Sheffer Polynomials and Linear Recurrences

by

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Introduction. In many special combinatorial problems the hardest part of the solution may be the discovery of an effective recursion - or of any recursion at all! Still, once a recursion has been established, Sheffer polynomials are often a simple and general tool for finding answers in closed form.

In chapters one and two we deal with the initial value problem for linear recurrences. In chapter three we study the boundary problem, where the solution is required to vanish above a given boundary. Under fortunate conditions these functions can be expressed as determinants.

1. Definitions.

Let K be a field with characteristic zero and V a K-vector space of functions $f: M \to K$, where M is any non empty set. A *recurrence* is a linear operator $R: V \to V$ which is zero only on constants:

$$\text{kernel } (R) = \{\text{constant functions}\}.$$

A *recursive sequence* $(r_n)_{n\in\mathbb{N}_0}$ for R is a sequence of functions in V such that

$$r_0 \text{ is a non-zero constant}$$

$$Rr_n = r_{n-1} \quad \forall n \geq 1 .$$

The following two examples will be continued in the next chapters:

Example S1: Let V be a vector space of polynomials in one variable. For real or complex variables, the derivative D is a recurrence, and $(x^n/n!)$ is a recursive sequence for D. The operator

$$E^a: f(x) \mapsto f(x+a)$$

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is called the shift operator. If an operator R has the expansion $R = L(E) := \sum_{i\geq 0} b_i E^{a_i}$, then R is a recurrence iff $\sum_{i\geq 0} b_i = 0$ and $\sum_{i\geq 0} b_i a_i \neq 0$.

Example E1: Replace the variable x in the polynomials above by q^x, if defined. Then $q^{-x}L(E)$ is a recurrence iff $\sum_{i\geq 0} b_i = 0$ and $L(q^n) = \sum_{i\geq 0} b_i q^{na_i} \neq 0 \ \forall n \geq 1$. $(q^{nx}/\prod_{j=1}^{n} L(q^j))_{n\in\mathbb{N}_0}$ is a recursive sequence for $q^{-x}L(E)$.

2. Representation. A recursive sequence (r_n) for R is uniquely defined, if any sequence of initial values $(r_n(\nu_n))_{n\in\mathbb{N}_0}$ is given. We proved in [7]:

Theorem 1: If (r_n) is a recursive sequence for R with initial values $(r_n(\nu_n))$, and if for all $i \geq 0$ $(t_{i,n})_{n\in\mathbb{N}_0}$ are recursive sequences for R such that $t_{i,n}(\nu_{i+n}) = \delta_{0,n}$, then

$$r_n(x) = \sum_{i=0}^{n} r_i(\nu_i)t_{i,n-i}(x) \qquad \forall n \in \mathbb{N}_0 .$$

In view of this theorem it is important to know recursive sequences with initial values equal to zero for $n > 0$. A recursive sequence for R is called the basic sequence (b_n) for R, iff $b_n(0) = \delta_{0,n}$.

Example S2: An operator Q is shift invariant, iff $QE^a = E^aQ$ for all shift operators E^a. A shift invariant recurrence is called a delta operator. Sheffer sequences are recursive sequences for delta operators. The recurrences D and $L(E)$ in example S1 are delta operators. $(x^n/n!)$ is the basic sequence for D. The backward difference operator $\nabla := E^0 - E^{-1}$ has the basic sequence $(\binom{x+n-1}{n})_{n\in\mathbb{N}_0}$. This recurrence, i.e.,

$$s_n(x) = s_n(x-1) + s_{n-1}(x)$$

is very important in path enumeration (see [8]). Applications of theorem 1 for D and ∇ yield the exact distribution of some Rényi-type statistics (see [9]). More examples and properties of delta operators can be found in the "Finite Operator Calculus" of G.-C. Rota, D. Kahaner and A. Odlyzko [10]. Especially useful for our purposes is the following property: If (b_n) is the basic sequence for a delta operator Q, then $((n+1)b_{n+1}(x)/x)_{n\in\mathbb{N}_0}$ is a Sheffer sequence for Q (see [10, p. 702]). Using this, it is elementary to prove that

$$(1)\qquad s_n(x) := (x-un-v)b_n(x-v)/(x-v)$$

defines the Sheffer sequence for Q with initial values $s_n(un+v) = \delta_{0,n}$. Therefore, we get in theorem 1 for the case $\nu_n = un+v$ the Sheffer sequences

$$t_{i,n}(x) = (x-u(n+i)-v)b_n(x-iu-v)/(x-iu-v) .$$

If (ν_n) is constant, theorem 1 reduces to the binomial theorem for Sheffer polynomials [10, p. 700].

As an example we prove the following:

<u>Lemma</u>: Define a double sequence $(a_{n,m})$ of numbers by the following recursion:

$$(2)\qquad a_{n,m} := a_{n,m-1} - a_{n-1,m-2} \quad \text{for all} \quad n \geq 0, \quad m \geq 2 ,$$

where $a_{n,m} := 0$ for all $n < 0$, and $a_{0,0} \neq 0$. If the initial values $a_{n,0} = a_{n,1}$ are given for all $n \geq 0$, then

$$(3)\qquad 2a_{n,i} = a_{n,i-1} + \sum_{k=(n-i+1)_+}^{N} a_{k,0} \binom{2(n-k)-i}{n-k} \frac{i}{i-2(n-k)}$$

where

$$(m)_+ = \begin{cases} m & \text{if} \quad m \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Proof: Let (s_n) be the Sheffer sequence for the delta operator E^1-E^2 with initial values $s_n(0) = a_{n,0}$ for $n \geq 0$.

Now it is straightforward to prove by induction over n and i that for all $n \geq 0$ and $i \geq 0$

(4) $$s_n(i) + s_{n-i}(-i) = 2a_{n,i} - a_{n,i-1} .$$

With $\nu_n = 0 \quad \forall n \in \mathbb{N}_0$ we get from theorem 1:

$$s_n(i) = \sum_{k=0}^{n} s_k(0) b_{n-k}(i) , \quad \text{and}$$

$$s_{n-i}(-i) = \sum_{k=0}^{n-i} s_k(0) b_{n-i-k}(-i) .$$

$b_n(x) := \frac{x}{x-2n} \binom{2n-x}{n}$ defines the basic sequence for $E^1 - E^2 = -E^2\nabla$ (see [10, p. 711]). From

$$b_{n-k}(i) = \frac{i}{i-2(n-k)} \binom{2(n-k)-i}{n-k} = -b_{n-i-k}(-i)$$

and (4) we obtain (3) ∎

Of course, a non-recursive expression for $a_{n,i}$ can be computed from (3). A recursion like (2) occurs in the "problème des ménages" (see I. Kaplansky and J. Riordan [5, (18)]). The initial values are $a_{n,0} = n!$. In this problem we have to determine the number $u_n := 2a_{n,2n} - a_{n,2n-1}$. (3) yields the well known solution $u_n = \sum_{k=0}^{n} k!(-1)^{n-k} \binom{n+k}{n-k} \frac{2n}{n+k}$.

Example E2: An operator τ is Eulerian shift invariant, iff $\tau E^a = q^a E^a \tau$ for all shift operators E^a. An Eulerian shift-invariant recurrence is called Eulerian delta operator, its recursive sequences are Eulerian Sheffer sequences. $q^{-x}L(E)$ in example E1 is an Eulerian delta operator. Aside from minor changes in the terminology the theory of Eulerian delta operators was developed by G.E. Andrews in [2]. The Eulerian operator $\Delta_q := q^{1-x}(E^0 - E^{-1})$ has applications in partition problems and path enumeration (paths with given area underneath, see [7]). The corresponding basic sequence is given by

(5) $$b_n(x) := \left[{n-1+x \atop n}\right] ,$$

where

$$\left[{x \atop n}\right] := \frac{(q^{x-n+1})_n}{(q)_n} \quad \text{and} \quad (z)_n := \prod_{i=0}^{n-1} (1-zq^i) ,$$

$$(z)_0 := 1 \text{ and } \left[{x \atop s}\right] := 0 \quad \forall s \notin \mathbb{N}_0 .$$

The Eulerian binomial theorem can be derived from theorem 1:

<u>Corollary 1</u>: If (s_n) is an Eulerian Sheffer sequence and (b_n) the basic sequence for τ, then

$$s_n(x+y) = \sum_{i=0}^{n} s_i(y) q^{y(n-i)} b_{n-i}(x) \qquad \forall n \geq 0 .$$

Proof: See [2] or [7].

The Eulerian Sheffer sequence (s_n) for τ with initial values $s_n(v-un) = \delta_{0,n}$ is given by

$$s_n(x) = q^{nv-u\binom{n}{2}} b_n^{(u)}(x-v+un) \qquad \forall n \geq 0 ,$$

if $(b_n^{(u)})$ is the basic sequence for τE^u (see [7]). There is no such simple relation between (b_n) and $(b_n^{(u)})$ as in the case of delta operators. Therefore, we don't have an analog of (1). Similar to Theorem 11 in [2] we find the generating function of $b_n^{(u)}(x)$ for special Eulerian operators:

<u>Proposition</u>: If $\tau = q^{-x}L(E)$ and $L(E) = \sum_{i \geq 0} b_i E^{a_i}$ with $\sum_{i \geq 0} b_i = 0$ and $\sum_{i \geq 0} b_i q^{na_i} \neq 0$ for all $n > 1$, then $(b_n^{(u)})$, the basic sequence of τE^u, has the generating function

$$\sum_{n \geq 0} b_n^{(u)}(x) t^n = f^{(u)}(tq^x)/f^{(u)}(t) ,$$

where

$$f^{(u)}(t) := \sum_{n \geq 0} t^n q^{-u\binom{n+1}{2}} / \prod_{j=1}^{n} L(q^j) .$$

Proof: In example E1 we introduced the Eulerian Sheffer sequence

$$s_n(x) := q^{nx} / \prod_{j=1}^{n} L(q^j)$$

for τ. Then $(q^{u\binom{n}{2}} s_n(x-un))_{n\in\mathbb{N}_0}$ is an Eulerian Sheffer sequence for τE^u. Corollary 1, applied to this sequence and $(b_n^{(u)})$ with $y = 0$, gives the desired result ■

For $\tau = \nabla_q$ we get

$$f^{(u)}(t) = \sum_{n\geq 0} (-1)^n t^n q^{\binom{n}{2} - u\binom{n+1}{2}} / (q)_n .$$

Then $f^{(u)}(-tq^{1+u})$ equals $F_{-u}(t)$ in the notation of G.E. Andrews in [1]. Applications to partition problems can also be found in [1].

3. The boundary problem. Now we assume that the set M in chapter 1 is completely ordered. Let $(\mu_n)_{n\in\mathbb{N}_0}$ be a monotone non-decreasing sequence in M. We look for a sequence (F_n) of functions, which are „recursive" with respect to R and vanish for $x > \mu_n$. Precisely,

$$F_n(x) = \begin{cases} r_{n,i}(x) & \text{for all } i \leq n \text{ and } \mu_{i-1} \leq x \leq \mu_i \\ 0 & \text{for all } x > \mu_n , \end{cases}$$

where $Rr_{n,i} = r_{n-1,i}$, and $r_{n,n}$ is a non-zero constant if $\mu_{n-1} < \mu_n$ (define $\mu_{-1} := \inf(M)$ for convenience). $r_{n,i}$ lies in V, but in general F_n does not. We call (F_n) a μ-recursive sequence for R.

Theorem 2: Let (F_n) be a μ-recursive sequence for R. If $\mu_n^{(i)} := \mu_i$ $\forall n \geq 0$, $i \geq 0$, and $(T_{i,n})_{n\in\mathbb{N}_0}$ is the $\mu^{(i)}$-recursive sequence for R with initial values $T_{i,n}(\mu_i) = \delta_{0,n}$, then

$$F_n(x) = \sum_{i\geq 0} F_i(\mu_i) T_{i,n-i}(x) \quad \forall n \geq 0 .$$

Proof: See [7].

Example S3: If R is a delta operator, we get in theorem 2

$$T_{i,n}(x) = b_n(x-\mu_i)_- ,$$

where

$$f(x)_- := \begin{cases} f(x) & \forall x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

If (F_n) has the initial values $F_n(\nu_n) = \delta_{0,n}$, we get from Cramer's rule

$$F_n(\mu_n) = (-1)^n \det(b_{i-j+1}(\nu_i - \mu_{j-1})_-)_{i,j=1,\dots,n} .$$

For $R = \nabla$ the determinant was derived first by G. Kreweras [6]. This case is important for path enumeration restricted by an upper boundary. For $R = D$, μ-recursive sequences occur in order statistics. G.P. Steck derived the determinant for this case in [11].

Example E3: If R is an Eulerian delta operator, we get $T_{i,n}(x) = q^{n\mu_i} b_n(x-\mu_i)_-$ in theorem 2. Consider the following restricted partition problem: For a given non-decreasing sequence $(\tilde{\mu}_n)$ of non-negative integers define the numbers

$$A_\ell(n,m) := \#\{(\lambda_1,\dots,\lambda_n) \mid \lambda_i \in \mathbb{N}_0; \lambda_1 \leq \dots \leq \lambda_n \leq m; \sum_{i=1}^{n} \lambda_i = \ell; \lambda_i \leq \tilde{\mu}_i \text{ for all } i=1,\dots,n\} .$$

Let $A_\ell(n,m)$ be zero if $m > \tilde{\mu}_{n+1}$ or $m < 0$. It is elementary to show that the generating function

$$a_{n,m}(q) := \sum_{\ell \geq 0} A_\ell(n,m) q^\ell$$

satisfies the recursion

$$a_{n,m}(q) = \begin{cases} a_{n,m-1}+q^m a_{n-1,m} & \text{if } 0 \le m \le \tilde{\mu}_{n+1} \\ 0 & \text{otherwise} \end{cases}$$

Define the boundary (μ_n) by $\mu_n := \tilde{\mu}_{n+1}$. For $n > 0$ and $m \geq 0$ we obtain $a_{n,m}(q) = F_n(m)$, if (F_n) is the μ-recursive sequence for $q^{-x}(E^0-E^{-1})$ with initial values $F_n(-1) = \delta_{0,n}$. The corresponding basic sequence is $(q^n[{x+n-1 \atop n}])_{n\in\mathbb{N}_0}$ (see (5)). Theorem 2 yields

$$a_{n,\tilde{\mu}_n}(q) = (-1)^n \det(q^{(i-j+1)(\tilde{\mu}_j+1)}[{i-j-1-\tilde{\mu}_j \atop i-j+1}])_{i,j=1,\ldots,n}$$

$$= \det(q^{\binom{i-j+1}{2}}[{\tilde{\mu}_j+1 \atop i-j+1}])_{i,j=1,\ldots,n} .$$

Theorem 2 has been inspired by the work of B.R. Handa and S.G. Mohanty in [3], who proved the case $R = \nabla_q$.

4. Multi-indexed sequences. Let $\underline{n} = (n_1,\ldots,n_\rho)\in\mathbb{N}_0^\rho$ be an index vector. We call $(r_{\underline{n}})_{\underline{n}\in\mathbb{N}_0^\rho}$ a *recursive ρ-sequence* for the recursion R, iff

$$r_{\underline{0}} \text{ is a non-zero constant, and}$$

$$Rr_{\underline{n}} = \sum_{k=1}^{\rho} r_{\underline{n}-\underline{e}_k} \qquad \forall \underline{n}\in\mathbb{N}_0^\rho ,$$

where $\underline{e}_k$ stands for the k-th unit vector in $\mathbb{Z}^\rho$, and $r_{\underline{n}-\underline{e}_k} \equiv 0$ if $\underline{n}-\underline{e}_k$ has any negative component.

Many of the results listed in the preceeding chapters are generalized to recursive ρ-sequences in [7]. For brevity, we give only two examples.

Example S4: If $\nu_{\underline{n}} = \underline{u}^T\underline{n}+\underline{v}$, then the multi-indexed version of theorem 1 yields for Sheffer ρ-sequences $(s_{\underline{n}})$ for Q:

$$s_{\underline{n}}(x) = \sum_{\underline{i} \geq \underline{0}} s_{\underline{i}}(\underline{u}^T\underline{i}+\underline{v}) \frac{x-\underline{u}^T\underline{n}-\underline{v}}{x-\underline{u}^T\underline{i}-\underline{v}} b_{\underline{n}-\underline{i}}(x-\underline{u}^T\underline{i}-\underline{v}) ,$$

where

$$b_{\underline{n}}(x) = \frac{(n_1+\ldots+n_\rho)!}{n_1!\cdot\ldots\cdot n_\rho!} b_{n_1+\ldots+n_\rho}(x) \quad \forall \underline{n} \in \mathbb{N}_0^\rho ,$$

and $(b_n)_{n\in\mathbb{N}_0}$ is the basic sequence for Q.

The use of Sheffer ρ-sequences in higher dimensional path counting is demonstrated in [8], where new proofs are given for some of the results of B.R. Handa and S.G. Mohanty in [4].

Example E4: As in example E3, we consider partitions of ℓ, but we index the parts of the partition by pairs:

$$0 \leq \lambda_{\underline{i}_1} \leq \cdots \leq \lambda_{\underline{i}_{s+t+1}}$$

such that $(0,0) = \underline{i}_1 < \underline{i}_2 < \cdots < \underline{i}_{s+t+1} < \underline{i}_{s+t+2} = (s,t)\in\mathbb{N}_0^2$ is a maximal chain, i.e., $\underline{i}_{k+1} - \underline{i}_k$ is either $(1,0)$ or $(0,1)$. In addition, we assume that $\lambda_{\underline{i}_k} \leq \mu_{\underline{i}_k}$ $\forall k = 1,\ldots,s+t+1$ for a given non-decreasing integer boundary $(\mu_{\underline{n}})$. Denote the number of such partitions by $H_\ell(s,t)$. We obtain from corollary 4.2 in [7]:

$$\sum_{\ell \geq 0} H_\ell(s,t)q^\ell =$$

$$(-1)^{st}\det\left(q^{\binom{v_{i+1}+w_{i+1}-v_j-w_j}{2}} \frac{(v_{i+1}+w_{i+1}-v_j-w_j)!}{(v_{i+1}-v_j)!(w_{i+1}-w_j)!} \begin{bmatrix} \mu_{v_j,w_j}+1 \\ v_{i+1}+w_{i+1}-v_j-w_j \end{bmatrix} \right)_{i,j=1,\ldots,st+s+t}$$

where the pairs $(0,0) = (v_1,w_1),\ldots,(v_{(s+1)(t+1)},w_{(s+1)(t+1)}) = (s,t)$ are the elements of the integer interval $[(0,0),(s,t)]$ in a quasi-order, such that from $i > j$ follows either v_j-v_i or w_j-w_i is negative. In the matrix above the entry i,j has to be zero if one of the differences is negative. A quasi-order of the required kind can be obtained, if $w_k := k-1 \bmod (t+1)$ and $v_k := (k-1-w_k)/(t+1)$ for all $k = 1,\ldots,(s+1)(t+1)$.

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