SHEFFER POLYNOMIALS IN PATH ENUMERATION

by

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1. Introduction.

A lattice path has unit step vectors, say, if its steps go only in the direction of the axis, one unit each. In other words: if the path has reached the point $(n_1,\ldots,n_{r+1})^T \in \mathbb{N}_0^{r+1}$, the next point is obtained by adding 1 to exactly one component n_i $(1 \leq i \leq r+1)$. For $m \in \mathbb{N}_0$ and $\underline{n} \in \mathbb{N}_0^r$ denote by $D(\underline{n},m)$ the number of paths, which

start at the origin,

have only unit step vectors,

stay in a certain specified region of $\mathbb{N}_0^{r+1}$, and

reach the point $(\underline{n},m) \in \mathbb{N}_0^{r+1}$.

The first kind of regions we look at are sets of points $(\underline{i},j)$ with $j > \nu(\underline{i})$, where ν is a given boundary function. We make the following obvious restrictions for the boundary ν:

ν is integer valued

$\nu(\underline{0}) = -1$ (each path starts at the origin)

$\nu(\underline{n}) \geq \nu(\underline{m})$ $\forall\ \underline{n} \geq \underline{m}$ (no path can go down).

Here $\underline{n} \geq \underline{m}$ means $n_i \geq m_i$ for all $i = 1,\ldots,r$. The following recursion and side conditions uniquely determine the numbers $D(\underline{n},m)$ for the one-sided boundary case: ($\underline{e}_k :=$ k^{th}-unit vector in $\mathbb{Z}^r$)

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$$(1) \qquad D(\underline{n},m) = \begin{cases} D(\underline{n},m-1) + \sum_{k=1}^{r} D(\underline{n}-\underline{e}_k,m) & \forall m > \nu(\underline{n}),\ \underline{n} > \underline{0} \\ 0 & \forall m \leq \nu(\underline{n}),\ \underline{n} \geq \underline{0} \end{cases}$$

and

$$(2) \qquad D(\underline{n},m) = \begin{cases} 1 \text{ if } \underline{n} = \underline{0} \text{ and } m \in \mathbb{N}_0 \\ 0 \quad \forall \underline{n} \in \mathbb{Z}^r \setminus \mathbb{N}_0^r\ ,\ m \in \mathbb{Z}\ . \end{cases}$$

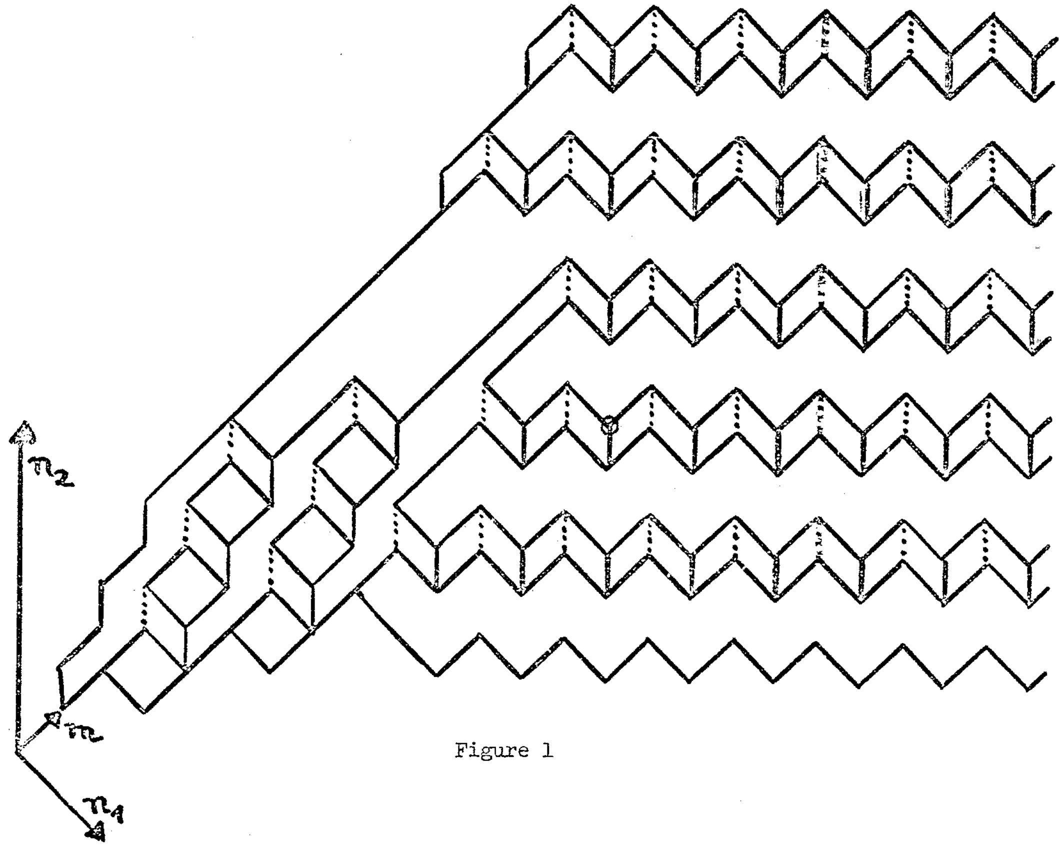


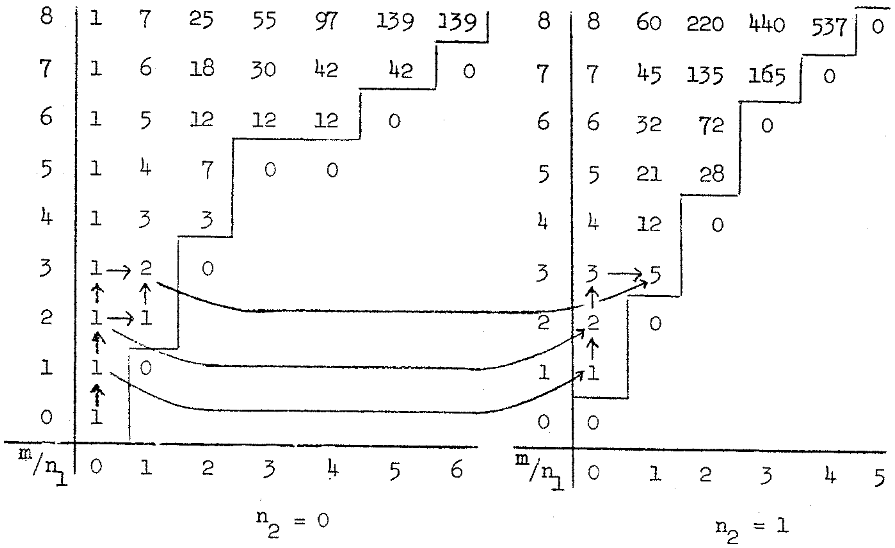
Figure 1

Example 1: Figure 1 shows the region

$$\{(i_1,i_2,j) \mid j > \nu(i_1,i_2)\} \text{ in } \mathbb{N}_0^3 \text{ , with}$$

$$\nu(i_1,i_2) = \begin{cases} 2i_1+i_2-1 & \forall (0,0) \le (i_1,i_2) \le (3,3) \\ i_1+2i_2+1 & \text{else.} \end{cases}$$

For $n_2 = 0$ and $n_2 = 1$ the computation of $D(n_1,n_2,m)$ is demonstrated in the next two tables.



m/n_1	0	1	2	3	4	5	6
8	1	7	25	55	97	139	139
7	1	6	18	30	42	42	0
6	1	5	12	12	12	0	
5	1	4	7	0	0		
4	1	3	3				
3	1	2	0				
2	1	1					
1	1	0					
0	1						

$n_2 = 0$

m/n_1	0	1	2	3	4	5
8	8	60	220	440	537	0
7	7	45	135	165	0	
6	6	32	72	0		
5	5	21	28			
4	4	12	0			
3	3	5				
2	2	0				
1	1					
0	0					

$n_2 = 1$

General results about path enumeration can be found in T.V. Narayana's book [6]. The two sided boundary case (see section 5) is solved with other methods by B. R. Handa and S.G. Mohanty [2].

2. Affine Boundaries.

From the recursion (1) and side conditions (2) follows the existence of a multi-indexed sequence of real polynomials $(d_{\underline{n}})_{\underline{n} \in \mathbb{N}_0}$ with the properties:

$$d_{\underline{n}}(m) = D(\underline{n},m) \quad \forall m \geq \nu(\underline{n}), \quad \underline{n} > 0$$

$$d_{\underline{0}}(x) = 1 \quad \forall x \in \mathbb{R}$$

$$(3) \quad d_{\underline{n}}(x) - d_{\underline{n}}(x-1) = \sum_{k=1}^{r} d_{\underline{n}-\underline{e}_k}(x) \quad \forall x \in \mathbb{R} .$$

Define $d_{\underline{n}}(x) \equiv 0$ if $\underline{n} \in \mathbb{Z}^r \setminus \mathbb{N}_0^r$. $(d_{\underline{n}})$ is a multi-indexed analogy of a Sheffer sequence. The theory of Sheffer sequences is highly developed in the „Finite Operator Calculus" [3]. An abstract of the definitions and theorems needed for two dimensional path counting can be found in [8], where we refer to in the following. We call a sequence of polynomials $(s_{\underline{n}})_{\underline{n} \in \mathbb{N}_0^r}$ a Sheffer r-sequence for the delta operator Q [8, section 3.1], iff $s_{\underline{0}}(x) = c \neq 0 \ \forall x \in \mathbb{R}$ and

$$(4) \qquad Q\, s_{\underline{n}} = \sum_{k=1}^{r} s_{\underline{n}-\underline{e}_k} \qquad \forall \underline{n} \in \mathbb{N}_0^r .$$

$(s_{\underline{n}})$ has roots in ν, say, iff $s_{\underline{n}}(\nu(\underline{n})) = \delta_{\underline{0},\underline{n}} := \begin{cases} 1 & \text{if } \underline{n} = \underline{0} \\ 0 & \text{else} \end{cases}$, where ν is any function from $\mathbb{N}_0^r$ onto $\mathbb{R}$ (not necessary a boundary). The Sheffer r-sequence $(d_{\underline{n}})$ corresponds to the backwards difference delta operator ∇: $\nabla d_{\underline{n}}(x) := d_{\underline{n}}(x) - d_{\underline{n}}(x-1)$, and (4) holds because of (3).

In generalization of [8, (4.1)], we obtain: if ν is any affine function, $\nu(i) = \underline{a}^T \underline{i} + b$ $(\underline{a} \in \mathbb{R}^r,\ b \in \mathbb{R})$, then the Sheffer r-sequence for ∇ with roots in ν is given by

$$(5)\qquad \frac{x-\underline{a}^{T}\underline{n}+b}{x-b}\binom{x-b+\overline{n}-1}{\underline{n}} = \frac{x-\underline{a}^{T}\underline{n}-b}{x-b+\overline{n}}\binom{x-b+\overline{n}}{\underline{n}}$$

For all $x \in \mathbb{R}$ and $\underline{n} \in \mathbb{N}_0^r$, where $\overline{n} := \sum_{k=1}^{r} n_k$ and

$$(6)\qquad \binom{y}{\underline{n}} := \frac{(y-\underline{n}+1)\cdot(y-\underline{n}+2)\cdot\ldots\cdot y}{n_1!\,\cdots\,n_r!}\,, \quad \binom{y}{\underline{0}} = 1\,.$$

For $\underline{a} \in \mathbb{N}_0^r$ and $b = -1$, ν is a boundary, and (5) is the number $D(\underline{n},m)$ of paths, staying strictly over the hyperplane $\{(\underline{n},m) \in \mathbb{Z}^{r+1} \mid m=\underline{a}^{T}\underline{n}-1\}$. See [5] for another proof.

3. Piecewise Affine Boundaries.

Analogously to lemma 4.1 in [8], the following main representation formula holds:

Lemma 1: If $(r_{\underline{n}})$ is a Sheffer r-sequence for Q, then

$$r_{\underline{n}}(x) = \sum_{\underline{j}=\underline{0}}^{\underline{n}} r_{\underline{j}}(\nu(\underline{j}))\,s_{\underline{j},\underline{n}-\underline{j}}(x) \quad \forall\, x \in \mathbb{R}\,,$$

where $(s_{\underline{j},\underline{n}})_{\underline{n} \in \mathbb{N}_0^r}$ is for each $\underline{j} \in \mathbb{N}_0^r$ the Sheffer r-sequence for Q with roots in $\nu_{\underline{j}}(\underline{i}) := \nu(\underline{i}+\underline{j})$.

This lemma and formula (5) yield a theorem which can be used to compute Sheffer r-sequences for ∇ in a recursive way:

Theorem 1: Let $(r_{\underline{n}})$ be the Sheffer r-sequence for ∇ with roots in

$$\nu(\underline{i}) := \begin{cases} \rho(\underline{i}) & \forall\, \underline{i} \in L \\ \underline{c}^{T}\underline{i}+d & \forall\, \underline{i} \in \mathbb{N}_0^r \setminus L\,, \end{cases}$$

where ρ is any function from $\mathbb{N}_0^r$ onto $\mathbb{R}$, $\underline{c} \in \mathbb{N}_0^r$, $d \in \mathbb{R}$, and L is a set in $\mathbb{N}_0^r$ with the property: if $\underline{i} \in L$ and $\underline{0} \leq \underline{j} \leq \underline{i}$, then $\underline{j} \in L$. If the numbers $r_{\underline{i}}\,(\underline{c}^{T}\underline{i}+d)$ are known for all $\underline{i}$ in L, then $r_{\underline{n}}(x)$ can be computed from

$$(7) \qquad r_{\underline{n}}(x) = \sum_{\underline{i} \in L} r_{\underline{i}}(\underline{c}^{T}\underline{i}+d)\, \frac{x-\underline{c}^{T}\underline{n}-d}{x-\underline{c}^{T}\underline{i}-d} \binom{x-d-\underline{c}^{T}\underline{i}+\bar{n}-\bar{i}-1}{\underline{n}-\underline{c}}$$

for all $x \in \mathbb{R}$, $\underline{n} \in \mathbb{N}_0^r$ (see (6) for the symbols $\bar{n}$ and $\binom{y}{\underline{n}}$).

If $\rho(\underline{i}) = \underline{a}^{T}\underline{i}+b$ in the theorem above ($\underline{a} \in \mathbb{R}^r$, $b \in \mathbb{R}$), we obtain from (5) and (7):

$$(8) \qquad r_{\underline{n}}(x) = \sum_{\underline{i} \in L} \frac{(\underline{c}-\underline{a})^{T}\underline{i}+d-b}{\underline{c}^{T}\underline{i}+d-b} \binom{\underline{c}^{T}\underline{i}+d-b+\bar{i}-1}{\underline{i}} \frac{x-\underline{c}^{T}\underline{n}-d}{x-\underline{c}^{T}\underline{i}-d} \binom{x-\underline{c}^{T}\underline{i}-d+\bar{n}-\bar{i}-1}{\underline{n}-\underline{i}}$$

Example 2: In continuation of example 1 we compute $D(4,2,10) = d_{4,2}(10)$. We find $\underline{a}^{T} = (2,1)$, $b = -1$, $\underline{c}^{T} = (1,2)$, $d = 1$ and $L = \{(i_1,i_2) \in \mathbb{N}_0^2\colon 0 \leq i_1 \leq 3,\ 0 \leq i_2 \leq 3\}$. Hence, (8) yields

$$D(4,2,10) = \sum_{i_1=0}^{3} \sum_{i_2=0}^{2} \frac{-i_1+i_2+2}{i_1+2i_2+2} \binom{2i_1+3i_2+1}{i_1,i_2} \frac{1}{9-i_1-2i_2} \binom{14-2i_1-3i_2}{4-i_1,2-i_2}$$

$$= 11541 \quad ((4,2,10) \text{ is the circled point in Figure 1}).$$

Repeated use of theorem 1 yields an explicit expression for each piecewise affine boundary function, if the affine pieces are defined on sets $L_1, L_2, \ldots$ with the property: if $\underline{i} \in L_m$, and $\underline{j} \leq \underline{i}$, then $\underline{j} \in \bigcup_{t=1}^{m} L_t$. But computing $r_{\underline{n}}(x)$ for an $\underline{n} \in L_m$ would need an $(m-1)$-fold summation.

4. A General Ballot Problem.

In a ballot with $r+1$ candidates C_k, the candidate C_{r+1}, say, gets the relative majority of m votes. Given n_k votes for each candidate C_k $(k = 1,\ldots,r)$, what is the probability that candidate C_{r+1} had the relative majority during the whole counting of votes? For $r = 1$ we obtain the ordinary ballot problem. It is well known (and follows from (5) with $\nu(i) := i-1$) that in this case $D(n_1,m) = \frac{m-n_1+1}{m+1}\binom{n_1+m}{n_1}$ is the number of countings (= paths) in question. For the general case we have to compute the Sheffer r-sequence $(d_{\underline{n}})$ for ∇ with roots in $\tilde{\nu}(\underline{i}) := \max\{i_1,\ldots,i_r\}-1$. Beside $r=1$ an explicit solution is only known for $r = 2$ [4]. We will show another proof for this case, and a recursion defining identity for the general case. The expansion

$$d_{\underline{n}}(x) = \sum_{\underline{i} \geq \underline{0}} d_{\underline{i}}(-\bar{i})\binom{x+\bar{n}}{\underline{n}-\underline{i}} \quad \forall\, x \in \mathbb{R} \tag{9}$$

holds for all Sheffer r-sequences for ∇ (choose $\nu(\underline{i}) = -\bar{i}$ in lemma 1). Decompose $\underline{n} \in \mathbb{N}_0^r$ into $(n,\underline{m})$, where $n := n_1$ and $\underline{m} = (n_2,\ldots,n_r)$. Write $d_{n,\underline{m}}$ instead of $d_{\underline{n}}$. Then (9) becomes

$$d_{n,\underline{m}}(x) = \sum_{i \geq 0}\, \sum_{\underline{j} \geq 0} d_{i,\underline{j}}(-i-\bar{j})\binom{x+n+\bar{m}}{n-i,\underline{m}-\underline{j}}\ .$$

Using the abbreviation $\varphi(i,\underline{j}) := d_{i,\underline{j}}(-i-j)$, we obtain from $\nu(\underline{i}) = \max\{i_1,\dots,i_r\}-1$:

$$(10) \qquad \delta_{0,n+\bar{m}} = \sum_{i \geq 0} \sum_{\underline{j} \geq 0} \varphi(i,\underline{j}) \binom{2n+\bar{m}-1}{n-i,\underline{m}-\underline{j}} \quad \forall\, n \geq \max\{m_1,\dots,m_{r-1}\} .$$

The boundary function ν is invariant under permutations of its argument. Therefore, $d_{\underline{n}}$ is invariant under permutations of its index vector $\underline{n}$, and it is sufficient to have an algorithm for computing $d_{n,\underline{m}}$ only for $n \geq \max\{m_1,\dots,m_{r-1}\}$. (10) defines a recursion of this kind. But (10) is unnecessary complicated. An elementary induction over $\bar{m}$ shows that

$$(11) \qquad \delta_{0,n+\bar{m}} = \sum_{i \geq 0} \varphi(i,\underline{m}) \binom{2n+\bar{m}-1}{n-i} \quad \forall\, n \geq \max\{m_1,\dots,m_{r-1}\} .$$

The numbers $\varphi(i,\underline{m})$ are not only invariant under permutations of the vector $(i,\underline{m})$, they are in addition independent of zeros in this vector. Therefore, the numbers computed for an r-dimensional problem can be used in an r+k-dimensional problem. For instance, $\varphi(i,k,j)$ equals $\varphi(i,0,j,0,0,k)$. Some special values (from (11)):

$$\varphi(0,0) = 1$$

$$\varphi(1,0) = -\varphi(0,0)\binom{2+0-1}{1} = -1$$

$$\varphi(n,0) = -\binom{2n-1}{n} + \binom{2n-1}{n-1} - \sum_{i=2}^{n-1} \varphi(i,0)\binom{2n-1}{n-1} = -\sum_{i=2}^{n-1} \varphi(i,0)\binom{2n-1}{n-i}$$

$$= 0 \quad \forall\, n \geq 2 \quad \text{(by induction)} .$$

The following formula for $\varphi(i,j)$ $\forall\, i,j \in \mathbb{N}_1$ is due to G. Kreweras [4, p. 83]:

$$\varphi(i,j) = 2(-1)^{i+j} \frac{(i+j-1)!\,(2(i+j-1)-1)!}{i!\,j!\,(2i-1)!\,(2j-1)!} .$$

Kreweras proved his result by inserting $\varphi(i,j)$ into (9), showing that the polynomial has the right zeros and yields the right recursion. We can insert $\varphi(i,j)$ into (11) and have to prove that

$$(12) \qquad 0 = \sum_{i=1}^{n} 2(-1)^{i+m} \frac{(i+m-1)!\,(2(i+m-1)-1)!}{i!\,m!\,(2i-1)!\,(2m-1)!} \binom{2n+m-1}{n-i} \quad \forall\, n \geq m \geq 2$$

(the case $m = 1$ is elementary).

The following proof of this identity is due to D. Stanton. Using generalized hypergeometric functions, (12) can be written as

$$0 = {}_4F_3\left[\begin{matrix} m+1,\ m,\ m+1/2,\ -n+1 \\ 2,\ 3/2,\ m+n+1 \end{matrix}\right] \quad \forall\, n \geq m \geq 2 .$$

Transform this well-poised ${}_4F_3$ into the Saalschützian
($\langle x\rangle_n := x\cdot(x+1)\cdot\ldots\cdot(x+n-1)$)

$$\frac{\langle n\rangle_{n-1}}{\langle m+n+1\rangle_{n-1}} \, {}_4F_3\left[\begin{matrix} -n+1,\ (m+1)/2,\ m/2+1,\ 3/2-m \\ 2,\ 3/2,\ 3/2-n \end{matrix}\right]$$

(See W.N. Bailey [1], formula 4.5.(1), p. 30).

With transformation 4.6.(2), p. 32 of [1] we obtain

$$ {}_4F_3\left[\begin{matrix}(m+1)/2,\ m/2+1,\ 3/2-m,\ -n+1\\ 3/2,\ 2,\ 3/2-n\end{matrix}\right] = $$

$$ = \frac{\langle 2-m\rangle_{n-1}\langle m+5/2\rangle_{n-1}}{\langle 4\rangle_{n-1}\langle 1/2\rangle_{n-1}}\ {}_4F_3\left[\begin{matrix}5/2,\ m+2,\ 3/2-m,\ -n+1\\ 3/2,\ m+5/2,\ 3+n\end{matrix}\right]. $$

In the nominator of the r.h.s., $\langle 2-m\rangle_{n-1}$ equals zero for all $n \geq m \geq 2$.

5. The Two-Sided Case.

Now we introduce a second (upper) boundary function $\mu\colon \mathbb{N}_0^r \to \mathbb{N}_0$ and count paths in the region $\{(\underline{n},m)\colon \nu(\underline{n}) < m \leq \mu(\underline{n})\}$. Excluding trivial cases, we make the assumptions

$$ \mu(\underline{n}) \geq \mu(\underline{m}) \quad \forall\ \underline{n} \geq \underline{m} \quad \text{and} $$

$$ \nu(\underline{n}) < \max\{\mu(\underline{n}-\underline{e}_k) \mid k = 1,\dots,r\} \quad \forall\ \underline{n} > \underline{0}. $$

The numbers $D(\underline{n},m)$ are now determined by

$$ D(\underline{n},m) = \begin{cases} D(\underline{n},m-1) + \sum\limits_{k=1}^{r} D(\underline{n}-\underline{e}_k,m) & \forall\ \nu(\underline{n}) < m \leq \mu(\underline{n}),\ \underline{n} > \underline{0} \\ 0 & \forall\ m \leq \nu(\underline{n}) \text{ or } m > \mu(\underline{n}),\ \underline{n} \geq \underline{0}, \end{cases} $$

and

$$D(\underline{n},m) = \begin{cases} 1 & \text{if } \underline{n} = \underline{0} \text{ and } 0 \leq m \leq \mu(\underline{0}) , \\ 0 & \text{if } \forall\, \underline{n} \in \mathbb{Z}^r \setminus \mathbb{N}_0^r,\ m \in \mathbb{Z} . \end{cases}$$

A generalization of piecewise polynomial Sheffer functions (Sheffer splines) to r dimensions leads to the following analog of corollary III.1 in [7]:

$$\text{(13)} \qquad \delta_{\underline{0},\underline{n}} = \sum_{\underline{j} \geq \underline{0}} (-1)^{\overline{n}-\overline{j}} \binom{\mu(\underline{j})-\nu(\underline{n})}{\underline{n}-\underline{j}}_{+} D(\underline{j},\mu(\underline{j})) ,$$

where

$$\binom{x}{\underline{n}}_{+} := \begin{cases} \binom{x}{\underline{m}} & \text{if } x \geq 0 , \\ 0 & \text{else} . \end{cases}$$

(13) defines a recursion for $D(\underline{n},\mu(\underline{n}))$. For any $\nu(\underline{n}) < m \leq \mu(\underline{n})$, $D(\underline{n},m)$ can be found by changing μ into $\mu'(\underline{i}) := \min\{m,\mu(\underline{i})\}$.

Let $N(\underline{n})$ be the set of all indices $\underline{m}$ with $\underline{0} \leq \underline{m} \leq \underline{n}$. N has $d := \prod_{k=1}^{r} (n_k+1)$ elements. Arrange this d elements in such a way that $\underline{m}_1 = \underline{0}$, $\underline{m}_d = \underline{n}$ and $\underline{m}_i - \underline{m}_j \in \mathbb{Z}^r \setminus \mathbb{N}_0^r$ for all $1 \leq i < j \leq d$. Using this quasi order (introduced by B. R. Handa and S. G. Mohanty [2] in the path counting context) and Crámer's rule, (13) yields

$$(14) \qquad D(\underline{n},\mu(\underline{n})) = (-1)^{d-1-\overline{n}} \det\binom{\mu(\underline{m}_j) - \nu(\underline{m}_{i+1})}{\underline{m}_{i+1} - \underline{m}_j}_+ \quad i,j = 1,\dots,d-1 \, .$$

For $r=1$, this result was obtained by G. Kreweras 1965 [4, 2.3.7], and by G.P. Steck 1969 [9]. For general r, (13) and (14) were first proved by B.R. Handa and S.G. Mohanty 1979 [2] with other methods.

Example 3 (from [2]). Suppose, $(\underline{n},m)$ is a fixed point in $\mathbb{N}_0^{r+1}$ and we want to compute $D(\underline{n},m)$. The following relations hold between our boundaries ν, μ, and the upper and lower restrictions a,b in [2]:

$$a(\underline{i}) = m-1-\nu(\underline{n}-\underline{i}) \ , \quad \text{and}$$

$$b(\underline{i}) = m-\mu(\underline{n}-\underline{i}) \qquad \forall\ \underline{0} \leq \underline{i} \leq \underline{n} \, .$$

Consider the number of paths which reach $(\underline{n},m) := (1,2,5)$ and cross only through points (i_1,i_2,j) where $j > \max\{3.7 i_1, i_2^2-1\}$. Then $D(1,2,5)$ is determined by the upper boundary $\mu \equiv 5$ and the lower boundary ν with values

$i_1 \backslash i_2$	0	1	2
0	-1	-1	2
1	3	3	3

(The other values of ν are not interesting for this example.) Now we bring $N = \{\underline{m} \in \mathbb{N}_0^2 \,|\, \underline{0} \leq \underline{m} \leq (1,2)\}$ in the order $\underline{m}_1 := (0,0)$, $\underline{m}_2 := (0,1)$, $\underline{m}_3 := (0,2)$, $\underline{m}_4 := (1,0)$, $\underline{m}_5 := (1,1)$, $\underline{m}_6 := (1,2)$ (see [2] for another ordering). From (14) we obtain

$$D(1,2,5) = (-1)^{6-1-3} \det\left(\binom{5-\nu(\underline{m}_{i+1})}{\underline{m}_{i+1}-\underline{m}_j}_+\right)_{i,j=1,\dots,5}$$

$$= \begin{vmatrix} \binom{6}{0,1} & 1 & 0 & 0 & 0 & 0 \\ \binom{3}{0,2} & \binom{3}{0,1} & 1 & 0 & 0 & 0 \\ \binom{2}{1,0} & 0 & 0 & 1 & 0 & 0 \\ \binom{2}{1,1} & \binom{2}{1,0} & 0 & \binom{2}{0,1} & 1 & 0 \\ \binom{2}{1,2} & \binom{2}{1,1} & \binom{2}{1,0} & \binom{2}{0,2} & \binom{2}{0,1} & \end{vmatrix} = 44 \; .$$

Of course, there is a faster method for computing this little example: obsere, that ν can be written as

$$\nu(\underline{i}) = \begin{cases} 4i_1-1 & \forall\ \underline{0} \le \underline{i} \le (1,1) \\ i_1+2 & \text{else} \; . \end{cases}$$

Now (8) yields

$$D(1,2,m) = \sum_{\underline{i}=(0,0)}^{(1,1)} \frac{-3i_1+2+1}{i_1+2+1} \binom{\bar{i}+i_1+2}{\underline{i}} \frac{m-1-2}{m-i_1-2} \binom{3-\bar{i}+m-i_1-2-1}{1-i_1,2-i_2}$$

$$= \binom{2}{0,0} \frac{m-3}{m-2} \binom{m}{1,2} + \binom{3}{0,1} \frac{m-3}{m-2} \binom{m-1}{1,1} = \frac{1}{2}(m-1)(m-3)(m+6) \; .$$

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