(j,k) \\
&= \sum_{j=1}^{m} \binom{j-1}{m-j} \sum_{i=1}^{2k-1} (-1)^{i-1} \frac{i-1}{i-j} \binom{i-j}{j-1} c(i,k)
\end{aligned}
$$

Combine the identity [1, (3.146)] $\sum_{j=1}^{m}\binom{j-1}{m-j}\frac{i-1}{i-j}\binom{i-j}{j-1}=\binom{i-1}{m-1}$ and the previous Lemma to get

$$c(m,k)=\sum_{i=1}^{2k-1}(-1)^{i-1}c(i,k)\sum_{j=1}^{m}\binom{j-1}{m-j}\frac{i-1}{i-j}\binom{i-j}{j-1}$$
$$=\sum_{j=1}^{m}\binom{j-1}{m-j}\sum_{i=1}^{2k-1}(-1)^{i-1}\frac{i-1}{i-j}\binom{i-j}{j-1}c(i,k)$$

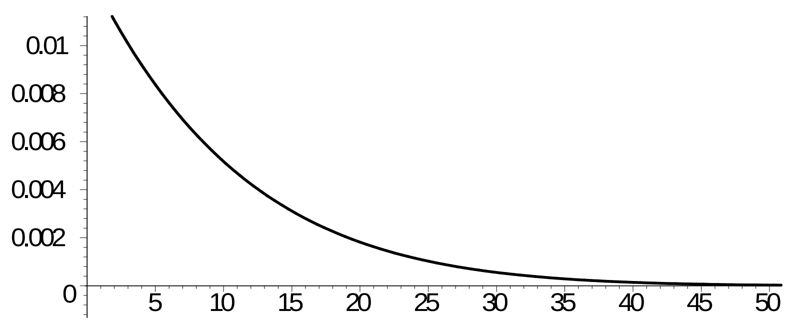
as desired. ■

# 6 Distribution of the Size of the Intersection

If we consider the cardinality of the intersection of a sequence of $k$ random (equally likely) subintervals of $[1,n]$ as a random variable $L_n$, say, then

$$\Pr(L_n=l)=\Pr(L_{n+1-l}=1)=\left|\Lambda_{n+1-l}^{k}\right|\binom{n+1}{2}^{-k}$$

for all positive integers $l$. For the graph below we applied Corollary 9, plotting a continuous approximation to this distribution ($n=100$ and $k=5$). In that example, $\Pr(L_n=0)=0.876$.



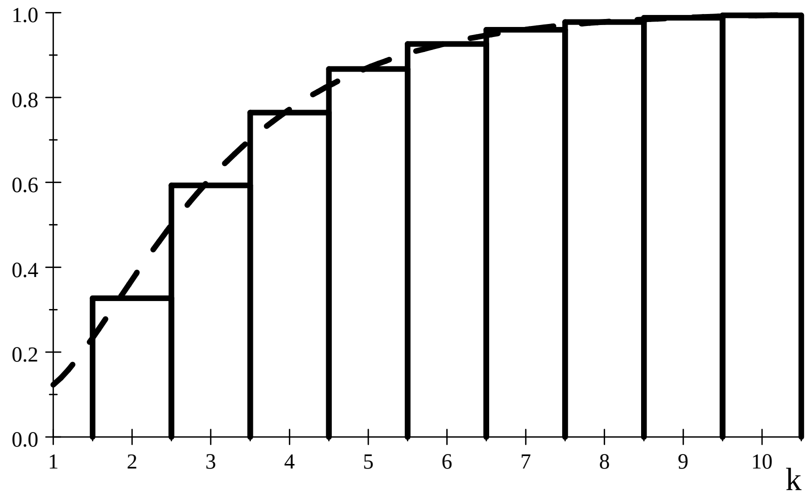

$\Pr(L_{100}=l)$ for $1\le l\le 50$ and $k=5$ (continuous approximation)

The probability of no intersection will increase with the number $k$ of intervals; by 2 and Corollary 9

$$\Pr(L_n=0)=1-\binom{n+1}{2}^{-k}\sum_{p=1}^{n}\left(p^k-(p-1)^k\right)(n+1-p)^k$$
$$\approx 1-2^k\frac{\left((n+1)^{2k+1}-n^{2k+1}\right)k!k!}{(2k+1)!n^k(n+1)^k}\approx 1-2^k\frac{k!k!}{(2k)!}$$
$$\approx 1-2^k\frac{k^{2k+1}\sqrt{2\pi}}{(2k)^{2k+1/2}}\approx 1-2^{-k}\sqrt{k\pi} \quad (10)$$

(applying first $\lim_{n\to\infty} \frac{(n+1)^{2k+1}-n^{2k+1}}{n^k(n+1)^k} = 2k+1$, and then Stirling's approximation formula).

The continuous curve in the following graph shows this approximation for $n = 10$ and $k$ from 2 to 10, together with the bar graph depicting the exact distribution.




The probability of selecting $k$ nonintersecting intervals

## 6.1 The First Two Moments

Calculating the expected value $\mu_{k,n}$ of $L_n$ is a straightforward exercise; finding the variance $\sigma^2_{k,n}$ may be easier via the second (falling) factorial moment (Lemma 13).

**Lemma 12** *The expected size $\mu_{k,n}$ of the intersection of a $k$-tupel of subintervals of $[1,n]$ equals*

$$\binom{n+1}{2}^{-k} \sum_{i=1}^{n} i^k (n+1-i)^k .$$

*Furthermore,* $\lim_{n\to\infty} \mu_{k,n} \approx \frac{2^k (n+1)^{k+1} k!k!}{n^k (2k+1)!} \approx (n+1)^{k+1} n^{-k} 2^{-k-1} \sqrt{\pi/k}$.

With the help of Stirling's formula we can further approximate $\mu_{k,n}$ by $(n+1)^{k+1} n^{-k} 2^{-k-1} \sqrt{\pi/k}$. There is however a significant loss in precision from the first to the second approximation. For example, if $k = n = 50$, the relative error $\left(\mu_{k,n} - \frac{2^k (n+1)^{k+1} k!k!}{n^k (2k+1)!}\right)/\mu_{k,n}$ equals $-2 \times 10^{-32}$, and $\left(\mu_{k,n} - (n+1)^{k+1} n^{-k} 2^{-k-1} \sqrt{\pi/k}\right)/\mu_{k,n} = -7 \times 10^{-3}$. In any case, we observe a slow increase of the expected cardinality of the intersection when $n$ increases, and an exponential decline, when $k$ increases.

**Proof.** From $\Pr(L_n = l) = \Pr(L_{n+1-l} = 1) = \left|\Lambda^k_{n+1-l}\right| \binom{n+1}{2}^{-k}$ for all positive integers $l$ follows $\mu_{k,n} = \binom{n+1}{2}^{-k} \sum_{l=1}^{n} l \left|\Lambda^k_{n+1-l}\right|$. In terms of beta polynomials we must show that

$$\beta_{2k+1}(n+1) = \sum_{l=1}^{n} l \nabla^2 \beta_{2k+1}(n+2-l) \tag{11}$$

(see (9)). This holds for $n=1$ because $\nabla^2\beta_{2k+1}(2)=\left|\Lambda_1^k\right|=1=1^k(2-1)^k=\beta_{2k+1}(2)$. Note that for $k>0$ the beta polynomials have roots at $n=0$ and $n=1$. By induction $\sum_{l=1}^{n+1} l\nabla^2\beta_{2k+1}(n+2-l)$

$$
\begin{aligned}
&=\sum_{l=0}^{n}(l+1)\nabla^2\beta_{2k+1}(n+1-l)=\beta_{2k+1}(n)+\nabla^2\sum_{l=0}^{n-1}\beta_{2k+1}(n+1-l)\\
&=\beta_{2k+1}(n)+\nabla\left(\beta_{2k+1}(n+1)-\beta_{2k+1}(1)\right)\\
&=\beta_{2k+1}(n)+\beta_{2k+1}(n+1)-\beta_{2k+1}(n)
\end{aligned}
$$

We saw in Lemma 7 that $\beta_{2k+1}(n+1)\approx\frac{(n+1)^{2k+1}k!k!}{(2k+1)!}$, explaining the approximation $\frac{2^k(n+1)^{k+1}k!k!}{n^k(2k+1)!}$ for $\mu_{k,n}$. ■

**Lemma 13**

$$E\left[L_n\left(L_n-1\right)\right]=2\binom{n+1}{2}^{-k}\sum_{i=1}^{n}\sum_{j=1}^{n-i}(ij)^k$$

**Proof.** Note that $L_1\equiv 1$, thus the Lemma holds for $n=1$, because $0=E\left[L_1\left(L_1-1\right)\right]=2\binom{2}{2}^{-k}\sum_{i=1}^{1}\sum_{j=1}^{1-i}(ij)^k$. Suppose it holds for $n\geq 1$. By induction, $\binom{n+2}{2}^kE\left[L_{n+1}\left(L_{n+1}-1\right)\right]$

$$
\begin{aligned}
&=\sum_{l=1}^{n+1}l(l-1)\left|\Lambda_{n+2-l}^k\right|=\sum_{l=1}^{n}(l-1)l\left|\Lambda_{n+1-l}^k\right|+\sum_{l=1}^{n}2l\left|\Lambda_{n+1-l}^k\right|\\
&=\binom{n+1}{2}^kE\left[L_n\left(L_n-1\right)\right]+2\beta_{2k+1}(n+1)\\
&=2\sum_{i=0}^{n}i^k\sum_{j=i}^{n}(j-i)^k+2\sum_{i=0}^{n+1}i^k(n+1-i)^k=2\sum_{i=0}^{n+1}i^k\sum_{j=i}^{n+1}(j-i)^k
\end{aligned}
$$

■

Hence $Var[L]=E\left[L^2\right]-E[L]^2=E\left[L(L-1)\right]+E[L]-E[L]^2$
$=\binom{n+2}{2}^{-k}\left(2\sum_{i=1}^{n}\sum_{j=1}^{n-i}(ij)^k-\beta_{2k+1}(n+1)\left(\beta_{2k+1}(n+1)-1\right)\right)$.

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