

# Recursive Initial Value Problems for Sheffer Sequences

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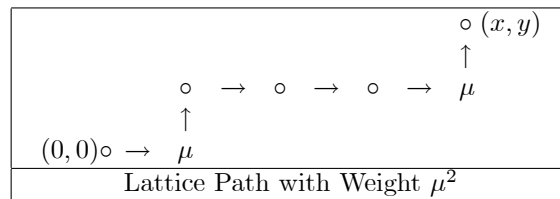
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## Abstract

We consider certain systems of linear operator equations where the  $n$ -th order solution is made unique by an initial value taken from the preceding solution. Such conditions arise in the enumeration of lattice paths with weighted left turns, if the paths are restricted by a lower linear barrier of integer slope.

## 1 Introduction

All our lattice paths take unit steps in the  $\uparrow$  North ( $= y$ ) and  $\rightarrow$  East ( $= x$ ) direction, starting at the origin. Symmetry lies at the center of most of the appealing formulas in lattice paths enumeration, and some symmetries are still preserved when we enumerate paths with weighted left turns  $\rightarrow \overset{\uparrow}{\circ}$ .



However, for such paths the two basic problems of staying below and above a line parallel to the diagonal are no longer symmetric (formulas (7) and (8)). The discrepancy between “below” and “above” gets considerably worse when the line has a slope different from one (formula (6) and Theorem 2). Yet the proof of Theorem 2 shows that there is still enough symmetry left in the recurrence relation to allow significant simplifications.

We enlist the help of the Finite Operator Calculus [3] to solve these problems. Suppose,  $B$  is a degree reducing linear operator on the algebra of polynomials,

like the derivative operator  $D$  or the (forward) difference operator  $\Delta$ . We are interested in polynomial sequences  $\{p_n(x)\}_{n \geq 0}$ ,  $\deg(p_n) = n$ , which solve the system of operator equations

$$Bp_n(x) = p_{n-1}(x)$$

for all positive integers  $n$ .  $\{p_n(x)\}_{n=0,1,\dots}$  is a *Sheffer sequence* if  $B$  is a *delta operator*, a power series  $B(D)$  of order 1 in  $D$ . For example,  $\Delta f(x) := f(x+1) - f(x) = (e^D - 1)f(x)$  (Taylor series) is a delta operator. Another important delta operator for the enumeration of lattice paths with weighted left turns is the generalized backwards difference operator

$$\Omega := \frac{\Delta}{\mu + \Delta} \quad (1)$$

where  $\mu$  is the (formal) weight parameter different from zero. An  $\Omega$ -Sheffer sequence  $\{d_n(x)\}$  must therefore satisfy the recurrence  $\Delta d_n(x) = (\Delta + \mu)\Omega d_n(x) = (\Delta + \mu)d_{n-1}(x)$ ,

$$d_n(x) = d_n(x-1) + d_{n-1}(x) + (\mu - 1)d_{n-1}(x-1). \quad (2)$$

For  $\mu = 1$  this is the well-known backwards difference recurrence for lattice paths with unit steps in the  $y$ - and  $x$ -direction. If the left turns  $\rightarrow \overset{\uparrow}{\circ}$  are weighted with  $\mu$ , the total weight (generating function of the number of left turns) of such paths follows the recurrence (2). If no further restrictions apply then

$$d_n(x) = \sum_{l \geq 0} \binom{n}{l} \binom{x}{l} \mu^l \quad (3)$$

is the total weight of all such paths from the origin to  $(x, n)$ . Note that this solution is symmetric in  $n$  and  $x$  for nonnegative integers, as required by the recursion (2).

The theory of delta operators and Sheffer sequences was developed in the Finite Operator Calculus [3]; an overview of applications to the initial value problem

$$\begin{aligned} Bp_n(x) &= p_{n-1}(x) \text{ for all } n = 1, 2, \dots \\ p_n(cn + \alpha) &= y_n \text{ for all } n = 0, 1, \dots \end{aligned} \quad (4)$$

in combinatorics can be found in [2]. In this paper we solve a *recursive initial value problem* (Theorem 1) where instead of given initial points  $(\alpha, y_0), (c + \alpha, y_1), (2c + \alpha, y_2), \dots$  we have conditions of the form

$$\begin{aligned} Bp_n(x) &= p_{n-1}(x) \text{ for all } n = 1, 2, \dots \\ p_n(cn + \alpha) &= p_{n-1}(cn + \alpha) \text{ for all } n = K, K + 1, \dots \end{aligned} \quad (5)$$

We assume that  $p_0(x) = s_0(x), \dots, p_{K-1}(x) = s_{K-1}(x)$  where  $\{s_n(x)\}$  is some given Sheffer sequence. This sequence may originate from similar initial conditions with parameters  $\tilde{c}, \tilde{\alpha}$ , and  $0 \leq \tilde{K} < K$ . In other words, we could handle

piecewise linear initial inputs at  $\tilde{c}n + \tilde{\alpha}$  for  $n = \tilde{K}, \dots, K - 1$ , and  $cn + \alpha$  for  $n \geq K$ , etc.

Finding the total weight  $t_v(u)$  of all lattice paths with weighted left turns ending at  $(u, v)$  and staying strictly below the line  $x = c(y - K)$  for positive integers  $c$  and  $K$  is an initial value problem of the type (4). We can expand  $t_v(u)$  by applying [2, (19.5)]

$$t_v(u) = \sum_{i=0}^{K-1} d_i(c(i-K)) \frac{u - (v-K)}{u - c(i-K)} b_{v-i}(u - c(i-K)) \quad (6)$$

where  $\{b_n(x)\}$  is the basic sequence (19). The case  $c = 1$  is very special; the symmetry of  $\{d_n(x)\}$  can be used to find the solution

$$t_v(u) = \sum_{l \geq 0} \left( \binom{u}{l} \binom{v}{l} - \binom{u+K}{l} \binom{v-K}{l} \right) \mu^l, \quad (7)$$

which enumerates the weighted paths below  $y = x + K$  for all  $0 \leq v \leq u + K$ .

It is a very different problem to find the total weight  $\{r_n(y)\}$  of paths with weighted left turns staying strictly **above** the line  $y = c(x - K)$ . We will see in Section 3 that we must solve a recursively defined initial value problem of the type (5). For  $m \geq c(n - K) \geq 0$  there are

$$\sum_{i=1}^K \sum_{k=0}^l \binom{1-ci}{l-k} \binom{K-i-1}{l-k-1} \binom{m+ci}{k} \binom{n-K+i+1}{k+1} \frac{ci + m + k(m - c(n - K + 1))}{(n - K + i + 1)(m + ci)}$$

paths with  $l$  left turns that end at  $(n, m + 1)$  staying strictly above the line  $y = c(x - K)$  (see (23)). This representation contains an alternating sum. Formula (22) is non-alternating, but requires more terms for large  $n$ . Again, the case  $c = 1$  is considerably easier (see [1]), giving

$$r_n(m) = \sum_{l \geq 0} \left( \binom{n}{l} \binom{m}{l} - \binom{n-K+2}{l+1} \binom{m+K-2}{l-1} \right) \mu^l \quad (8)$$

for  $m > n - K$ .

## 2 Recursive Initial Values

We want to “solve” the system of delta operator equations

$$Bp_n(x) = p_{n-1}(x) \text{ for all } n = 1, 2, \dots$$

under the initial conditions

$$\begin{aligned} p_n(x) &= t_n(x) \text{ for all } n = 0, 1, \dots, K - 1 \\ p_n(cn + \alpha) &= p_{n-1}(cn + \alpha) \text{ for all } n = K, K + 1, \dots \end{aligned} \quad (9)$$

where  $\{t_n(x)\}$  is some given  $B$ -Sheffer sequence,  $c, \alpha$ , and  $K$  are constants. “Solving” the systems means expanding  $p_n(x)$  in terms of the *basic sequence*  $\{b_n(x)\}$  associated with the delta operator  $B$ . This is the  $B$ -Sheffer sequence with initial values  $b_n(0) = \delta_{0,n}$ , which is usually explicitly known. If  $\beta(t)$  is the compositional inverse of  $B(t)$  then  $\sum_{n \geq 0} b_n(x)t^n = \exp(x\beta(t))$  (see [3]). For the proof of our representation theorem we need the following facts from the Finite Operator Calculus [3].

- The linear operator  $E^{-c}f(x) := f(x - c)$  is an example of a translation operator. Note that  $E^{-c} = e^{-cD}$ ; hence all delta operators are translation invariant,

$$BE^{-c} = E^{-c}B,$$

and  $BE^{-c}$  is again a delta operator.

- If  $\{f_n(x)\}$  is a  $B$ -Sheffer sequence and  $c$  and  $\alpha$  are constants, then

$$\{f_n(cn + \alpha + x)\} \text{ is a Sheffer sequence for } BE^{-c}. \quad (10)$$

- If  $\{b_n(x)\}$  is the basic sequence for  $B$  then

$$\left\{ \frac{x}{x + cn} b_n(x + cn) \right\} \text{ is the basic sequence for } BE^{-c}. \quad (11)$$

- It follows from the previous two results that

$$\left\{ \frac{x - cn}{x} b_n(x) \right\} \text{ is a } B\text{-Sheffer sequence} \quad (12)$$

if  $\{b_n(x)\}$  is the basic sequence for  $B$ .

- *Binomial Theorem for Sheffer Sequences:*

$$f_n(x + y) = \sum_{i=0}^n f_i(x) b_{n-i}(y). \quad (13)$$

### Theorem 1

$$p_n(x) = t_n(x) - \sum_{i=K}^n (t_i(ci + \alpha) - t_{i-1}(ci + \alpha)) \sum_{j=0}^{n-i} \frac{x - \alpha - c(i + j)}{x - \alpha - ci} b_j(x - \alpha - ci)$$

is the solution to the recursive initial value problem (9).

**Proof.** If we write

$$p_n(x) = t_n(x) - q_{n-K}(x - \alpha)$$

then  $\{q_n(x)\}$  is also a Sheffer sequence for the delta operator  $B$ , satisfying the condition

$$q_n(cn + cK) - q_{n-1}(cn + cK) = t_{n+K}(cn + cK + \alpha) - t_{n+K-1}(cn + cK + \alpha) \quad (14)$$

for all  $n \geq 0$ .

If we define

$$\begin{aligned} w_n(x) &= q_n(cn + cK + x) - q_{n-1}(cn + cK + x) \\ &= (1 - B)q_n(cn + cK + x) \text{ for all } n \geq 0 \end{aligned} \quad (15)$$

then  $\{w_n(x)\}$  is a Sheffer sequence for  $BE^{-c}$  because of (10).  $\{w_n(x)\}$  has the initial values

$$w_n(0) = t_{n+K}(cn + cK + \alpha) - t_{n+K-1}(cn + cK + \alpha). \quad (16)$$

We can now apply the Binomial Theorem (13) to expand  $w_n(x)$  in terms of  $\left\{\frac{x}{x+cn}b_n(x+cn)\right\}$ , the basic sequence of  $BE^{-c}$  (see (11))

$$w_n(x) = \sum_{i=0}^n w_{n-i}(0) \frac{x}{x+ci} b_i(x+ci).$$

Because of (15),

$$\begin{aligned} q_{n-K}(cn+x) &= \frac{1}{1-B} w_{n-K}(x) \\ &= \sum_{i=0}^{n-K} w_{n-K-i}(0) \frac{1}{1-B} \frac{x}{x+ci} b_i(x+ci) \\ &= \sum_{i=0}^{n-K} w_{n-K-i}(0) \left( \sum_{j \geq 0} B^j \right) \frac{x}{x+ci} b_i(x+ci) \\ &= \sum_{i=0}^{n-K} w_{n-K-i}(0) \sum_{j=0}^i \frac{x+cj}{x+ci} b_{i-j}(x+ci) \end{aligned} \quad (17)$$

$$(18)$$

using the fact that  $\frac{x}{x+ci} b_i(x+ci)$  can also be interpreted as the  $B$ -Sheffer polynomial  $\frac{x-ci}{x} b_i(x)$  (see (12)) evaluated at  $x+ci$ .

Finally

$$\begin{aligned} p_n(x) &= t_n(x) - q_{n-K}(cn+x-\alpha-cn) \\ &= t_n(x) - \sum_{i=0}^{n-K} w_{n-K-i}(0) \sum_{j=0}^i \frac{x-\alpha-c(n-j)}{x-\alpha-c(n-i)} b_{i-j}(x-\alpha-c(n-i)). \end{aligned}$$

■

### 3 Lattice Paths with Weighted Left Turns above a Line

In order to find the weight  $r_n(y)$  of the ballot paths above the line  $y = c(x - K)$  it is no longer admissible to use the initial value  $r_n(c(n - K)) = 0$  for all  $n \geq K$ ,

because the recurrence  $r_n(y+1) = r_n(y) + r_{n-1}(y+1) + (\mu-1)r_{n-1}(y)$  requires that we can “walk through” the point  $(n, y)$  – we cannot block this passage with an initial value  $r_n(c(n-K)) = 0$ . Instead, we must work with the recursive initial condition (5) where  $\alpha = 1 - cK$

$$r_n(c(n-K) + 1) = r_{n-1}(c(n-K) + 1)$$

for all  $n \geq K$  (bold entries in the table below).

Total weights of paths with $\mu$ -weighted left turns above the line $y = 2(x - 4)$ (the columns show the values of the polynomials $r_n(y)$ )							
6	1	$1 + 6\mu$	$1 + 12\mu + 15\mu^2$	$1 + \dots 20\mu^3$	$1 + \dots 15\mu^4$	$1 + \dots 6\mu^5$	$1 + \dots \mu^6$
5	1	$1 + 5\mu$	$1 + 10\mu + 10\mu^2$	$1 + \dots 10\mu^3$	$1 + \dots 5\mu^4$	<b><math>1 + \dots \mu^5</math></b>	<b><math>1 + \dots \mu^5</math></b>
4	1	$1 + 4\mu$	$1 + 8\mu + 6\mu^2$	$1 + \dots 4\mu^3$	$1 + \dots \mu^4$	$1 + \dots 5\mu^4$	
3	1	$1 + 3\mu$	$1 + 6\mu + 3\mu^2$	$1 + \dots \mu^3$	<b><math>1 + \dots 4\mu^3</math></b>	<b><math>1 + \dots 4\mu^3</math></b>	
2	1	$1 + 2\mu$	$1 + 4\mu + \mu^2$	$1 + \dots 3\mu^2$	$1 + \dots 6\mu^2$		
1	1	$1 + \mu$	$1 + 2\mu$	<b><math>1 + 3\mu</math></b>	<b><math>1 + 3\mu</math></b>		
0	1	1	1	1			
$y \uparrow$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$

For the first few degrees the lower bound has no effect,

$$r_n(y) = d_n(y) = \sum_{l=0}^n \binom{n}{l} \binom{y}{l} \mu^l$$

for all  $n = 0, \dots, K - 1$ . We can apply Theorem 1 to expand  $r_n(y)$ ; however, the delta operator allows a simplified expansion.

**Theorem 2**

$$\begin{aligned} r_n(y+1) &= d_n(y+1) - \frac{1}{\mu} \sum_{i=0}^{n-K} b_{K+i}(ci+1) \Delta \frac{y-c(n-K)}{y-ci} b_{n-K-i+1}(y-ci) \\ &= \frac{1}{\mu} \sum_{i=1}^K b_{K-i}(1-ci) \Delta \frac{y-c(n-K)}{y+ci} b_{n-K+i+1}(y+ci) \end{aligned}$$

where

$$d_n(y) = \sum_{l=0}^n \binom{n}{l} \binom{y}{l} \mu^l \quad \text{and} \quad b_n(y) = \sum_{l=0}^n \binom{n-1}{n-l} \binom{y}{l} \mu^l.$$

**Proof.** We begin by substituting  $\{d_n\}$  for  $\{t_n\}$  in (16).

$$\begin{aligned} w_n(0) &= d_{n+K}(cn + cK + \alpha) - d_{n+K-1}(cn + cK + \alpha) \\ &= \sum_{l=0}^{n+K} \binom{n+K}{l} \binom{cn+1}{l} \mu^l - \sum_{l=0}^{n+K-1} \binom{n+K-1}{l} \binom{cn+1}{l} \mu^l \\ &= \sum_{l=1}^{n+K} \binom{n+K-1}{l-1} \binom{cn+1}{l} \mu^l. \end{aligned}$$

$\Omega$  now plays the role of the delta operator  $B$  in Theorem 1, and  $\Omega$  has the basic sequence

$$b_n(x) = \sum_{l=0}^n \binom{x}{l} \binom{n-1}{n-l} \mu^l \text{ for } n = 0, 1, \dots \quad (19)$$

Hence  $\{w_n\}$  has the initial values

$$w_n(0) = b_{n+K}(cn + 1).$$

The enumeration of (unrestricted) lattice paths with  $\mu$ -weighted left turns is very special, because they are counted by Sheffer polynomials  $d_n(x)$  (see (3)) which are symmetric,  $d_n(m) = d_m(n)$ , for all nonnegative integers  $m$  and  $n$ . We have shown [2] that there is no other Sheffer sequence with this property! The operator equation  $\Delta = \mu\Omega/(1-\Omega)$  (see (1)) is a consequence of that symmetry, and it enables us to circumvent the expansion of  $1/(1-B)$  in (17). Note that

$$\begin{aligned} \Omega \frac{x-c}{x+ci} b_{i+1}(x+ci) &= \Omega E^{ci} \frac{x-c(i+1)}{x} b_{i+1}(x) \\ &= E^{ci} \frac{x-ci}{x} b_i(x) = \frac{x}{x+ci} b_i(x+ci). \end{aligned}$$

Hence we evaluate (17) as

$$\begin{aligned} q_{n-K}(cn+x) &= \frac{1}{\mu} \sum_{i=0}^{n-K} w_{n-K-i}(0) \frac{\mu\Omega}{1-\Omega} \frac{x-c}{x+ci} b_{i+1}(x+ci) \\ &= \frac{\Delta}{\mu} \sum_{i=0}^{n-K} b_{n-i}(c(n-K-i)+1) \frac{x-c}{x+ci} b_{i+1}(x+ci). \quad (20) \end{aligned}$$

We can apply the Binomial Theorem (13) with  $B := \Omega E^{-c}$  for faster calculation of this formula if  $K$  is small,

$$\begin{aligned} &\frac{\Delta}{\mu} \sum_{i=1}^{n+1} b_{n+1-i}(c(n+1-K-i)+1) \frac{x-c}{x-c+ci} b_i(x-c+ci) \\ &= \frac{\Delta}{\mu} b_{n+1}(c(n+1)-cK+1+x-c) \\ &= d_n(c(n-K)+1+x). \quad (21) \end{aligned}$$

Apply (20) or (21) to  $q_{n-K}(y+cK)$  in

$$r_n(y+1) = d_n(y+1) - q_{n-K}(y+1-\alpha) = d_n(y+1) - q_{n-K}(y+cK)$$

to finish the proof. ■

We obtain a more detailed expansion of  $r_n(y)$  if we explicitly calculate the difference in Theorem 2. Substitution of

$$\begin{aligned} \Delta \frac{x-cj}{x} b_j(x) &= \frac{x+1-cj}{x+1} b_j(x+1) - \frac{x-cj}{x} b_j(x) \\ &= \frac{1}{jx} \sum_{l=1}^j \binom{x}{l-1} \binom{j}{l} \mu^l (cj+l(x-cj)) \end{aligned}$$

into (20) gives

$$q_{n-K}(y + cK) = \sum_{l=0}^n \mu^l \sum_{i=0}^{n-K} \sum_{k=0}^l \binom{ci+1}{l-k} \binom{K+i-1}{l-k-1} \binom{y-ci}{k} \binom{n-K-i+1}{k+1} \frac{(y-c(n-K+1))k-ic+y}{(n-K-i+1)(y-ci)}$$

and

$$\begin{aligned} r_n(y+1) &= \sum_{l=0}^n \mu^l \binom{n}{l} \binom{y+1}{l} - \sum_{l=0}^n \mu^l \sum_{i=0}^{n-K} \sum_{k=0}^l \binom{ci+1}{l-k} \binom{K+i-1}{l-k-1} \times \\ &\quad \binom{y-ci}{k} \binom{n-K-i+1}{k+1} \frac{(y-c(n-K+1))k-ic+y}{(n-K-i+1)(y-ci)}. \end{aligned} \quad (22)$$

With the help of (21) we find a second expansion in alternating terms

$$\begin{aligned} r_n(y+1) &= \sum_{l=0}^n \mu^l \sum_{i=1}^K \sum_{k=0}^l \binom{1-ci}{l-k} \binom{K-i-1}{l-k-1} \times \\ &\quad \binom{y+ci}{k} \binom{n-K+i+1}{k+1} \frac{ci+y+k(y-c(n-K+1))}{(n-K+i+1)(y+ci)}. \end{aligned} \quad (23)$$

Note that we have to interpret  $\frac{0}{0} \binom{0}{0}$  as 1 in both formulae.

Formula (22) is of the form *number of paths above = all paths minus number of crossing paths*. Extracting the coefficient of  $\mu^l$  we get

$$\sum_{i=0}^{n-K} \sum_{k=0}^l \binom{ci+1}{l-k} \binom{K+i-1}{l-k-1} \binom{m-ci}{k} \binom{n-K-i+1}{k+1} \frac{(m-c(n-K+1))k-ic+m}{(n-K-i+1)(m-ci)}$$

paths with  $l$  left turns that end at  $(n, m+1)$  after reaching the line  $y = c(x-K)$  somewhere.

## References

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