Symmetric Sheffer sequences, and their applications to lattice path counting

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Abstract

A sequence of Sheffer polynomials is symmetric, if the value of the n-th degree polynomial at any natural number m agrees with the m-th degree polynomial at n. While the above property sounds rather esoteric, symmetric Sheffer sequence frequently describe the elegant results of standard lattice path enumeration. We characterize all symmetric Sheffer sequences, and explain their role from the initial value problem point of view. Applications occur in the enumeration of nonintersecting weighted lattice paths, and the discussion of certain correlated random walks.

1 Introduction

A sequence of horizontal $\rightarrow$ and vertical $\uparrow$ unit steps will be called a *(standard) lattice path* in this paper. Denote by $D(m,n)$ the number of such paths starting at $(0,0)$ and reaching (m,n), perhaps under some constraints. Obviously, $D(m,n) = D(m-1,n) + D(m,n-1)$. We are mainly interested in constraints that allow us to view $D(m,n)$ as the solution of the system of difference equations

$$\nabla_m D(m,n) := D(m,n) - D(m-1,n) = D(m,n-1)$$

with some boundary conditions, reflecting the constraints. The condition

$$D(m,n) = 0 \quad \text{for negative } n$$

will always be imposed in the sequel. The "unrestricted" case can be described by the initial values $D(-1,n) = \delta_{0,n}$; of course $D(m,n) = \binom{n+m}{n}$. Figure 1 shows some values of $D(m,n)$ under two different restrictions. The bold numbers are initial values, describing the (redundant) vertical boundary $m = -1$ in the first case, and the two-piece boundary $m = \max\{-1, n-5\}$ for $n = 0, 1, ...$ in the second case. The latter boundary prohibits the paths from touching the line $x = y - 5$. The numbers in italics are obtained from $D(m,n)$ by polynomial extension – they occur in an area of the lattice that is off-limits for the paths and therefore do not represent counts. We call these polynomials $d_n(m)$ and say they *support* the count $D(m,n)$.

$m:$	-1	0	1	2	3	4
$n=4$	**0**	1	5	15	35	70
$n=3$	**0**	1	4	10	20	35
$n=2$	**0**	1	3	6	10	15
$n=1$	**0**	1	2	3	4	5
$n=0$	**1**	1	1	1	1	1

"Unrestricted" case.

$m:$	-1	0	1	2	3	4
$n=7$	*-15*	*-20*	*-20*	**0**	75	275
$n=6$	*-5*	*-5*	**0**	20	75	200
$n=5$	*-1*	**0**	5	20	55	125
$n=4$	**0**	1	5	15	35	70
$n=3$	**0**	1	4	10	20	35
$n=2$	**0**	1	3	6	10	15
$n=1$	**0**	1	2	3	4	5
$n=0$	**1**	1	1	1	1	1

Boundary $m=\max\{-1,n-5\}$.

Figure 1

The above presentation in terms of supporting polynomials and difference equations de-emphasizes specific combinatorics tools in favor of a more algebraic approach. We reduce the problem to a system of certain operator equations on polynomial sequences $\{d_n(x)\}$, say,

$$Bd_n(x)=d_{n-1}(x) \text{ for all } n=0,1,... \tag{1}$$

where the *recurrence* B is a degree reducing and translation invariant linear operator like ∇ (and like differentiation). Translation invariant means that $B(p(x+a))=(Bp)(x+a)$ for all polynomials p and constants a. Boundary conditions make the solutions unique. In Figure 1 they are of the form

$$\begin{aligned} d_n(an+b)&=\delta_{0,n} && \text{for all } n=0,...,L \\ d_n(cn+d)&=0 && \text{for all } n>L, \end{aligned}$$

where $L\geq 0$. The (umbral) calculus that handles such systems for translation invariant recurrences prints out the solution

$$d_n(x)=\sum_{k=0}^{L}\frac{ck+d-ak-b}{ck+d-b}b_k(ck+d-b)\frac{x-cn-d}{x-ck-d}b_{n-k}(x-ck-d), \tag{2}$$

where $\{b_n(x)\}$ solves the system (1) with initial values $b_n(0)=\delta_{0,n}$ (the *basic* polynomial sequence for B, see Niederhausen (1980), Corollary 2.3). In the context of path enumeration we substitute ∇ for B, and consequently $\binom{n+x-1}{n}$ for $b_n(x)$. In the examples above we used boundaries of the form $d_n(\max\{-1,n-L\})=\delta_{0,n}$, where $L>0$. Hence, there are

$$D(m,n)=d_n(m)=\sum_{k=0}^{L-1}\binom{2k-L}{k}\frac{m-n+L}{m-k+L}\binom{n-2k+m+L-1}{n-k} \tag{3}$$

paths that start at the origin and reach the point (m,n), $m > n-L$, without ever touching the line $x = y - L$.

But this is not the way the problem is attacked in Combinatorics. In Combinatorics, paths are reflected (André!) and rotated, and after a few elegant arguments the much easier answer

$$D(m,n) = \binom{n+m}{n} - \binom{n+m}{n-L} \tag{4}$$

is obtained. (The disappointed problem solver may find some consolation in the lack of combinatorial tools for *improving* on formula (2) for slopes different from 1 and 0.) Proving algebraically that the long form (3) equals the short (4) does not unveil the mystery. Looking at (4) from the boundary value problem point of view, it is obvious that $d_n(x) = \binom{n+x}{n} - \binom{n+x}{n-L}$ satisfies the right recurrence $\nabla d_n(x) = d_{n-1}(x)$. But the correct boundary values are only taken on 'by accident':

For all $n \geq L$, $\quad d_n(n-L) = \binom{n+n-L}{n} - \binom{n+n-L}{n-L} = \binom{n+n-L}{n} - \binom{n+n-L}{n} = 0.$
In other words, the polynomial sequence $\left\{w_n(x) = \binom{n+x}{n}\right\}$ has the amazing property

$$w_n(m) = w_m(n) \text{ for all } m, n = 0, 1, ...$$

Argument and degree can be interchanged! We call this the *symmetry property*.

Solutions to (1) are called Sheffer sequences. As a tool, Sheffer sequences stand between purely combinatorial arguments and generating function methods. Symmetric Sheffer sequences are destined to help solving boundary problems of the form $d_n(\max\{-1, n-L\}) = \delta_{0,n}$ in the most elegant way. But where are they in the literature? Or is $\left\{\binom{n+x}{n}\right\}$ the only one? We show in this paper (section 2) that the class of symmetric Sheffer sequences is not large, as expected. They count the number of unrestricted standard lattice paths with weighted left turns. In other words, such lattice paths enumeration is characterized by its use of symmetric Sheffer sequences! In a probabilistic interpretation, the general symmetric Sheffer sequences describe (in section 3.1) the probability that a certain Markov-chain reaches the point (m,n), generalizing the concept of a random walk to correlated random walks. We show in section 3 how symmetric Sheffer sequences efficiently generate the standard enumeration results (c.f. Mohanty (1979)) in this area.

For a bivariate application of symmetric Sheffer polynomials let $A(i, j+1; k+1, l)$ be the number of all pairs of paths that go from $(0,1)$ to $(i, j+1)$, and from $(1,0)$ to $(k+1, l)$, respectively. Obviously

$$A(i, j+1; k+1, l) = \binom{j+i}{j}\binom{k+l}{k} = w_i(j)w_l(k).$$

Thus the number of pairs is supported by bivariate Sheffer polynomials, which are again symmetric in the unrestricted case. This symmetry can be exploited if we determine the cardinality $R(i, j+1; k+1, l)$ of the subset of *nonintersecting* pairs of paths.

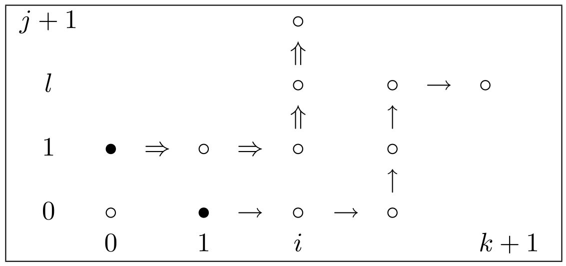

Figure 2: Nonintersecting lattice paths

If $j+1 \geq l \geq 0$ and $k+1 \geq i \geq 0$ one can show that $R(i, j+1; k+1, l)$ is supported by a bivariate Sheffer sequence $\{r_{i,l}(x,y)\}$ satisfying both partial difference equations

$$\begin{aligned} &\nabla_j r_{i,l}(j,k) = r_{i,l}(j,k) - r_{i,l}(j-1,k) = r_{i-1,l}(j,k) \\ \text{and} \quad & \\ &\nabla_k r_{i,l}(j,k) = r_{i,l}(j,k) - r_{i,l}(j,k-1) = r_{i,l-1}(j,k). \end{aligned} \tag{5}$$

Using the symmetry of $\{w_n(x)\}$ and the boundary condition

$$R(k+1, j+1; k+1, j+1) = 0 = r_{j+1,k+1}(k,j)$$

for all nonnegative integers j and k, we get

$$R(i, j+1; k+1, l) = r_{i,l}(j,k) = \binom{i+j}{i}\binom{l+k}{l} - \binom{i+j}{i-1}\binom{l+k}{l-1} \tag{6}$$

for all $j+1 \geq l \geq 0$ and $k+1 \geq i \geq 0$ (Narayana's formula, see Narayana (1955) for a special case). The case $i = k$, $j = l$ has already been solved by MacMahon (1916), Vol 2, No.242.

The above derivation of Narayana's formula is sketchy; there is a combinatorial proof in Sulanke (1993). But the result is only a special case of nonintersecting lattice paths with weighted turns. In section 4 we give a detailed proof for the weighted case, which requires a different approach. In "Counting pairs of nonintersecting lattice paths with respect to weighted turns" by Krattenthaler and Sulanke (1993) this result was derived using the "rotation method", and more references are given there. It was their work that renewed my interest in symmetric Sheffer sequences and motivated the search for their complete characterization. A q-analog of Narayana's formula is also considered in Krattenthaler and Sulanke (1993). In that case, a turn at (i,j) gets the weight q^{i+j}. The method we use in section 4 can be extended to such weight functions, but there are some additional technical difficulties to deal with. We leave this case for a later paper (Niederhausen (1994)).

2 Classification of symmetric Sheffer sequences

A *Sheffer sequence for* β *and* ρ is a polynomial sequence $\{a_n(x)\}_{n=0,1,...}$ in $\mathbb{R}[x]$, i.e. $\deg(a_n(x)) = n$, such that

$$\sum_{n\geq 0} a_n(x)t^n = \rho(t)e^{x\beta(t)}, \tag{7}$$

where

- $\rho(t) = \rho_0 + \rho_1 t + \rho_2 t^2 + \ldots$ is a power series in t of order 0 (i.e., $\rho_0 \neq 0$),
- $\beta(t) = \beta_1 t + \beta_2 t^2 + \ldots$ is a power series of order 1 with $\beta_1 = 1$.

The recurrence B on $\mathbb{R}[x]$ that maps a_n into a_{n-1} ($n = 1, 2, ...$) is called a delta operator, and has the power series expansion $B = \beta^{-1}(D)$, where D is the derivative (delta) operator $D : x^n \mapsto nx^{n-1}$ and β^{-1} is the compositional inverse of β.

The polynomial sequence $\{w_n(x)\}$ is *symmetric* iff

$$w_m(n) = w_n(m) \tag{8}$$

for all n and m in $\mathbb{N}_0$, the set of nonnegative integers. The following theorem is trivial but useful.

Theorem 1 *If* $\{w_n(x)\}$ *is a symmetric Sheffer sequence for* β *and* ρ, *then* $\rho(t) = \alpha/(1-t)$ *for some constant* $\alpha \neq 0$.

Proof. Let $n = 0$. From (8) follows $w_m(0) = w_0(m)$ for all $m \in \mathbb{N}_0$. But $w_0(m)$ is constant, hence all polynomials $w_m(x)$ have the same constant term α, say, and therefore

$$\frac{\alpha}{1-t} = \sum_{n\geq 0} w_n(0)t^n = \rho(t)$$

(see (7)) ■

We want to determine the power series $\omega(t)$ in the generating function

$$\sum_{n\geq 0} w_n(x)t^n = \frac{\alpha}{1-t}e^{x\omega(t)}$$

of symmetric Sheffer sequences. We find the following functional equation for ω:

$$\begin{aligned}
\frac{\alpha}{1-\omega^{-1}(t)}\frac{1}{1-se^t} &= \sum_{n\geq 0} s^n \frac{\alpha}{1-\omega^{-1}(t)} e^{n\omega(\omega^{-1}(t))} = \sum_{n\geq 0} s^n \sum_{k\geq 0} w_k(n)\omega^{-1}(t)^k \\
&= \sum_{k\geq 0}\sum_{n\geq 0} s^n w_n(k)\omega^{-1}(t)^k = \sum_{k\geq 0} \omega^{-k}(t)\frac{\alpha}{1-s}e^{k\omega(s)} \\
&= \frac{\alpha}{1-s}\frac{1}{1-\omega^{-1}(t)e^{\omega(s)}}
\end{aligned}$$

Replacing s by $\omega^{-1}(s)$ gives

$$(1-\omega^{-1}(s))(1-\omega^{-1}(t)e^s) = (1-\omega^{-1}(t))(1-\omega^{-1}(s)e^t).$$

Therefore we can separate the variables s and t into factors

$$\omega^{-1}(t)(1-e^s+\omega^{-1}(s)e^s) = \omega^{-1}(s)(1-e^t+\omega^{-1}(t)e^t).$$

Hence, either $1-e^t+\omega^{-1}(t)e^t=0$, i.e., $\omega^{-1}(t)=1-e^{-t}$, or $\omega^{-1}(t)$ is proportional to $1-e^t+\omega^{-1}(t)e^t$ for some proportionality factor $\frac{1}{1-\mu}$, say, $0\neq\mu\neq 1$. Solving for $\omega^{-1}(t)$ gives in the latter case

$$\omega^{-1}(t) = \frac{\frac{1}{1-\mu}(1-e^t)}{1-\frac{1}{1-\mu}e^t} = \frac{1-e^t}{1-\mu-e^t}.$$

$\mu=1$ covers the first case. We call the delta operator $\Omega^{(\mu)} := \omega^{-1}(D)$ the *weighted path recurrence*, because it is associated with Sheffer sequences (see Rota et al. (1973)) that play a role in the enumeration of lattice paths with weighted turns (see section 3). The weighted path recurrence can be expressed in terms of the familiar forwards difference operator $\Delta = E^1-1=e^D-1$,

$$\Omega^{(\mu)} = \omega^{-1}(D) = \frac{e^D-1}{\mu+e^D-1} = \frac{1}{\mu}\frac{\Delta}{1+\Delta/\mu} = -\sum_{n\geq 1}(-\mu)^{-n}\Delta^n.$$

$\mu=1$ gives $\Omega^{(1)}=1-e^{-D}=\nabla$, the backwards difference operator (in this case, $w_n(x)=\alpha\binom{n+x}{n}$). It is interesting to reverse the relationship and express Δ and ∇ in terms of $\Omega^{(\mu)}$,

$$\begin{aligned}\Delta &= \mu\Omega^{(\mu)}/(1-\Omega^{(\mu)}) &(9)\\ \nabla &= (\mu e^{-D}+\nabla)\Omega^{(\mu)} = \mu e^{-D}\sum_{i\geq 1}\left(\Omega^{(\mu)}\right)^i = \mu e^{-D}\Omega^{(\mu)}/(1-\Omega^{(\mu)}).\end{aligned}$$

2.1 Explicit symmetric Sheffer polynomials

Above our original symmetric Sheffer polynomials $\{\binom{n+x}{n}\}$ we found a one-parameter class of Sheffer sequences$\{w_n(x)\}$ with generating functions $\frac{\alpha}{1-t}e^{x\omega(t)}$ where $\omega(t)$ is the compositional inverse of $\omega^{-1}(t) = \frac{1-e^t}{1-\mu-e^t}$. Thus $e^{x\omega(t)} = 1+\mu t/(1-t)$. We expand for all $k\in\mathbb{R}$

$$\frac{\alpha}{(1-t)^k}\left(1+\frac{\mu t}{1-t}\right)^x = \alpha\sum_{n\geq 0}t^n\sum_{l=0}^{n}\binom{x}{l}\binom{n+k-1}{l+k-1}\mu^l, \tag{10}$$

and obtain for $k=1$ the symmetric Sheffer polynomials

$$w_n^{(\mu)}(x) = \alpha\sum_{l=0}^{n}\binom{n}{l}\binom{x}{l}\mu^l. \tag{11}$$

The symmetry is obvious. If $k = 0$ we get from (10) the basic sequence $\{b_n(x)\}$ for the weighted path recurrence $\Omega^{(\mu)}$

$$b_n(x) = \sum_{i=1}^{n} \binom{x}{i} \binom{n-1}{i-1} \mu^i \quad \text{for positive integers } n. \tag{12}$$

We expressed in (9) the backwards difference operator as a power series in $\Omega^{(\mu)}$. This shows that any Sheffer sequence $\{s_n(x)\}$ is associated with the weighted path recurrence iff it satisfies one or both of the expansions

$$s_n(x) - s_n(x-1) = (\mu - 1)s_{n-1}(x-1) + s_{n-1}(x), \tag{13}$$

$$s_n(x) - s_n(x-1) = \mu \sum_{i=0}^{n-1} s_i(x-1). \tag{14}$$

In the next section we present a counting problem that is governed by the above recurrence.

3 Lattice paths with weighted left turns

A lattice path makes a *left turn* if a vertical step follows a horizontal step: $\underset{\rightarrow\circ}{\uparrow}$. A path starting at $(0, 1)$ is uniquely determined by its end point $(m, n+1)$, say, and its sequence $(\xi_1, \eta_1), ..., (\xi_l, \eta_l)$ of left turning points

$$1 \leq \xi_1 < ... < \xi_l \leq m, \quad 1 \leq \eta_1 < ... < \eta_l \leq n. \tag{15}$$

A lattice paths that starts at $(0, 1)$, and reaches the point $(m, n+1)$ with exactly l left turns gets the weight μ^l. Let $D_\mu(m, n+1; l)$ denote the sum of the weights of all such paths. In the unrestricted case,

$$D_\mu(m, n+1; l) = \binom{m}{l} \binom{n}{l} \mu^l. \tag{16}$$

$D_\mu(m, n+1) := \sum_{l \geq 0} D_\mu(m, n+1; l)$ is the number of paths with *μ-weighted left turns* that start at $(0, 1)$ and reach $(m, n+1)$. In the unrestricted case, $D_\mu(m, n+1)$ can be represented by the symmetric Sheffer sequence $\left\{ w_n^{(\mu)}(x) \right\}$ (see (11)) with $\alpha = 1$. We check the weighted path recurrence relation (13) in Figure 3:

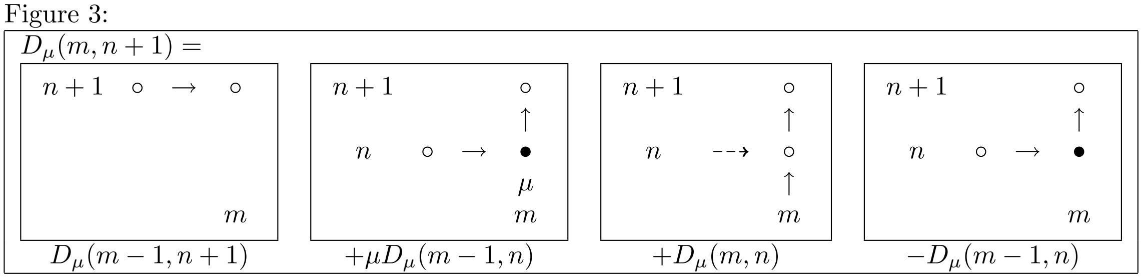

Figure 3

In the backwards difference notation,

$$\nabla_m D_\mu(m, n+1) = (\mu - 1) D_\mu(m-1, n) + D_\mu(m, n). \tag{17}$$

It is important to realize that this recurrence allows us to find $D_\mu(m, n+1)$ even if some boundary values are given. For example, if the paths are restricted as in Figure 4, the numbers $D_\mu(m, n+1)$ can still be recursively calculated from the above formula. Underneath a monotone increasing boundary, $D_\mu(m, n+1)$ is supported by a polynomial $d_n^{(\mu)}(x)$ with $\deg d_n^{(\mu)}(x) = n$. The sequence $\left\{ d_n^{(\mu)}(x) \right\}$ is a Sheffer sequence for the weighted path recurrence $\Omega^{(\mu)}$ because of (17) and (13).

n					0	•	•
6				0	•	•	•
5				0	•	•	•
4		0	0	•	•	•	•
3	0	•	•	•	•	•	•
2	0	•	•	•	•	•	•
1	•	•	•	•	•	•	•
m :	0	1	2	3	4	5	6

Figure 4

3.1 The μ–weighted path as a correlated random walk

We can view a μ-weighted lattice path as a "Correlated random walk" W (Mohanty (1979), Chapter 5.2, Renshaw and Henderson (1981)), a Markov chain with states 'horizontal' (=h) and 'vertical' (=v), and transition matrix

from \ to:	h	v
h	$p_{h\|h}$	$p_{v\|h}$
v	$p_{h\|v}$	$p_{v\|v}$

(18)

For example, a vertical step is taken with probability $p_{v|h}$ if the previous step was horizontal (a left turn). In Renshaw and Henderson (1981), $p_{h|h} = p_{v|v} = p$, and hence $p_{v|h} = p_{h|v} = 1 - p =: q$.

The analysis gets easier if we restrict our attention to random walks W that start with a vertical step (from $(0,0)$ to $(0,1)$, with probability p_0), and end with a horizontal step (from $(m, n+1)$ to $(m+1, n+1)$).

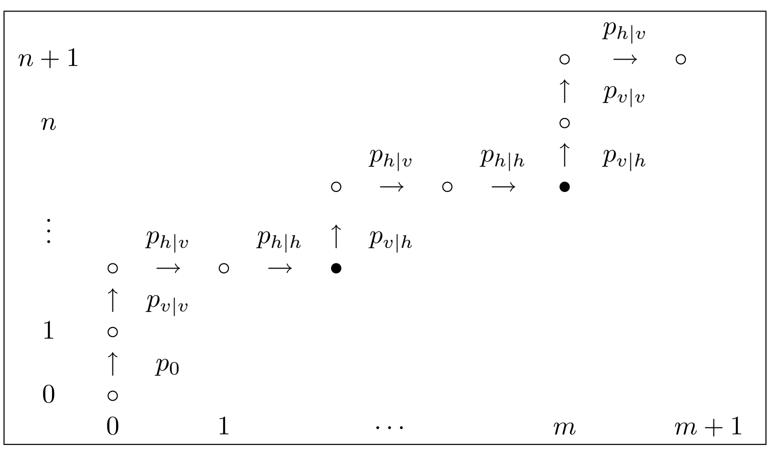

Figure 5: $m = 4,\ n = 4,\ l = 2:\quad \Pr(W) = p_0 p_{v|h}^2 p_{h|v}^3 p_{h|h}^2 p_{v|v}^2$

Now the first and last turn must be right turns, and therefore we have $l+1$ right turns if the path takes l left turns:

$$\Pr(W) = p_0 p_{v|h}^{l} p_{h|v}^{l+1} p_{h|h}^{m-l} p_{v|v}^{n-l} = p_0 p_{h|v} p_{h|h}^{m} p_{v|v}^{n} \left(\frac{p_{v|h} p_{h|v}}{p_{h|h} p_{v|v}} \right)^{l}.$$

There are of course $\binom{m}{l}\binom{n}{l}$ such paths. Hence,

P_0 := Pr(Random walk starts at the origin with a vertical step and reaches $(m+1, n+1)$ in a horizontal step) $= p_0 p_{h|v} p_{h|h}^{m} p_{v|v}^{n} w_n^{(\mu)}(m)$

if we define $\mu = \frac{p_{v|h} p_{h|v}}{p_{h|h} p_{v|v}}$, and $w_n^{(\mu)}(x)$ as in (11). P_0 can be used as a starting point for further calculations. For example,

P_k := Pr(Random walk starts at the origin with exactly k horizontal steps before the first vertical step and reaches $(m+1, n+1)$ in a horizontal step) $= (1-p_0) p_{v|h} p_{h|v} p_{h|h}^{m-1} p_{v|v}^{n} w_n^{(\mu)}(m-k)$.

This simplifies if we let $p_0 = p_{v|h}$, and hence $p_{h|h} = 1 - p_0$. Then

$$P_k = p_{v|h} p_{h|v} p_{h|h}^{m} p_{v|v}^{n} w_n^{(\mu)}(m-k)$$

for all $k = 0, \ldots, m$. Summing over k gives

Pr(Random walk starts at the origin and reaches $(m+1, n+1)$ in a horizontal step) $= p_{h|h}^{m+1} p_{v|v}^{n+1} \sum_{l \geq 0} \binom{n}{l}\binom{m+1}{l+1} \mu^{l+1}$.

3.2 Linear boundaries

We saw that unrestricted paths with weight μ for each left turn are counted by $D_\mu(m, n+1) = w_n^{(\mu)}(m) = \sum_{l=0}^{n} \binom{n}{l}\binom{m}{l} \mu^l$. Now we keep the paths strictly underneath the line $y = x + L + 1$ for some integer $L \geq 0$. The weighted counts $\bar{D}_\mu(m, n+1)$ of these paths are represented by Sheffer polynomials $\{\bar{w}_n\}$ that satisfy the same weighted path recurrence $\Omega^{(\mu)} \bar{w}_n(m) = \bar{w}_{n-1}(m)$ as before, but have the initial values $\bar{D}_\mu(0, n+1) = \bar{w}_n(0) = 1$ for all $n = 0, \ldots L-1$, and $\bar{D}_\mu(n-L, n+1) = \bar{w}_n(\ n-L) = 0$ for all $n \geq L$. In the same way as in the unweighted case (4), the symmetry of $\left\{ w_n^{(\mu)}(x) \right\}$ immediately leads to the answer $\bar{w}_n(x) = w_n^{(\mu)}(x) - w_{n-L}^{(\mu)}(x+L)$. So we find the probability $\bar{P}_{m,n}$ that a Markov chain with transition probabilities (18) starts at the origin with a vertical step, reaches $(m+1, n+1)$ in a horizontal step, and does not touch the line $y = x + L + 1$:

$$\begin{aligned} p_0^{-1} p_{h|v}^{-1} p_{h|h}^{-m} p_{v|v}^{-n} \bar{P}_{m,n} &= \bar{w}_n(m) = w_n^{(\mu)}(m) - w_{n-L}^{(\mu)}(m+L) \\ &= \sum_{l=0}^{n} \left[\binom{n}{l}\binom{m}{l} - \binom{n-L}{l}\binom{m+L}{l} \right] \mu^l \end{aligned}$$

for all $n \geq 1,\ m \geq \max\{0, n-L\}$ $\quad (\mu = p_{v|h} p_{h|v} / (p_{h|h} p_{v|v})\ )$. A detailed discussion of this problem is given in Mohanty (1979, Chapter 5.2, for random walks reaching the boundary line for the first time after n steps (first passage probabilities). Formula (2) gives 'closed expressions' for boundary lines with positive integer slope larger than one (see also Mohanty (1966)). Use the basic sequence

$$b_n(x) = \sum_{i=1}^{n} \binom{x}{i} \binom{n-1}{i-1} \mu^i$$

in formula (2). A generating function approach to such problems can be found in Krattenthaler (1989). Piecewise linear boundaries can be treated by repeated application of formula (2), as described in Niederhausen (1980).

4 Intersecting weighted lattice paths

In this section we will work with bivariate Sheffer sequences $\{r_{m,n}(x,y)\}$ associated with partial delta operators (recurrences) $B = \beta^{-1}(D)$ and $G = \gamma^{-1}(D)$, say, on $\mathbb{R}[x]$ such that

$$B_x r_{m,n}(x,y) = r_{m-1,n}(x,y) \quad \text{and} \quad G_y r_{m,n}(x,y) = r_{m,n-1}(x,y)$$

for all positive integers m and n. $\{r_{m,n}(x,y)\}$ has the generating function

$$\sum_{m,n \geq 0} r_{m,n}(x,y) s^m t^n = \rho(s,t) e^{x\beta(t)+y\gamma(t)}, \tag{19}$$

where $\rho(s,t)$ is a bivariate power series of order 0, and β and γ have order 1 as in section 2. Bivariate Sheffer sequences are uniquely defined if they satisfy the above partial operator equations and take on a given initial value $r_{m,n}(x_m, y_n)$ for each $m, n \in \mathbb{N}_0$. Of course, the explicit representation of such polynomials can be complicated, even if the boundaries are piecewise linear. More about multivariate Umbral Calculus can be found in Roman (1979).

We mentioned nonintersecting pairs of lattice paths in the introduction, and indicated how their enumeration can be seen as a bivariate boundary value problem. This approach is no longer valid if we give weights μ to the left turns of the upper path, and weights ν, say, to the right turns of the lower path. To see that let $N(a, m+1; n+1, d)$ denote the number of such pairs of paths reaching $(a, m+1)$ from $(0, 1)$, and $(n+1, d)$ from $(1, 0)$. The weighted path recurrence relation (17) holds for the upper and for the lower path if their end points are far enough apart:

$$N(a,m+1;n+1,d) = N(a-1,m+1;n+1,d)+(\mu-1)N(a-1,m;n+1,d)+N(a,m;n+1,d)$$

for all $m > d \geq 0$ on the interval $1 \leq a \leq n$, and

$$N(a,m+1;n+1,d) = N(a,m+1;n+1,d-1)+(\nu-1)N(a,m+1;n,d-1)+N(a,m+1;n,d)$$

on the interval $1 \leq d \leq m,\ n > a \geq 0$.

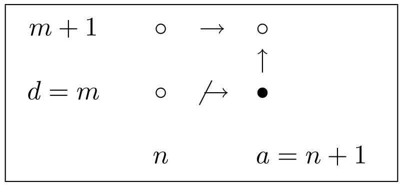

Figure 6

But Figure 6 shows that the obvious boundary value $N(n+1, m; n+1, m) = 0$ can not validate the recurrence for the upper path, if it comes too close to the lower path:

$$N(n+1, m+1; n+1, m) \\ \neq N(n, m+1; n+1, m) + (\mu - 1)N(n, m; n+1, m) + N(n+1, m; n+1, m)$$

because we cannot pass from (n, m) through the (black) end point $(n+1, m)$ of the lower path. We will obtain more useful initial values from the following lemma. Let $I(a, m+1; n+1, d)$ denote the number of weighted *intersecting* pairs of paths reaching $(a, m+1)$ from $(0, 1)$ with μ-weighted left turns, and $(n+1, d)$ from $(1, 0)$ with ν-weighted right turns, where $0 \leq a \leq n$ and $0 \leq d \leq m$. Of course, $I(a, m+1; n+1, d)$ and $N(a, m+1; n+1, d)$ sum up to the number of 'unrestricted' paths,

$$\sum_{l,r \geq 0} \binom{a}{l}\binom{m}{l}\binom{n}{r}\binom{d}{r}\mu^l \nu^r,$$

(see (16)).

Lemma 2 *Let* $\left\{g_n^{(\mu)}\right\}$ *denote the Sheffer sequence for* $\Omega^{(\mu)}$ *with polynomials*

$$g_n(x) := \sum_{l=1}^{n-1} \mu^l \binom{n-2}{l-1}\binom{x}{l+1} \text{ for all } n \geq 2, \quad g_1 := \Omega^{(\mu)} g_2 \quad \text{and} \quad g_0 := \Omega^{(\mu)} g_1.$$

(they are obtained from (10) by setting $k = -1$ and $\alpha = 1/\mu$).

For all $m \geq d \geq 0$ and $n \geq a \geq 0$

$$I(1, m+1; n+1, d) = \mu g_{n+1}^{(\nu)}(d+1) \quad \text{and} \quad I(a, m+1; n+1, 1) = \nu g_{m+1}^{(\mu)}(a+1).$$

Proof.

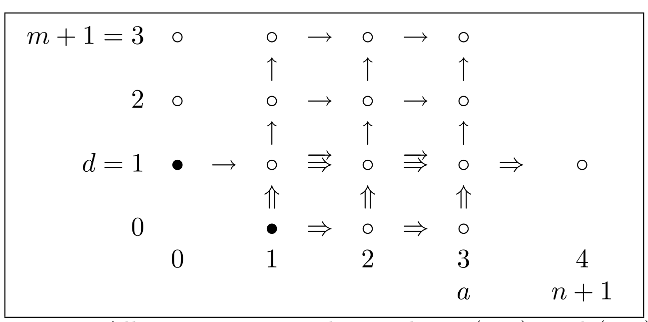

Figure 7: All intersecting paths reaching $(3, 3)$ and $(4, 1)$.

We show the second identity (the first follows in the same way). $I(0, m+1; n+1, 1) = 0$ proves the case $a = 0$. Let $a > 0$. The lower path must turn right at some point $(k, 1)$ for $k = 1, 2, ..., a \le n$ (see figure above). In order to intersect, the upper path must first go right, meeting the lower path, and then leave the lower path in a left turn at some point $(j, 1)$ for $j = k, ..., a$. Thereafter, no restrictions apply. Hence,

$$I(a, m+1; n+1, 1) = \sum_{k=1}^{a} v \sum_{j=k}^{a} \mu G(j, 2 \,|\, a, m+1), \tag{20}$$

where $G(j, 2 \,|a, m+1)$ is the number of unrestricted paths with μ-weighted left turns from $(j, 2)$ to $(a, m+1)$:

$$G(j, 2 \,|a, m+1) \ = \sum_{l=0}^{m} \mu^l \binom{m-1}{l} \binom{a-j}{l}.$$

Substituting this into (20) gives

$$I(a, m+1; n+1, 1) = \nu \sum_{l=1}^{m} \mu^l \binom{m-1}{l-1} \binom{a+1}{l+1}$$

after two trivial summations ■

We want to show the existence of a Sheffer sequence $\{s_{m,n}(x, y)\}$ for $\left(\Omega_x^{(\mu)}, \Omega_y^{(\nu)}\right)$ such that

$$I(a, m+1; n+1, d) = s_{m+1,n+1}(a, d) \text{ for all } m \ge d > 0 \text{ and } n \ge a > 0.$$

Lemma 2 can be written in the very suggestive form

$$I(1, m{+}1; n{+}1, d) = g_{m+1}^{(\mu)}(2) g_{n+1}^{(\nu)}(d{+}1) \quad \text{and} \quad I(a, m{+}1; n{+}1, 1) = g_{m+1}^{(\mu)}(a{+}1) g_{n+1}^{(\nu)}(2)$$

for all $m \ge d \ge 0$ and $n \ge a \ge 0$. Obviously, $s_{2,2}(a, d) := g_2^{(\mu)}(a{+}1) g_2^{(\nu)}(d{+}1) = \binom{a+1}{2}\binom{d+1}{2}\mu^2\nu^2$ supports $I(a, 2; 2, d)$, because $a = 1 = d$ is the only admissible argument. From this starting point, the existence of $\{s_{m,n}(x, y)\}$ follows by induction if we apply the next theorem with $Q_x = \Omega_x^{(\mu)}$ and $R_y = \Omega_y^{(\nu)}$ using the initial polynomials from Lemma 2.

Theorem 3 *Let n and m be positive integers, $u < u_{n-1} \le u_n$ and $b < b_{m-1} \le b_m$ be real numbers, and let Q and R be linear operators on $\mathbb{R}[x]$ such that $\deg(Qx^n) = \deg(Rx^n) = n{-}1$ for all positive integers n, and $\ker(Q) = \ker(R) = \{$constant functions$\}$. Suppose, $p_{m,n}(x, y)$ is a polynomial in x of degree m and in y of degree n, and $f(x, y)$ a piecewise polynomial function in both variables such that*

(a) $Q_x f(x, y) = Q_x p_{m,n}(x, y)$ on the interval $u \le x \le u_n$, $b \le y \le b_{m-1}$, and

(b) $R_y f(x,y) = R_y p_{m,n}(x,y)$ *on the interval* $u \le x \le u_{n-1},\ b \le y \le b_m$.

If there are numbers a *and* d, $u \le a \le u_n,\ b \le d \le b_m$, *such that*

(c) $f(a,y)$ *is a polynomial of degree* n *in* y *for all* $b \le y \le b_m$, *and*

(d) $f(x,d)$ *is a polynomial of degree* m *in* x *for all* $u \le x \le u_n$, *then* $f(x,y)$ *is a polynomial in* x *of degree* m *and in* y *of degree* n *on* $u \le x \le u_n,\ b \le y \le b_m$.

The initial polynomial parts of f *in (c) and (d)* uniquely *determine the solution of the partial operator equations (a) and (b).*

Proof. Only constants are in the kernels of Q and R. It follows from (a) that $f(x,y) - p_{m,n}(x,y)$ is a constant in x that could be a piecewise polynomial function in y on $u \le x \le u_n,\ b \le y \le b_{m-1}$. The condition (c) shows that this term is a (unique) polynomial piece, even on the larger interval $b \le y \le b_m$. Also, $f(x,y)$ and $p_{m,n}(x,y)$ must have the same degree m in x on $u \le x \le u_n$. In the same way, $f(x,y) - p_{m,n}(x,y)$ is a (unique) polynomial piece in x by conditions (b) and (d) on $u \le x \le u_n$, and $f(x,y)$ and $p_{m,n}(x,y)$ must have the same degree n in y on $b \le y \le b_m$ ■

Only now after we have shown that $I(a, m+1; n+1, d)$ is supported by a Sheffer sequence, we can use that $g^{(\mu)}_{m+1}(a+1) g^{(\nu)}_{n+1}(d+1)$ has the right initial values to deduce from the uniqueness part of Theorem 3

$$I(a,m+1;n+1,d) = g^{(\mu)}_{m+1}(a+1) g^{(\nu)}_{n+1}(d+1) = \sum_{l=0}^{m}\sum_{r=0}^{n} \mu^l v^r \binom{m-1}{l-1}\binom{a+1}{l+1}\binom{n-1}{r-1}\binom{d+1}{r+1}$$

for all $m \ge d \ge 0$ and $n \ge a \ge 0$. Sulanke and Krattenthaler (1993) obtained this result using the "rotation method". If $\mu = \nu = 1$ then the formula simplifies to

$$I(a, m+1; n+1, d) = \binom{m+a}{m+1}\binom{n+d}{n+1}$$

(see the Narayana numbers (6)). This is the intersecting contribution to the "Refinement of Narayana numbers" constructed in Sulanke (1993).

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