Rota's Umbral Calculus and Recursions

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Abstract. Umbral Calculus can provide exact solutions to a wide range of linear recursions. We summarize the relevant theory and give a variety of examples from combinatorics in one, two and three variables.

1. Introduction

What kind of recursions can we hope to solve with Rota's Umbral Calculus (UC)? We like to call them *delta recursions*; however this cannot be explained without some UC terminology. Therefore the answer is postponed until the end of this introduction.

As a first approximation we may say that Rota's Umbral Calculus is about an isomorphism between formal power series (f), linear functionals (L), and a certain class of (shift invariant) linear operators (S) on polynomials (p).

The isomorphisms (and an example)			
	$f(t)$	$=$	$\sum_{m\geq 0} a_m t^m$
(Exponential	e^{at}	$=$	$\sum_{m\geq 0} \frac{a^m}{m!} t^m$)
	$\nearrow\!\!\swarrow$		$\nwarrow\!\!\searrow$
	$\left\langle L \mid \frac{x^m}{m!} \right\rangle = a_m$	$\longleftrightarrow$	$Sp(x) = \sum_{m\geq 0} a_m \frac{d^m}{dx^m} p(x)$
(Evaluation:	$\left\langle \mathrm{Eval}_a \mid \frac{x^m}{m!} \right\rangle = \frac{a^m}{m!}$		$E^a p(x) = p(x+a)$ Shift operator)

Common ground to the three concepts are special polynomial sequences, called Sheffer sequences. A polynomial sequence $(s_m(x))_{m\in\mathsf{N}_0}$ is a sequence of polynomials $s_m(x) \in \mathsf{K}[x]$ such that $\deg s_m = m$, $s_0 \neq 0$. It is convenient to define $s_m = 0$ for negative m. The coefficient ring K is assumed to be of characteristic 0. For this paper it suffices to choose K as $\mathsf{R}[\omega]$, the ring of polynomials in some weight parameter ω. A formal power series $g(t)$ of order 1, i.e., $g(0) = 0$ and $g'(0)$ is a unit in K, will be called a *delta series*. We substitute the derivative operator $\mathcal{D}$ for t in a delta series, and obtain a shift-invariant linear operator Q called a *delta operator*. The derivative $\mathcal{D}$ itself is a delta operator, and like $\mathcal{D}$ every delta operator Q reduces degrees by one, and has its null-space equal to the constant polynomials. The solutions to the system of operator equations

$$Qs_m(x) = s_{m-1}(x)$$

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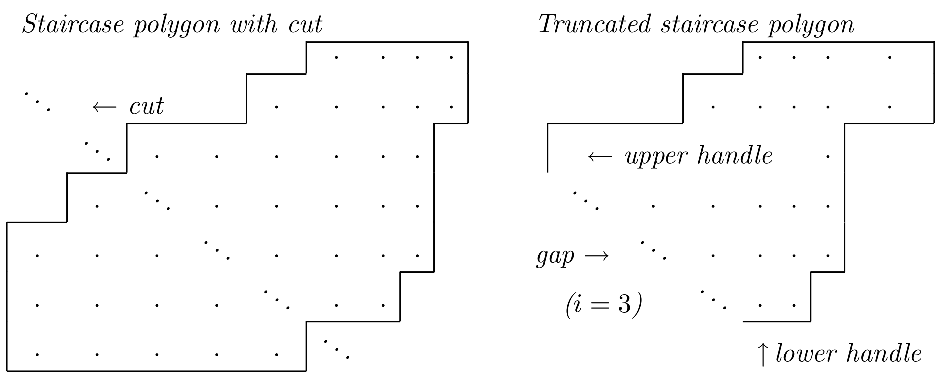
are therefore only determined up to constants; every polynomial sequence (s_m) solving that system is a Q - *Sheffer sequence* (note that $m!s_m(x)$ is a Sheffer polynomial in the sense of Rota, Kahaner and Odlyzko [18]). Initial values determine the constants, and if the initial values are $s_m(x) = {}_{0,m}$, this special Sheffer sequence is called the Q - *basic sequence*. Umbral Calculus can be used as a tool for solving recursions, if the exact solutions to such recursions are Sheffer sequences. In that case, the recursion will lead to an operator equation involving the (unknown) delta operator Q, and the Q-basic sequence is expanded with the help of the Transfer Formulas (2.1) or (2.3) below. It is rare that an interesting combinatorial problem can be solved by a basic sequence, but the following example is an exception.

Example 1. *A staircase polygon (parallelogram polyominoe) can be viewed as a pair of lattice paths with steps $\rightarrow$ and $\uparrow$, starting and ending at a common point without intersections in between, thus forming a polygon. In order to count the number c_{2n} of staircase polygons with circumference $2n$, we actually enumerate truncated (by a line with slope -1) staircase polygons which have a diagonal gap $i > 0$ (from truncation) and call their number $b_m(i)$ if the remaining piece of the original staircase circumference is $2i + 2m$. Note that $2i$ is the smallest possible remaining circumference of a truncated staircase polygon with gap i, thus $b_0(i) = 1$. The picture below explains the terminology and shows what we mean by the "handles" of the truncated polygon.*



If we move the truncation line one unit to the right, we lose the old handles, and the gap changes according to their direction,

$$b_m(i) = \underbrace{b_m(i-1)}_{\text{both handles inwards}} + \underbrace{b_{m-2}(i+1)}_{\text{both handles outwards}} + 2\underbrace{b_{m-1}(i)}_{\text{handles parallel}}$$

(Conway, Delest, and Guttmann [2]). The initial values $b_m(0) = 0$ for $m > 0$ guarantee that there always is a gap. The above recursion can be interpreted as a system of difference equations, and from $b_0(i) = 1$ follows that the solution can be extended to a polynomial sequence $(b_m(x))_{m \in \mathsf{N}_0}$. Define the operator B by linear extension of its action on the basis (b_m), $Bb_m(x) := b_{m-1}(x)$ for all m. The recursion $b_m(x) = b_m(x-1) + b_{m-2}(x+1) + 2b_{m-1}(x)$ is equivalent to the operator

identities

$$1 = E^{-1} + E^1 B^2 + 2B \quad \text{or}$$
$$\nabla = E^1 B^2 + 2B$$

where $\nabla = 1 - E^{-1}$ *is the backwards difference operator, a delta operator with basic sequence* $\binom{x-1}{0}, \binom{x}{1}, \binom{x+1}{2}, \ldots$. *The Transfer Theorem 2 shows that* B *is also a delta operator, and its basic sequence is our solution sequence* $(b_m(x))$ *because of the initial values* $b_m(0) = {}_{0,m}$. *By formula (2.3)*

$$b_m(x) = x \sum_{i=1}^{m} [B^m] \left(E^1 B^2 + 2B\right)^i \frac{1}{x} \binom{i-1+x}{i} = \sum_{j=0}^{m} \binom{j}{m-j} \frac{2^{2j-m} x}{m+x} \binom{m+x}{j}$$

($[t^m] f(t)$ *stands for the coefficient of* t^m *in* $f(t)$*), and by (2.5)*

$$\sum_{m \geq 0} b_m(x) t^m = \left(\frac{1}{2} - t + \sqrt{\frac{1}{4} - t}\right)^{-x}.$$

The total number c_{2n} *of staircase polygons of circumference* $2n$ *equals the number of truncated staircase polygons of circumference* $2n-2$ *and gap* 1*,*

$$c_{2n} = b_{n-2}(1) = \sum_{j=0}^{n-2} \binom{j}{n-2-j} \frac{2^{2j-n+2}}{n-1} \binom{n-1}{j} = \frac{1}{n} \binom{2n-2}{n-1}$$

(Levine [7]*), a Catalan number.*

If necessary and possible, the initial conditions are formulated as a functional on $\mathsf{K}[x]$, and the Functional Expansion Theorem 3 expands the Sheffer polynomials s_m in terms of the basic sequence. We call this the "bottom up" approach: Starting with simple "building blocks" (in combinatorics they usually are binomial coefficients) we construct the basic sequence, and from this more advanced material the exact solution is put together such that the initial conditions are satisfied. In the more commonly applied "top down" approach a generating function identity is solved, and the exact solution may be obtained by finding coefficients. We will show that in many applications UC delivers the generating function too; the more interesting cases may be those where that is not the case (Theorem 2). While the theory behind the Umbral Calculus needed for this type of applications is so simple that we could as well omit it, the "bottom up" approach itself needs some examples to demonstrate its power. As always, the elegant examples are easy in most approaches, and the hard applications are too long to present. We hope that the rather long example in Section 4.1 gives a glimpse of the possible applications. The following examples are discussed in this paper.

# of variables	Topic	Section
One	Generalized elevated Schröder paths	3.1
Two	Staircase polygons	1 and 2.1
Two	Tiling a $4 \times n$ rectangle with m squares of size 2×2	2.2
Two	Lattice paths with infinite step set and special access	4.1
Three	A lattice walk with frequent stops	5.2

All the above examples came from combinatorics and are discrete. However, recursions like

$$\frac{d}{dx} g_m(x) = g_{m-1}(x - \varepsilon) + \omega \frac{d}{dx} g_{m-2}(x - \quad) \tag{1.1}$$

are also in the scope of this paper. Theorem 2 expands the basic solution (initial values $g_m(0) = \delta_{0,m}$) as

$$g_m(x) = \sum_{k=0}^{(m-1)/2} \binom{m-k-1}{k} \frac{x\omega^k (x - (m-2k)\varepsilon - k \quad)^{m-2k-1}}{(m-2k)!}.$$

Returning to the question at the beginning of this introduction we can say that UC tools can be applied to delta recursions, i.e., recursions that are solved by polynomial sequences, and lead to operator identities where some (known) delta operator is expressed as a delta series   in some other (unknown) delta operator. The coefficients of the delta series may be linear operators themselves, as long as they can be expanded as power series in $\mathcal{D}$, and the first term in the delta series is invertible. For example, in the differential equation (1.1) the derivative operator $\mathcal{D}$ is expressed as the delta series $\ (Q) = E^{-\varepsilon} Q / \left(1 - \omega E^{-\ } Q^2\right) = E^{-\varepsilon} Q + \dots$, where $Q g_m = g_{m-1}$. We do not need to know more about Q to find its basic sequence! If the initial values are more involved than $g_m(0) = \delta_{0,m}$, we will in general no longer be able to solve the system. However, Theorem 3 can be used for expansions of the solution if the initial values can be expressed in terms of a functional L that does not vanish on constants.

2. Theory

The Finite Operator Calculus (Rota, Kahaner, and Odlyzko [18]) contains almost all the Umbral Calculus we need for this paper, except for Theorem 3. We abbreviate this "classical" umbral calculus by UC. There are many generalizations (for example [1],[5], and [22]) and variants (e.g. Zeilberger [26], and subsequent papers). The dynamic survey on umbral calculus, www.combinatorics.org/Surveys/ds3.pdf, by Di Bucchianico and Loeb [3], has more than 500 references.

We cite from [18] that a Q-Sheffer sequence $(s_m)_{m \in \mathbb{N}_0}$ has a generating function of the form

$$\sum_{m \geq 0} s_m(x) t^m = \rho(t) e^{x\beta(t)}$$

where $\rho(t) = \sum_{m \geq 0} s_m(0) t^m$ is a power series of order 0, $\beta(t)$ is a delta series, and the delta operator Q can be expanded in terms of the derivative $\mathcal{D}$ as $Q = \beta^{-1}(\mathcal{D})$, where $\beta^{-1}(t)$ is the compositional inverse of β, thus $\beta^{-1}(\beta(t)) = t$. If the Q - basic sequence (b_m) is known, we can obtain $\beta(t)$ directly from

$$\beta(t) = \ln\left(\sum_{m\geq 0} b_m(1)\, t^m\right).$$

However in many applications none of these terms are explicitly given; only certain relationships among them are known.

2.1. The Basic Sequence. The backwards difference operator ∇ and the derivative $\mathcal{D}$ are so simple that their respective basic sequences can be guessed. If we cannot guess the B-basic sequence $(b_m)_{m\in\mathsf{N}_0}$ of some delta operator B it may be possible to connect (b_m) with a known Q-basic sequence (q_m), say. This is helpful, because we need not expand delta operators in powers of $\mathcal{D}$; a linear operator is a delta operator iff it can be written as a delta series in terms of any other delta operator.

Theorem 1 (Transfer Theorem). *Let Q be a delta operator with Q-basic sequence $(q_m)_{m\in\mathsf{N}_0}$. If $Q = \ (B)$ for some delta series $\ \in \mathsf{K}[[B]]$ and linear operator B, then B is also a delta operator, and the B - basic sequence $(b_m)_{m\in\mathsf{N}_0}$ has for all $m \in \mathsf{N}_0$ the expansion*

$$b_m(x) = \sum_{i=0}^{m} c_{m,i}\, q_i(x) \tag{2.1}$$

where $c_{m,i}$ is the coefficient of B^m in $\ (B)^i$. For the generating function of the basic sequence (b_m) holds

$$\sum_{m\geq 0} b_m(x)\, t^m = \sum_{i\geq 0} q_i(x) \quad (t)^i. \tag{2.2}$$

The proof of this theorem is implicitly contained in [18]; because it is so short we reproduce it here.

Proof. There exists a delta series $\gamma(t)$ such that $\sum_{m\geq 0} q_m(x)\, t^m = e^{x\gamma(t)}$ and $Q = \gamma^{-1}(\mathcal{D})$. Note that B is a delta operator because B can be expanded as a delta series in $\mathcal{D}$, i.e., $B = \ ^{-1}(Q) = \ ^{-1}\left(\gamma^{-1}(\mathcal{D})\right)$. Hence

$$\sum_{m\geq 0} b_m(x)\, t^m = e^{x\gamma(\ (t))} = \sum_{i\geq 0} q_i(x) \quad (t)^i = \sum_{m\geq 0} t^m \sum_{i=0}^{m} q_{m-i}(x)\left([t^m] \quad (t)^{m-i}\right).$$

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Denote the ring of shift invariant operators by Σ. Delta operators can be expanded as delta series $\ \in \Sigma[[t]]$ if $[t] \quad (t)$ is invertible; however this expansion is no longer unique and it often generates interesting identities. With the help of Lagrange-Bürmann inversion the expansion (2.1) generalizes to the following theorem.

Theorem 2 (Generalized Transfer Formula). *Let Q be a delta operator with Q - basic sequence $(q_m)_{m\in\mathsf{N}_0}$. If $C_{m,i}$ is the coefficient of B^m in $\quad(B)^i$, where $\quad$ is the delta series $Q = \quad(B) \in \Sigma[\![B]\!]$, then B is also a delta operator. The B - basic sequence $(b_m)_{m\in\mathsf{N}_0}$ has for positive m the expansion*

$$b_m(x) = x\sum_{i=1}^{m} C_{m,i}\frac{1}{x}q_i(x). \tag{2.3}$$

See [11] for a proof. The case $\quad(B) = BT^{-1}$, for some invertible shift-invariant operator T, leads to the Transfer Formula in [17, p.133], also called Rodrigues formula. The factor x in this formula will not cancel in general, because $C_{m,i}$ is an operator. For example, we can write the forward difference operator $\Delta : p(x) \mapsto p(x+1) - p(x)$ in terms of the backwards difference ∇,

$$\Delta = E^1\left(1 - E^{-1}\right) = E^1\nabla \in \Sigma[\![\nabla]\!].$$

The Δ-basic sequence can be guessed as $\binom{x}{0}, \binom{x}{1}, \ldots$. By (2.3)

$$x\sum_{i=1}^{m}\left([\nabla^m]\left(E^1\nabla\right)^i\right)\frac{1}{x}\binom{x}{i} = x\frac{1}{x+m}\binom{x+m}{m} = \binom{m-1+x}{m}$$

is the m-th basic polynomial, which we already used in the staircase example in the Introduction. This example is a special case of a more general result. If $Q = E^aB$ and $\sum_{m\geq 0} q_m(x)t^m = e^{x\gamma(t)}$, then for all $m > 0$

$$b_m(x) = \sum_{i=1}^{m} x\left([B^m]\quad\left(E^aB\right)^{m-i}\right)\frac{1}{x}q_i(x) = xE^{am}\frac{1}{x}q_m(x) = \frac{x}{x+\alpha m}q_m(x+\alpha m). \tag{2.4}$$

Remark 1. *If $Q_{(a)} := E^{-a}Q$, then $q_m^{(a)}(x) = xq_m(x+am)/(x+am)$ is the $Q_{(a)}$ - basic polynomial (see (2.4)). The Abel polynomials $x(x+am)^{m-1}/m!$ are the $\mathcal{D}_{(a)}$ - basic polynomials. The process of moving from Q to $Q_{(a)}$ generates many identities; we call this process "Abelization". Note that $(s_m(x+am))_{m\in\mathsf{N}_0}$ is a $Q_{(a)}$-Sheffer sequence iff (s_m) is a Q - Sheffer sequence. We write $s_m^{[a]}(x)$ for $s_m(am+x)$. In general, if $q_m^{(a)}(x) = xq_m(x+am)/(x+am)$ is the $Q_{(a)}$-basic polynomial, then*

$$t_m(x) := q_m^{(a)}(x-am) = \frac{x-am}{x}q_m(x)$$

is a Sheffer polynomial for the delta operator $E^aQ_{(a)} = Q$. Note that $t_m(am) = \delta_{0,m}$. In symbols, $\left(q_m^{(a)[-a]}\right)$ is a Q - Sheffer sequence.

The notation $Q_{(a)}, q_m^{(a)}$ and $s_m^{[a]}$ will be used later in the paper. If (q_m) has the generating function $e^{x\beta(t)}$ we will denote the generating function of $\left(q_m^{(a)}\right)$ by $e^{x\beta_{(a)}(t)}$.

Unfortunately, Theorem 2 does not tell us the generating function $\sum_{m\geq 0} b_m(x)t^m$ as Theorem 1 does. However, in many applications the only operator occurring among the coefficients of are shift operators, and it may be possible to express E^1 as $\gamma(B)$, say, using only scalar coefficients. In that case $\nabla = 1 - 1/\gamma(B)$, and

$$\sum_{m\geq 0} b_m(x)\,t^m = \sum_{i\geq 0}\binom{i-1+x}{i}\left(1-\frac{1}{\gamma(t)}\right)^i = \gamma(t)^x \tag{2.5}$$

by Theorem 1. For example, in the introductory staircase example we found $1 = E^{-1} + E^1B^2 + 2B$, hence $E^{-1} = \frac{1}{2}\left(1 - 2B + \sqrt{1-4B}\right)$.

2.2. The Particular Functional.

We are looking for the particular Sheffer sequence $(s_m)_{m\in\mathbb{N}_0}$ which satisfies some given "initial" conditions. Different methods have been applied in the past (for example [4], [13], [14], and [15]). In our approach we assume that the conditions can be expressed as values of a particular functional L, so that $\langle L \mid s_m\rangle$ is known for all m. There may be several such functionals. With every choice of a particular functional L comes a unique shift invariant (particular) operator

$$\mu_L := \sum_{m\geq 0}\langle L \mid b_m\rangle B^m. \tag{2.6}$$

If $\langle L \mid 1\rangle \neq 0$, then μ_L is invertible. It is shown in [17] that the operator μ_L is invariant under the choice of the delta operator B and its basic sequence (b_m). For example, if $L = \mathrm{Eval}_a$ it is convenient to choose the pair $\mathcal{D}$ and $(x^m/m!)$ to show that

$$\mu_{\mathrm{Eval}_a} = \sum_{m\geq 0}\frac{a^m}{m!}\mathcal{D}^m = e^{a\mathcal{D}} = E^a, \tag{2.7}$$

the shift operator by a. It is also easy to show that a functional defined as $\langle L \mid s_m\rangle = \left\langle \mathrm{Eval}_a \mid B^k s_m\right\rangle$ for all $m \geq 0$ and given a and k, has the associated operator $\mu_L = E^aB^k$.

Now we are ready to state the Functional Expansion Theorem [12].

Theorem 3. *Suppose $(s_m)_{m\in\mathbb{N}_0}$ is a B - Sheffer sequence and L a functional such that $\langle L \mid 1\rangle \neq 0$. The polynomials $s_m(x)$ can be expanded in terms of the B - basic sequence $(b_m)_{m\in\mathbb{N}_0}$ as*

$$s_m(x) = \sum_{k=0}^{m}\langle L \mid s_k\rangle\,\mu_L^{-1} b_{m-k}(x).$$

They have the generating function

$$\sum_{m=0}^{\infty} s_m(x)\,t^m = \frac{\sum_{k=0}^{\infty}\langle L \mid s_k\rangle t^k}{\sum_{j=0}^{\infty}\langle L \mid b_j\rangle t^j}\sum_{m=0}^{\infty} b_m(x)\,t^m.$$

The *Binomial Theorem for Sheffer Sequences* [18]

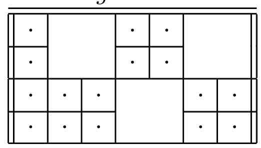
$$s_m(x+a) = \sum_{k=0}^{m} s_k(a)\,b_{m-k}(x) \tag{2.8}$$

is a special case of the Functional Expansion Theorem if we choose $L = \mathrm{Eval}_a$,

$$E^a s_m(x) = E^a \sum_{k=0}^{m} \langle \mathrm{Eval}_a \mid s_k \rangle \mu_{\mathrm{Eval}_a}^{-1} b_{m-k}(x) = E^a \sum_{k=0}^{m} s_k(a) E^{-a} b_{m-k}(x).$$

Example 2 (Tiling a rectangle with 2×2 squares). *Denote by $p_m(k)$ the number of tilings of a $4 \times m + k$ rectangle with m squares of size 2×2. Obviously $p_0(k) = 1$ for all $k \geq 0$, and $p_m(0) = 1$ for even $k \geq 0$, and 0 else.*

Tiling a 4×7 rectangle with three 2×2 squares



Heubach [6] found the recursion

$$p_k(x+1) = p_k(x) + 3p_{k-1}(x) + p_{k-2}(x) + 2\sum_{r=2}^{k} p_{k-r}(x).$$

If B is the delta operator mapping p_m into p_{m-1}, the above recursion is equivalent to the operator identity

$$E^1 = 1 + 3B + B^2 + 2\sum_{r \geq 2} B^r = \frac{1 + 2B - B^3}{1 - B}.$$

By (2.5) the B-basic sequence (b_m) has the generating function

$$\sum_{m \geq 0} b_m(x) t^m = \left(\frac{1 + 2t - t^3}{1 - t}\right)^x.$$

We choose $L = \mathrm{Eval}_0$ as the particular functional and obtain from the Functional Expansion Theorem 3

$$\sum_{m=0}^{\infty} p_m(x) t^m = \frac{\sum_{k=0}^{\infty} \langle L \mid p_k \rangle t^k}{\sum_{j=0}^{\infty} \langle L \mid b_j \rangle t^j} \left(\frac{1 + 2t - t^3}{1 - t}\right)^x = \frac{1}{1 - t^2}\left(\frac{1 + 2t - t^3}{1 - t}\right)^x.$$

Merlini, Sprugnoli, and Verri [9] developed an algorithm transforming tiling problems like this into a regular grammar, and "automatically" produce the generating function from the tiling problem with the help of a computer algebra package.

3. Recursions in one variable

Suppose $\sigma_0, \sigma_1, \ldots, \sigma_{\ell-1}$ are given initial values for the recursion

$$\sigma_m = \sum_{j=1}^{m} \alpha_j \sigma_{m-j}$$

for $m \geq \ell$, where $\alpha_1, \alpha_2, \ldots$ is a sequence of given factors.

Proposition 1.

$$\sum_{m\geq 0}\sigma_m t^m = \frac{\sum_{k=0}^{\ell-1}\left(\sigma_k - \sum_{j=1}^{k}\alpha_j\sigma_{k-j}\right)t^k}{1-\sum_{j\geq 1}\alpha_j t^j}$$

Proof. We have to distinguish two cases, $\alpha_1 \neq 0$ and $\alpha_1 = 0$. Suppose $\alpha_1 \neq 0$. To solve this type of problem with UC we define a polynomial sequence $(s_m(x))$ such that

$$s_m(x) - s_m(x-1) = \sum_{j\geq 1}\alpha_j s_{m-j}(x) \tag{3.1}$$

and

$$s_m(-1) = \begin{cases} \sigma_m - \sum_{j\geq 1}\alpha_j\sigma_{m-j} & \text{for } 0 \leq m < \ell \\ 0 & \text{for } m \geq \ell. \end{cases}$$

It is easy to check that $\sigma_m = s_m(0)$ for all $m \in \mathsf{N}_0$. Denote the linear operator which maps $s_m(x)$ into $s_{m-1}(x)$ by B. The recursion (3.1) shows that the backwards difference operator ∇ is a delta series in B (remember that $\alpha_1 \neq 0$), $\nabla = \ (B) = \sum_{j=1}^{\infty}\alpha_j B^j$, and (s_m) is a B-Sheffer sequence. Let (b_m) be the B-basic sequence. We apply the binomial formula (2.8) and find

$$\begin{aligned}\sum_{m\geq 0}\sigma_m t^m &= \sum_{m\geq 0}s_m(0)\,t^m = \sum_{m\geq 0}t^m\sum_{k=0}^{m}s_k(-1)\,b_{m-k}(1) &(3.2)\\ &= \frac{1}{1-\ (t)}\sum_{k=0}^{\ell-1}\left(\sigma_k - \sum_{j\geq 1}\alpha_j\sigma_{k-j}\right)t^k &(3.3)\end{aligned}$$

(see (2.2) for $b_{m-k}(1)$). If $\alpha_1 = 0$, we define $(s_m(x))$ such that

$$s_m(x) - s_m(x-1) = s_{m-1}(x-1) + \sum_{j=2}^{m}\alpha_j s_{m-j}(x).$$

For the new associated delta operator B holds $\nabla = E^{-1}B + \ (B)$, using the same as above, thus $1-\nabla = E^{-1} = (1-\ (B))/(1+B)$ and

$$\sum_{m\geq 0}b_m(1)\,t^m = \frac{1+t}{1-\ (t)} \tag{3.4}$$

(see (2.5)). Let

$$s_m(-1) = \begin{cases} \tau_m - \sum_{j=2}^{m}\alpha_j\tau_{m-j} & \text{for } 0 \leq m < \ell \\ (-1)^{m-1-\ell}\left(\tau_{\ell-1} - \sum_{j=2}^{\ell-1}\alpha_j\tau_{\ell-1-j}\right) & \text{for } m \geq \ell \end{cases} \tag{3.5}$$

where $\tau_m := \sum_{i=0}^{m} (-1)^i \sigma_{m-i}$. Note that $\tau_m + \tau_{m-1} = \sigma_m$. It follows that $s_m(0) = \sigma_m$ for $0 \leq m < \ell$ because of

$$\begin{aligned} s_m(0) &= s_m(-1) + s_{m-1}(-1) + \sum_{j\geq 2} \alpha_j s_{m-j}(0) \\ &= \tau_m - \sum_{j=2}^{m} \alpha_j \tau_{m-j} + \tau_{m-1} - \sum_{j=2}^{m-1} \alpha_j \tau_{m-1-j} \\ &= \sigma_m - \sum_{j\geq 2} \alpha_j \sigma_{m-j} + \sum_{j\geq 2} \alpha_j s_{m-j}(0). \end{aligned}$$

For $m \geq \ell$ holds $s_m(-1) = -s_m(-1)$, and therefore by induction

$$s_m(0) = \sum_{j=2}^{m} \alpha_j s_{m-j}(x) = \sum_{j=2}^{m} \alpha_j \sigma_{m-j} = \sigma_m.$$

As before, the numbers σ_m are expanded with the help of the binomial formula for Sheffer sequences,

$$\sum_{m\geq 0} \sigma_m t^m = \sum_{m\geq 0} \sum_{k=0}^{m} s_k(-1)\, b_{m-k}(1)\, t^m.$$

Substitution for $s_k(-1)$ and applying (3.4) shows that

$$\sum_{m\geq 0} \sigma_m t^m = \frac{\sum_{k=0}^{\ell-1} \left(\sigma_k - \sum_{j=2}^{k} \alpha_j \sigma_{k-j}\right) t^k}{1 - \sum_{j\geq 2} \alpha_j t^j}. \tag{3.6}$$

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3.1. Example: Area under elevated Schröder paths.

The three different step vectors on a generalized Schröder path are the horizontal step $(w, 0)$ of length $w \geq 2$, and the diagonal steps $(1, 1)$ and $(1, -1)$. The $(w, 0)$-steps get the (multiplicative) weight ω. The paths start at the origin, end on the x - axis, and never go below the x - axis. *Elevated* Schröder paths stay strictly above the x - axis, with the exception of the first and last step. Sulanke [20],[21] found the following recursion for the total weighted area σ_m under all elevated Schröder paths from $(0, 0)$ to $(m + 2, 0)$;

$$\sigma_m = \begin{cases} 2^m & \text{if } m \text{ is even and } 0 \leq m \leq w-1 \\ 0 & \text{if } m \text{ is odd and } 0 < m \leq w-1 \\ 2^w + \omega(w+1) & \text{if } m = w \text{ and } w \text{ is even} \\ \omega(w+1) & \text{if } m = w \text{ and } w \text{ is odd} \\ 4\sigma_{m-2} + 2\omega\sigma_{m-w} - \omega^2\sigma_{m-2w} & \text{if } m > w. \end{cases}$$

The factors α_j are therefore $\alpha_1 = 0, \alpha_2 = 4, \alpha_w = 2\omega$, and $\alpha_{2w} = -\omega^2$.

Elevated Schröder paths ($w = 2$, step vectors $\to\to, \nearrow, \searrow$)

n	9 = area			·↘	weight = 1		
2			·↗	¥	·↘		
1		·↗	¥	¥	¥	·↘	
0	·↗	¥	¥	¥	¥	¥	·
	0	1	2	3	4	5	m

n	8 = area				weight = ω		
2			·→→		·↘		
1		·↗	¥	¥	¥	·↘	
0	·↗	¥	¥	¥	¥	¥	·
	0	1	2	3	4	5	m

n	7		·↘		·↘		1
1		·↗	¥	·↗	¥	·↘	
0	·↗	¥	¥	¥	¥	¥	·
	0	1	2	3	4	5	m

n	6				·↘		ω
1		·→→		·↗	¥	·↘	
0	·↗	¥	¥	¥	¥	¥	·
	0	1	2	3	4	5	m

n	9		·↘				ω
1		·↗	¥	·→→		·↘	
0	·↗	¥	¥	¥	¥	¥	·
	0	1	2	3	4	5	m

n	5						ω^2
1		·→→		·→→		·↘	
0	·↗	¥	¥	¥	¥	¥	·
	0	1	2	3	4	5	m

$\sigma_4 = 16 + 20\omega + 5\omega^2$

For $0 \leq m < w$ we are given the initial values $\sigma_m = (1 + (-1)^m)\, 2^{m-1}$, and also $\sigma_w = (1 + (-1)^w)\, 2^{w-1} + \omega\,(w + 1)$. By (3.6)

$$\sum_{m \geq 0} \sigma_m t^m = \frac{\sum_{k=0}^{w-1} (\sigma_k - 4\sigma_{k-2})\, t^k + (\sigma_w - 4\sigma_{w-2} - 2\omega\sigma_0)\, t^w}{1 + \sum_{j \geq 2} \alpha_j t^j}$$
$$= \frac{1 + (\omega\,(w+1) - 2\omega)\, t^w}{(1 - \omega t^w)^2 - 4t^2}$$
$$\sigma_m = \sum_{j=0}^{m/w} \binom{m - j\,(w-1) + 1}{j} \frac{2^{m-1-jw}\,(m+1)\left(1 + (-1)^{m-jw}\right)\omega^j}{m - j\,(w-1) + 1}.$$

For "ordinary" elevated Schröder paths ($w = 2$) the sum simplifies,

$$\sigma_{2m} = \frac{1}{2}\left(1 + \sqrt{(1+\omega)}\right)^{2m+1} + \frac{1}{2}\left(1 - \sqrt{(1+\omega)}\right)^{2m+1}.$$

This (generating) function is familiar: The numbers ρ_m of non-selfintersecting $\{(1,0), (-1,0), (0,1)\}$ - paths starting at $(0,0)$ and having length m (Stanley [19, Example 4.1.2]) are enumerated by $\frac{1}{2}\left(1 + \sqrt{(1+\omega)}\right)^{m+1} + \frac{1}{2}\left(1 - \sqrt{(1+\omega)}\right)^{m+1}$, when the weight ω is given to the runs (maximal subwalks) of $\to$ - steps. They solve the recursion $\rho_m = 2\rho_{m-1} + \omega\rho_{m-2}$.

ρ_k and σ_k when $\omega = 1$

$k =$	0	1	2	3	4	5	6
$\rho_k =$	1	3	7	17	41	99	239
$\sigma_k =$	1	0	7	0	41	0	239

4. Applying the Functional Expansion Theorem

None of the above examples needed the power of the Functional Expansion Theorem 3, because the binomial formula suffices when explicit initial values are given. In [12] we investigated Sheffer sequences (s_m) satisfying initial conditions of a different form, starting with $s_m(x) = t_m(x)$ for all $m = 0, 1, \ldots, \ell - 1$, where $t_0(x), \ldots t_{\ell-1}(x)$ are given (initial) polynomials, continuing with conditions like $\int_0^1 s_m(y + cm)dy = 0$ or $s_m(cm) = \sum_{i=0}^{m-1} (-1)^i c^{i+1} s_{m-i-1}(cm)$ for all $m \geq \ell$. Another common feature of the above examples is that the existence of polynomial solutions to the recursion and initial values is rather obvious. The following example differs on both counts. It belongs to a type of restricted lattice paths problem with special access, introduced by Merlini, Rogers, Sprugnoli, and Verri [8, 1997].

4.1. Example: Privileged Access. Suppose a lattice walker can choose among the infinite number of steps in the step set $\mathsf{S} := \{(1,1), (1,-1)\} \cup \{(i,0) \mid i \in \mathsf{N}_1\}$. The horizontal steps $(i, 0)$ are weighted with $\ ^i$. Such weights are useful if we want to study the corresponding random walk with geometric probabilities for the horizontal steps. We also require that the ↘ - step keeps the walk in the first quadrant; it cannot be chosen when the path has reached the x - axis. However, after leaving the origin the walker has *special access* to the boundary $y = -1$ by selecting the fitting access step vector. In this example we let $\{(0, -n) \mid n \in \mathsf{N}_1\}$ be the set of special access steps, and we give them the weight α. The walker can vertically "jump down" to the boundary from any position. It is implied that the path must leave the boundary in the next step.

Sample path to $(13, 1)$ with access steps $\{(0, -n) \mid n > 0\}$

n														
2							○↓						○↘	
1				○	99K	○↗	⋮					○↗		○
0	○ →	○↓	○↗				⋮	○	99K		○↗			
-1	¥	○↗	¤	¤	¤	¤	○↗	¤	¤	¤	¤	¤	¤	¤
	0	1	2	3	4	5	6	7	8	9	10	11	12	$\rightarrow m$

The (weighted) numbers $A(m, n)$ of such paths from the origin to (m, n) have no polynomial extension such that $a_m(n) = A(m, n)$ on the support of A for any polynomial sequence $(a_m(x))$.

n			$A(m,n)$	
3				1
2			1	3
1		1	2	$5\ ^2 + \alpha\ \ + 2 + \alpha$
0	1		$2\ ^2 + \alpha\ \ + 1 + \alpha$	$4\ ^3 + 3\alpha\ ^2 + (3 + 3\alpha + \alpha^2)\ \ + 2\alpha + \alpha^2$
-1		$\alpha + \alpha$	$\alpha\left(2\ ^2 + (\ \ + 1)(2 + \alpha)\right)$	$\alpha(4\ ^3 + (5 + 3\alpha)\ ^2 + (6 + 4\alpha + \alpha^2)\ \ + 3 + 3\alpha + \alpha^2)$
	0	1	2	3 $\quad \rightarrow m$

However the transformation $M := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ maps the given problem into an equivalent walk where such an extension exists. The steps ↘, ↗, $(i,0)$ are mapped into $(2,1),(0,1)$ and $(i,i), i \in \mathsf{N}_1$, the latter taken with weight $\ ^i$. The transformed special access steps $(0,-n)$ are the horizontal steps $(n,0)$, $n>0$, with weight α. Let $D(m,n)$ be the weighted number of transformed paths to (m,n), $D(m,n) = A(n,n-m)$. We denote the polynomial extension of $D(m,n)$ by $d_m(n)$,

$$d_m(n) = D(m,n) \quad \text{for all } n \geq m-1.$$

The polynomial extension $d_m(n)$ (special access values in boxes)

n	1	3	$5\ ^2+\alpha\ \ +2+\alpha$	$4\ ^3+3\alpha\ ^2+\left(3+3\alpha+\alpha^2\right)\ \ +2\alpha+\alpha^2$
2	1	2	$2\ ^2+\alpha\ \ +1+\alpha$	$\boxed{0\ ^3+2\alpha\ ^2+\ \ \left(0+2\alpha+\alpha^2\right)+2\alpha+\alpha^2}$
1	1		$\boxed{0\ ^2+\alpha\ \ +0+\alpha}$	$-\ ^3+\alpha\ ^2+\ \left(-\mathbf{1}+\alpha+\alpha^2\right)+\mathbf{2}\alpha+\alpha^2$
0	1	0	$-\ ^2+\alpha\ \ -\mathbf{1}+\alpha$	$\mathbf{0}\ ^3+\mathbf{0}\ ^2+\ \left(\mathbf{0}+\mathbf{0}\alpha+\alpha^2\right)+\mathbf{2}\alpha+\alpha^2$
-1	$\mathbf{1}$	$-$	$-\ ^2+\alpha\ \ -\mathbf{2}+\alpha$	$\vdots$
	0	1	2	3 $\to m$

(the bold face values are extensions outside the support of D)

The polynomials are solutions to the system of difference equations

$$d_m(x) - d_m(x-1) = d_{m-2}(x-1) + \sum_{j \geq 1} \ ^j d_{m-j}(x-j)$$

and must satisfy the initial conditions $d_0(x) = 1$, $d_1(0) = 0$, and the transformed access condition

$$d_m(m-1) = \sum_{i \geq 1} \alpha d_{m-i}(m-1) \tag{4.1}$$

for all $m>1$. Denote by B the linear operator that maps $d_m(x)$ into $d_{m-1}(x)$. The recursion for (d_m) shows that

$$\nabla = E^{-1}B^2 + \sum_{j \geq 1} E^{-j}\ ^j B^j = E^{-1}B^2 + \frac{E^{-1}\ \ B}{1-E^{-1}\ \ B}. \tag{4.2}$$

The basic sequence $(b_m(x))$ can be expanded according to the transform formula (2.3)

$$\begin{aligned} b_m(x) &= \sum_{i=1}^{m} x\left[B^m\right]\left(E^{-1}B^2 + \frac{E^{-1}\ \ B}{1+E^{-1}\ \ B}\right)^i \frac{1}{x}\binom{i+x-1}{i} \\ &= \sum_{i=1}^{m}\sum_{j=0}^{i}\binom{i}{j}\binom{m-2j-1}{m-i-j}\ ^{m-2j}\frac{x}{i}\binom{x+i-m+j-1}{i-1} \end{aligned}$$

for $m>1$. The operator identity (4.2) "simplifies" to the finite recursion

$$\nabla = \ \ E^{-1}B\left(2-E^{-1}\right) + E^{-1}B^2\left(1-E^{-1}\ \ B\right).$$

After solving for E^1 we obtain from (2.5)

$$\sum_{m\geq 0} b_m(x)\, t^m = \left(\frac{1}{2}\left(t^2+2\ \ t+1\right)+\sqrt{\frac{1}{4}\left(t^2+2\ \ t+1\right)^2-\ \ t\left(1+t^2\right)}\right)^x$$

$$\sum_{m\geq 0} b_m^{(1)}(x)\, t^m = \left(\frac{1-2\ \ t-\sqrt{(1-2\ \ t)^2-4t^2\left(\ \ t-1\right)^2}}{2t^2\left(1-\ \ t\right)}\right)^x.$$

It is convenient to switch to the polynomials $d_n^{[1]}(x) = d_n(n+x)$ with delta operator $B_{(1)} = E^{-1}B$ and basic polynomials $b_m^{(1)} = xb_m(x+m)/(x+m)$ (see Remark 1). In terms of $\left(d_m^{[1]}\right)$ the privilege access condition (4.1) becomes $d_m^{[1]}(-1) = \sum_{i\geq 1}\alpha d_{m-i}^{[1]}(i-1)$. We define the particular functional L by linear extension of

$$\left\langle L \mid d_m^{[1]}\right\rangle = \left\langle \mathrm{Eval}_{-1} \mid d_m^{[1]}\right\rangle - \sum_{i\geq 1}\alpha\left\langle \mathrm{Eval}_{i-1} \mid B_{(1)}^i d_m^{[1]}\right\rangle \tag{4.3}$$

and obtain the particular operator

$$\mu_L = E^{-1} - \sum_{i\geq 1}\alpha E^{i-1}B_{(1)}^i = E^{-1}\left(1-\frac{\alpha E^1 B_{(1)}}{1-E^1B_{(1)}}\right).$$

Note that $\left\langle L \mid d_0^{[1]}\right\rangle = 1$, $\left\langle L \mid d_1^{[1]}\right\rangle = d_1^{[1]}(-1)-\alpha = -\alpha$, and $\left\langle L \mid d_m^{[1]}\right\rangle = 0$ for all $m > 1$. From the Functional Expansion Theorem 3 we obtain for $d_m(x)$ an explicit formula, $d_m(x) = d_m^{[1]}(x-m)$

$$\begin{aligned}
&= \mu(L)^{-1}\left(b_m^{(1)}(x-m) - \alpha b_{m-1}^{(1)}(x-m)\right)\\
&= \left(1+\frac{\alpha E^1 B_{(1)}}{1-(1+\alpha)E^1B_{(1)}}\right)\left(b_m^{(1)}(x-m+1)-\alpha b_{m-1}^{(1)}(x-m+1)\right)\\
&= (x-m+1)\left(\frac{b_m(x+1)}{x+1}-\frac{\alpha b_{m-1}(x)}{(1+\alpha)x}\right)\\
&+\alpha\sum_{l=1}^{m}(1+\alpha)^{l-1}\left(\frac{x+l-m+1}{x+1}b_{m-l}(x+1)-\frac{\alpha(x+l-m)}{(1+\alpha)x}b_{m-l}(x)\right).
\end{aligned}$$

5. Bivariate Umbral Calculus

For the remaining part of the paper we will focus on the bivariate case; generalizations of the following theorems to higher dimensions are straight forward. More on multivariate umbral calculus can be found in [16] and [23], [24]. A *bivariate Sheffer sequence* $(p_{m,n}(x,y))_{m,n\in\mathbb{N}_0}$ has a generating function of the form

$$\sum_{m,n\geq 0} p_{m,n}(x,y)\, s^m t^n = \rho(s,t)\, e^{x\beta_1(s,t)+y\beta_2(s,t)}$$

where $\rho(s,t) \in \mathsf{K}[[s,t]]$ has order 0, i.e., $\rho(0,0) \neq 0$, and $\beta_1(s,t), \beta_2(s,t)$ is a *pair of delta series* in $\mathsf{K}[[s,t]]$, which means that $\beta_1(s,t)/s$ and $\beta_2(s,t)/t$ are both power series of order 0 in $\mathsf{K}[[s,t]]$. If $\rho(s,t) = 1$ the resulting Sheffer sequence is a *bivariate basic sequence*. For every pair of delta series $\beta_1(s,t), \beta_2(s,t)$ there exists a compositional inverse $\gamma_1(s,t), \gamma_2(s,t)$, also a delta pair, such that

$$\beta_1(\gamma_1(s,t), \gamma_2(s,t)) = s \quad \text{and} \quad \beta_2(\gamma_1(s,t), \gamma_2(s,t)) = t.$$

Let $\mathcal{D}_1 := \partial/\partial x$ and $\mathcal{D}_2 := \partial/\partial y$. Using the above notation it can be shown that

$$\gamma_1(\mathcal{D}_1, \mathcal{D}_2) p_{m,n}(x,y) = p_{m-1,n}(x,y) \quad \text{and} \quad \gamma_2(\mathcal{D}_1, \mathcal{D}_2) p_{m,n}(x,y) = p_{m,n-1}(x,y).$$

We call $\gamma_1(\mathcal{D}_1, \mathcal{D}_2), \gamma_2(\mathcal{D}_1, \mathcal{D}_2)$ a *pair of delta operators*, and $(p_{m,n})$ the associated Sheffer sequence. Of course, all operators in $\mathsf{K}[[\mathcal{D}_1, \mathcal{D}_2]]$ commute. Again we will state two transfer theorems.

Theorem 4 (Bivariate Transfer Theorem I). *Let Q_1, Q_2 be a delta operator with Q_1, Q_2 - basic sequence $(q_{m,n}(x,y))_{m,n\in\mathsf{N}_0}$. If $Q_1 = {}_1(B_1, B_2)$ and $Q_2 = {}_2(B_1, B_2)$ for some pair ${}_1, {}_2$ of delta series and linear operator B_1, B_2, then B_1, B_2 is also a pair of delta operators, and the B_1, B_2-basic sequence $(b_{m,n})_{m,n\in\mathsf{N}_0}$ has for all $m, n \in \mathsf{N}_0$ the expansion*

$$b_{m,n}(x,y) = \sum_{i=0}^{m}\sum_{j=0}^{n} c_{m,n,i,j}\, q_{i,j}(x,y) \tag{5.1}$$

where $c_{m,n,i,j}$ is the coefficient of $B_1^m B_2^n$ in ${}_1(B_1, B_2)^i \; {}_2(B_1, B_2)^j$. For the generating function of the basic sequence $(b_{m,n})$ holds

$$\sum_{m,n\geq 0} b_{m,n}(x,y)\, s^m t^n = \sum_{i,j\geq 0} q_{i,j}(x,y) \; {}_1(s,t)^i \; {}_2(s,t)^j. \tag{5.2}$$

The proof is analogous to the proof of Theorem 1.

The Jacobian $J(\rho_1, \rho_2)$ of a pair $\rho_1(s,t), \rho_2(s,t)$ of power series is defined as

$$J(\rho_1, \rho_2)(s,t) := \begin{vmatrix} \partial P_1/\partial s & \partial P_2/\partial s \\ \partial P_1/\partial t & \partial P_2/\partial t \end{vmatrix}.$$

Theorem 5 (Bivariate Transfer Theorem II). *Let Q_1, Q_2 be a delta operator with Q_1, Q_2 - basic sequence $(q_{m,n}(x,y))_{m,n\in\mathsf{N}_0}$. If $B_1 = \rho_1(Q_1, Q_2) Q_1$ and $B_2 = \rho_2(Q_1, Q_2) Q_2$ for some pair ρ_1, ρ_2 of power series of order 0, then B_1, B_2 is also a pair of delta operators, and the B_1, B_2-basic sequence $(b_{m,n})_{m\in\mathsf{N}_0}$ has for all $m, n \in \mathsf{N}_0$ the expansion*

$$b_{m,n}(x,y) = \rho_1(Q_1, Q_2)^{-m-1} \rho_2(Q_1, Q_2)^{-n-1} J(Q_1\rho_1, Q_2\rho_2)(Q_1, Q_2)\, q_{m,n}(x,y) \tag{5.3}$$

Proof. Suppose $Q_i = \sigma_i^{-1}(\mathcal{D}_1, \mathcal{D}_2)$ and $B_i = \beta_i^{-1}(\mathcal{D}_1, \mathcal{D}_2)$ for some pairs of delta series σ_1, σ_2 and β_1, β_2. For notational simplicity we define

$$\omega_i(s,t) := \sigma_i^{-1}(\beta_1(s,t), \beta_2(s,t)).$$

Hence $\sum_{m,n\geq 0} b_{m,n}(x,y)\, s^m t^n$

$$\begin{aligned}
&= e^{x\beta_1(s,t)+y\beta_2(s,t)} = \sum_{m,n\geq 0} q_{m,n}(x,y)\,\omega_1(s,t)^m\,\omega_2(s,t)^n \\
&= \sum_{i\geq 0, j\geq 0} s^i t^j \sum_{0\leq m\leq i, 0\leq n\leq j} \left[s^i t^j\right] \omega_1(s,t)^m\,\omega_2(s,t)^n\, q_{m,n}(x,y) \\
&= \sum_{i\geq 0, j\geq 0} s^i t^j \sum_{m\geq 0, n\geq 0} \left[s^i t^j\right] \omega_1(s,t)^{i-m}\,\omega_2(s,t)^{j-n}\, Q_1^m Q_2^n q_{i,j}(x,y)\,.
\end{aligned}$$

By Lagrange - Bürmann inversion
$\left[s^i t^j\right]\omega_1(s,t)^{i-m}\,\omega_2(s,t)^{j-n} = \left[s^m t^n\right]\left(\frac{\omega_1^{-1}(s,t)}{s}\right)^{-i-1}\left(\frac{\omega_2^{-1}(s,t)}{t}\right)^{-j-1} J\left(\omega_1^{-1},\omega_2^{-1}\right).$
Noting that

$$\omega_1^{-1}(s,t) = s\rho_1(s,t) \quad \text{and} \quad \omega_2^{-1}(s,t) = t\rho_2(s,t)$$

finishes the proof. ¤

Example 3. *Suppose we have the pair of operator identities*

$$\begin{aligned}
\nabla_1 &= E_1^{-1} B_1 \\
\Delta_2 &= E_1^{-1} B_2 + \omega B_1 E_1^{-1} B_2.
\end{aligned}$$

We apply the above theorem with $B_1 = \Delta_1$ *and* $B_2 = \Delta_2 / \left(E_1^{-1} + \omega B_1 E_1^{-1}\right) = \Delta_2(1+\Delta_1)/(1+\omega\Delta_1)$. *Hence* $\rho_1(s,t) = 1$, $\rho_2(s,t) = (1+s)/(1+\omega s)$, *and*

$$\begin{aligned}
b_{m,n}(x,y) &= \rho_1(\Delta_1,\Delta_2)^{-m-1}\,\rho_2(\Delta_1,\Delta_2)^{-n-1}\, J(\Delta_1\rho_1,\Delta_2\rho_2)(\Delta_1,\Delta_2)\binom{x}{m}\binom{y}{n} \\
&= \left(\frac{1+\Delta_1}{1+\omega\Delta_1}\right)^{-n-1}\begin{vmatrix} 1 & \Delta_2\frac{\partial\rho_2}{\partial\Delta_1} \\ 0 & \frac{1+\Delta_1}{1+\omega\Delta_1}\end{vmatrix}\binom{x}{m}\binom{y}{n} \\
&= E_1^{-n}(1+\omega\Delta_1)^n\binom{x}{m}\binom{y}{n} = \sum_{i=0}^{\infty}\binom{n}{i}\omega^i\binom{x-n}{m-i}\binom{y}{n}.
\end{aligned}$$

For comparison we apply the first Transfer Theorem to the same problem, with $_1(B_1,B_2) = B_1$ *and*

$$_2(B_1,B_2) = \frac{B_2(1+\omega B_1)}{E_1^1} = \frac{B_2(1+\omega B_1)}{1+\Delta_1} = \frac{B_2(1+\omega B_1)}{1+B_1}.$$

Hence, the B *- basic sequence has the generating function* $\sum_{m,n\geq 0} b_{m,n}(x,y)\, s^m t^n$

$$= \sum_{m,n\geq 0}\binom{x}{m}\binom{y}{n} s^m\left(\frac{t(1+\omega s)}{1+s}\right)^n = (1+s)^x\left(1+t\frac{1+\omega s}{1+s}\right)^y.$$

If we apply the bivariate transfer formula (5.2) we get the expression

$$\begin{aligned} b_{m,n}(x,y) &= \sum_{i=0}^{m}\sum_{j=0}^{n}\left(\left[B_1^m B_2^n\right] B_1^i \left(\frac{B_2\left(1+\omega B_1\right)}{1+B_1}\right)^j\right)\binom{x}{i}\binom{y}{j} \\ &= \binom{y}{n}\sum_{i=0}^{m}\left(\left[B_1^{m-i}\right]\sum_{k=0}^{n}\binom{n}{k}\left(\frac{\left(\omega-1\right)B_1}{1+B_1}\right)^k\right)\binom{x}{i} \\ &= \binom{y}{n}\sum_{k=0}^{\infty}\binom{n}{k}\left(\omega-1\right)^k\binom{x-k}{m-k}. \end{aligned}$$

The two solutions we found for $b_{m,n}(x,y)$ reflect the well-known hypergeometric function identity

$$\binom{x}{m}{}_2F_1\left(\begin{matrix}-m,-n\\-x\end{matrix};1-\omega\right)=\binom{x-n}{m}{}_2F_1\left(\begin{matrix}-m,-n\\1-n-m-x\end{matrix};\omega\right).$$

The Functional Expansion Theorem 3 generalizes as expected, with μ_L defined as

$$\mu_L := \sum_{m,n\geq 0}\left\langle L \mid b_{m,n}\right\rangle B_1^m B_2^n \tag{5.4}$$

for any B_1, B_2 - basic sequence $(b_{m,n})$.

Theorem 6. *Suppose $(s_{m,n})_{m,n\in\mathsf{N}_0}$ is a B_1, B_2 - Sheffer sequence and L a linear functional on $\mathsf{K}[[s,t]]$ such that $\langle L \mid 1\rangle \neq 0$. The polynomials $s_{m,n}(x,y)$ can be expanded in terms of the B_1, B_2 - basic sequence $(b_{m,n})$ as*

$$s_{m,n}(x)=\sum_{j=0}^{m}\sum_{k=0}^{n}\left\langle L \mid s_{j,k}\right\rangle \mu_L^{-1} b_{m-j,n-k}(x,y).$$

They have the generating function

$$\sum_{m,n\geq 0}s_{m,n}(x,y)\,s^m t^n=\frac{\sum_{j,k\geq 0}\left\langle L \mid s_{j,k}\right\rangle s^j t^k}{\sum_{j,k\geq 0}\left\langle L \mid b_{j,k}\right\rangle s^j t^k}\sum_{m,n\geq 0}b_{m,n}(x,y)\,s^m t^n.$$

5.1. Bi-indexed Umbral Calculus. Three variable recursions like $p_{m,n}(\xi) = p_{m,n}(\xi-1) + p_{m-1,n}(\xi) + p_{m,n-1}(\xi)$ are not part of the above bivariate theory. Very basic combinatorial objects, like multinomial coefficients, follow recursions of this kind. An obvious approach to a polynomial theory for this type of recursions equates the variable x and y in a bivariate polynomial sequence to give $(p_{m,n}(\xi))_{m,n\in\mathsf{N}_0}$. Abusing the notation we write $p_{m,n}(\xi) = p_{m,n}(\xi,\xi)$, and we introduce the *diagonalization* operator $\Xi : \mathsf{K}[x,y] \to \mathsf{K}[\xi]$

$$\Xi p(x,y) = p(\xi).$$

Suppose $(s_{m,n})$ is a Sheffer sequence with generating function $\rho(s,t)\,e^{x\beta_1(s,t)+y\beta_2(s,t)}$. By $\mathsf{K}[[s,t]]$ - linear extension we define

$$\Xi\sum_{m,n\geq 0}s_{m,n}(x,y)\,s^m t^n := \sum_{m,n\geq 0}\Xi s_{m,n}(x,y)\,s^m t^n=\rho(s,t)\,e^{\xi(\beta_1(s,t)+\beta_2(s,t))}.$$

The following lemma is easy to verify.

Lemma 1. *For all $k \in \mathsf{N}_0$ holds*

$$\Xi \left(\mathcal{D}_1 + \mathcal{D}_2\right)^k = \mathcal{D}^k \Xi$$

A bi-indexed Sheffer sequence $(s_{m,n}(\xi))$ is a double sequence of univariate polynomials such that $\deg s_{m,n}(\xi) = m+n$ for all $m,n \in \mathsf{N}_0$, $s_{0,0}(\xi) \neq 0$, and

$$\sum_{m,n\geq 0} s_{m,n}(\xi)\, s^m t^n = \rho(s,t)\, e^{\xi\beta(s,t)}$$

where $\rho(s,t)$ is of order 0, and $\beta(s,t) = \beta_1(s,t) + \beta_2(s,t)$ for some pair of bivariate delta series. Obviously this pair is not unique. If $(s_{m,n}(x,y))$ is a bivariate Q_1, Q_2 - Sheffer sequence, then $(\Xi s_{m,n}(x,y))$ is a bi-indexed Sheffer sequence, and

$$\Xi \left(Q_1 + Q_2\right) s_{m,n}(x,y) = \Xi \left(s_{m-1,n}(x,y) + s_{m,n-1}(x,y)\right).$$

We say that $(s_{m,n}(\xi))$ is a Q - Sheffer sequence iff $Q s_{m,n}(\xi) = s_{m-1,n}(\xi) + s_{m,n-1}(\xi)$, and call Q a delta operator associated to $(s_{m,n}(\xi))$. Trivial examples of bi-indexed Sheffer polynomials are products of two univariate Sheffer polynomials. The following example shows another important class.

Example 4. *Let γ be a univariate delta series, and let $(q_m)_{m\in\mathsf{N}_0}$ be the ordinary univariate $Q = \gamma^{-1}(\mathcal{D})$ - basic sequence. Define $p_{m,n}(\xi) := \binom{m+n}{m} q_{m+n}(\xi)$ for all $m,n \in \mathsf{N}_0$. From $p_{n,m}(0) = \delta_{m,0}\delta_{n,0}$ and $Q p_{m,n}(\xi) = p_{m-1,n}(\xi) + p_{m,n-1}(\xi)$ follows that $(p_{m,n}(\xi))$ is a bi-indexed P - basic sequence with $P = Q$. What are P_1, P_2 such that $P\Xi = \Xi(P_1 + P_2)$? How does the associated basic sequence $(p_{m,n}(x,y))$ look like? The generating function*

$$\sum_{m,n\in\mathsf{N}_0} p_{m,n}(\xi)\, s^m t^n = \sum_{k\in\mathsf{N}_0} q_k(\xi) \sum_{m=0}^{k} \binom{k}{m} s^m t^{k-m} = e^{\xi\gamma(s+t)}$$

motivates us to define

$$\beta_1(s,t) = \gamma(s+t) - \gamma(t) \quad \text{and} \quad \beta_2(s,t) = \gamma(t)$$

because $\beta_1(s,t), \beta_2(s,t)$ is a delta pair, and $\beta_1(s,t) + \beta_2(s,t) = \gamma(t)$. Note that $P\Xi = \Xi(P_1 + P_2)$ if

$$P_1 = \gamma^{-1}(\mathcal{D}_1 + \mathcal{D}_2) - \gamma^{-1}(\mathcal{D}_2) \quad \text{and} \quad P_2 = \gamma^{-1}(\mathcal{D}_2).$$

We can check this by applying Lemma 1: $\Xi(P_1 + P_2) = \Xi\gamma^{-1}(\mathcal{D}_1 + \mathcal{D}_2) = \gamma^{-1}(\mathcal{D})$. The P_1, P_2 - basic polynomials diagonalize to $p_{m,n}(\xi) = \Xi p_{m,n}(x,y)$, and

$$\sum_{m,n\geq 0} p_{m,n}(x,y)\, s^m t^n = e^{x\gamma(s+t) + (y-x)\gamma(t)} \tag{5.5}$$

$$p_{m,n}(x,y) = \sum_{i=0}^{n} \binom{m+n-i}{m} p_{m+n-i}(x)\, p_i(y-x).$$

Properties of the bivariate behind the bi-indexed Sheffer sequences can be used to solve certain recursions. For example, if $(q_{m,n}(\xi))$ is a Q - basic sequence, then $f_{n,m}(\xi) := (m+1)\, q_{m+1,n}(\xi)/\xi$ and $g_{n,m}(\xi) := (n+1)\, q_{m,n+1}(\xi)/\xi$ are Q - Sheffer polynomials, and so is the shifted linear combination

$$\begin{aligned} s_{m,n}(\xi) &:= q_{m,n}(\xi-\eta) - a f_{m-1,n}(\xi-\eta) - b g_{m,n-1}(\xi-\eta) \\ &= (\xi-\eta-am-bn)\,\frac{q_{m,n}(\xi-\eta)}{\xi-\eta} \end{aligned} \tag{5.6}$$

for some given constants a, b and η. This Q - Sheffer sequence satisfies the initial condition $s_{m,n}(\eta+am+bn) = \delta_{0,m}\delta_{0,n}$. If $(r_{m,n}(\xi))$ is any Q - Sheffer sequence, and $Q\Xi = \Xi(Q_1+Q_2)$, then $(r_{m,n}(\xi+am+bn))$ is a $Q_{(a,b)}$ - Sheffer sequence, where $Q_{(a,b)}\Xi = \Xi\left(E_1^{-a}E_2^{-a}Q_1 + E_1^{-b}E_2^{-b}Q_2\right)$. The bi-indexed polynomial sequence $\left(\xi\frac{q_{m,n}(\xi+am+bn)}{\xi+am+bn}\right)_{m,n\in\mathbb{N}_0}$ is the $Q_{(a,b)}$ - basic sequence, the Abelization of $(q_{m,n}(\xi))$. More about multi-indexed UC can be found in [10] and [25].

5.2. Example: A Slow Walk. Suppose the lattice walker can choose at every tick of the clock one of the steps $\{\rightarrow, \uparrow, \nearrow, ©\}$, where $© = (0,0)$ is the "pause step". Give weight ω to the $\nearrow$ - step vector. Let $D(m,n;k)$ be the number of weighted paths from $(0,0)$ to (m,n) in k steps (seconds) under the restriction $D(i,j;l) = 0$ if (time) $l \leqslant ai+bj$ where a and b are positive integers, and $(i,j) \neq (0,0)$. In other words, the lattice point (i,j) can not be reached in a short time l; the walk is slow.

$D(n,m;k)$ for $a=2$ and $b=1$; $\omega=1$													
$\cdot_4$	$\cdot_6$	$\cdot_4$	$\cdot_6$	$\cdot_4$	$\cdot_6$	$\cdot_8$	$\cdot_4$	$\cdot_6$	$\cdot_8$	$\mathbf{0}_4$	0_6	0_8	0_{10}
$\cdot_3$	$\cdot_5$	$\cdot_3$	$\cdot_5$	$\cdot_3$	$\cdot_5$	$\cdot_7$	$\mathbf{0}_3$	0_5	0_7	1_3	0_5	0_7	0_9
$\cdot_2$	$\cdot_4$	$\cdot_2$	$\cdot_4$	$\mathbf{0}_2$	0_4	0_6	1_2	0_4	0_6	3_2	$\mathbf{0}_4$	0_6	0_8
$\cdot_1$	$\cdot_3$	$\mathbf{0}_1$	0_3	1_1	0_3	0_5	2_1	$\mathbf{0}_3$	0_5	3_1	4_3	0_5	0_7
$1\overset{\uparrow\nearrow}{\longrightarrow}$	$\cdot_2$	1_0	0_2	1_0	$\mathbf{0}_2$	0_4	1_0	1_2	0_4	1_0	2_2	$\mathbf{0}_4$	0_6
$k=0$		$k=1$		$k=2$			$k=3$			$k=4$			
The subscripts show the values of $ai+bj$. The levels $k=ai+bj$ are in bold.													

The solution to the difference equation

$$\begin{aligned} &D(n,m;k) - D(n,m;k-1) \\ &= D(n-1,m;k-1) + D(n,m-1;k-1) + \omega D(n-1,m-1;k-1) \end{aligned}$$

can be extended to a polynomial sequence $(d_{m,n}(\xi))$ because $D(0,0;k) = 1$ for all $k \geq 0$. Of course, $D(m,n;k) = 0$ if any of the three parameters is negative. In order to calculate the *nonzero* values of $D(m,n;k)$ the only relevant initial condition is $D(m,n;am+bn) = \delta_{m,0}\delta_{n,0}$. The recursion and operator equation for the polynomial extension are

$$\begin{aligned} d_{n,m}(\xi+1) &= d_{n,m}(\xi) + d_{n-1,m}(\xi) + d_{n,m-1}(\xi) + \omega d_{n-1,m-1}(\xi) \\ \Delta\Xi &= \Xi(B_1 + B_2 + \omega B_1 B_2) \end{aligned}$$

where $Bd_{n,m}(\xi) = d_{n,m}(\xi-1) + d_{n-1,m}(\xi-1)$ and $B\Xi = \Xi(B_1+B_2)$. In addition to $d_{0,0}(\xi) = 1$ the solution must have the initial values $d_{n,m}(an+bm) =$

$\delta_{m,0}\delta_{n,0}$. It follows from Lemma 1 that $\Delta\Xi = \Xi\left(E_1^1 E_2^1 - 1\right)$. Hence, it suffices to determine B such that $B_1 + B_2 + \omega B_1 B_2 = E_1^1 E_2^1 - 1$, or

$$E_1^{-1}\left(B_1 + B_2 + \omega B_1 B_2\right) = E_2^1 - E_1^{-1} = \nabla_1 + \Delta_2.$$

We decide to split this sum of operators into $\nabla_1 = E_1^{-1} B_1$ and $\Delta_2 = E_1^{-1} B_2 + \omega B_1 E_1^{-1} B_2$, because we found already the B_1, B_2 - basic sequence $(b_{m,n}(x,y))$ in Example 3, $b_{m,n}(x,y) = \binom{y}{n}\sum_{i=0}^{m}\binom{n}{i}\binom{x-n}{m-i}\omega^i$. The numbers $U(m,n;k)$ of *unrestricted* paths to (m,n) in k steps have initial values $U(m,n;0) = 0$ except for $U(0,0;0) = 1$. Hence $U(m,n;k) = b_{m,n}(k,k)$ on the support of U. The extensions of $D(m,n;k)$ have initial values $d_{n,m}(an + bm) = \delta_{(n,m),(0,0)}$. We apply formula (5.6) and get

$$D(m,n;k) = d_{n,m}(k) = \frac{k - an - bm}{k}\binom{k}{n}\sum_{i=0}^{m}\binom{n}{i}\binom{k-n}{m-i}\omega^i$$

on the support of D.

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