Proof of a Lattice Paths Conjecture Connected to the Tennis Ball Problem

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Abstract

The authors give a history of the so-called tennis ball problem, and discuss its relation to lattice path enumeration. We also prove a conjecture related to a solution of the symmetric case, namely when the number of balls removed each turn is exactly half the number inserted.

Key words:
Tennis Ball Problem, lattice path, Finite Operator Calculus

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1 The Tennis Ball Problem-History

The tennis ball problem was presented on pages 304 − 305 of the book by T.Tymoczko and J.Henle [12] in 1995. Their presentation deals with adding numbered books to a stack on a table, then removing some, infinitely many times. Motivated by that presentation, Ralph P. Grimaldi and Joseph G. Moser deal with performing the process a finite number of times [6].

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In their tennis ball problem, Colin L. Mallows and Lou Shapiro are given successive pairs of balls numbered $(1, 2)$, $(3, 4)$,... . At each stage they throw one ball out of the window. After n stages some set of n balls is on the lawn. They find a generating function and a explicit formula for the sequence $3, 23, 131, 664, 3166, 14545, 65187, 287060, 1247690$,..., the $n-$th term of which gives the sum over all possible arrangements of the total of the numbers on the balls on the lawn. They gave connections of the tennis ball problem with bicolored Motzkin paths and the ballot problem [9] .

The (s, t) tennis ball problem goes as follows. At turn one, balls numbered 1 through s are thrown into a basket and a gnome in the basket picks t of them and throws them onto the lawn. At turn two, balls numbered $s+1$ through $2s$ are thrown into the basket and the gnome, now having $2s - t$ balls to choose from, throws t of them onto the lawn. At the n-th turn balls $(n-1)s+1$ to sn go into the basket and the gnome throws out one the lawn t of the $ns-(n-1)t$ balls available to him.

Question 1. Looking at the balls out on the lawn, how many different ball sequences, $b_1 < b_2 < \cdots < b_{tn}$ are there?

Question 2. What is the sum of the numbers of these ball sequences over all ball sequences of length tn?

The most studied case is the (2,1) tennis ball problem. In this case Grimaldi and Moser showed that the answer to Question 1 is C_{n+1}, where $C_n = \binom{2n}{n}/(n+1)$ is the n-th Catalan number. Colin L. Mallows and Lou Shapiro answered the second question two years later, showing the result to be

$$\frac{2n^2+5n+4}{n+2}\binom{2n+1}{n} - 2^{2n+1}.$$

In [10], both questions are answered for the cases $(s, 1)$ and $(4, 2)$. The methods involved are generating trees, recursion by level in these trees, Lagrange inversion, and the Riordan group.The $(4, 2)$ case, treated in an appendix, is interesting because of connections to "Catalan trigonometry" [11] and to some problems involving minimally generated matroids and Tutte polynomials [1].

Mahendra Jani and Melkamu Zeleke give an alternative bijective proof of the generalized Tennis Ball Problem using k-trees. They also obtain a summation formula for all possible arrangements of balls out on the lawn using k-trees and lattice paths [7].

In the generalized tennis ball problem, sn balls are placed into a basket in n batches of s, and t balls are removed at every turn $(0 < t < s)$. The sn balls are numbered, and we ask for the number of different sets of tn balls that are removed. With $k = t$ and $l = s - t$, this number is equal to the number of lattice paths with unit North (N) and East (E) steps that start at the origin, go to the point (ln, kn), and never go above the boundary path $N^k E^l N^k E^l \cdots N^k E^l = (N^k E^l)^n$. For any boundary path P, Anna de Mier and Marc Noy [4] introduce the polynomial $t(M[P], x, y) = \sum_\pi x^{i(\pi)} y^{e(\pi)}$, where the summation runs through all paths (weakly) below P, and $i(\pi)$ (resp. $e(\pi)$) counts the number of N steps (resp. E steps) that π and P have in common, i.e., occur at the same place when the paths are written as a sequence of steps. If PN stands for P appended by N, then it is clear that $t(M[PN], x, y) = xt(M[P], x, y)$. The authors prove that

$$t(M[PE], x, y) = t(M[PE], x, y)x/(x-1) + t(M[PE], 1, y)(y - x/(x-1)).$$

Of course we want to determine $t(M[PN], 1, 1)$, the number of paths below P. By applying the kernel method, a theorem more general than the above tennis ball problem, the following is shown: If k_i and l_i are r positive integers each, then the number q_n of lattice paths from the origin to $(n\sum l_i, n\sum k_i)$ never crossing $(N^{k_1}E^{l_1} \cdots N^{k_r}E^{l_r})$ has generating function $Q(z) = \sum_{n\geq 0} q_n z^n = R(w_1, \ldots, w_l)/z$, where $w_1, \ldots, w_l$ are the unique solutions of $(w-1)^l - zw^{k+l} = 0$, and R is a computable symmetric rational function. In the case of the tennis ball problem all k_i are equal to k, and all l_i are equal to l. In this case, $Q(z) = -(1-w_1)\cdots(1-w_l)/z$.

It should be noted that the problem of the number of paths dominated by a fixed path goes back at least to G. Kreweras [8]. The numbers f_i of dominated paths reaching the boundary path at the i-th E step (so that $f_{ln} = q_n$) can be recursively calculated from $f_0 = 1$, and

$$f_i = \sum_{j=0}^{i-1} f_j (-1)^{i-1-j} \binom{(\lfloor j/l \rfloor + 1)k + 1}{i-j}.$$

A flag matroid can be viewed as a chain of matroids linked by quotients. Flag matroids, of which relatively few interesting families have previously been known, are a particular class of Coxeter matroids. Anna de Mier gives a family of flag matroids arising from an enumeration problem that is a generalization of the tennis ball problem [3]. These flag matroids can also be defined in terms of lattice paths, and they provide a generalization of the lattice path matroids of [1].

2 Proof of a Conjecture about Tennis Ball Numbers

In [5], the authors present the (s, t) tennis ball problem as a lattice paths enumeration problem, along with a conjecture, which we prove here.

Theorem 2.1 *(Stated as a conjecture in [5]) Let $t_k(n)$ represent the number of (E, N) paths from the origin to the point (k, n), staying strictly above $E\left(E^bN^b\right)^{\infty}$, and C_n the n-th Catalan number, then $\sum_{i=0}^{b-1} t_{bn}(bn - i) = b \cdot C_{bn}$.*

In [5], the tennis ball numbers are at the points $((s - t)n + 1, tn + 1)$, and the sum in the conjecture is used in the derivation of the generating function for these numbers. Note that the conjecture says the numbers $t_{bn}(b(n-1)+1)$, $t_{bn}(b(n-1)+2)$,. . . ,$t_{bn}(b(n-1)+n)$ in the average are equal to C_{bn}. This is evident for $b = 1$, but also holds for larger values of b.

Proof. For the number of (E, N) paths from the origin to the point $(sn + 1, tn)$ that stay strictly above the infinite staircase S_1, where S_1 is the path starting at $(0, t)$ then following $(E^sN^t)^{\infty}$, Chapman et al. [2] gave

$$|S_1| = t\binom{sn+tn}{tn} - s\binom{sn+tn}{tn-1}.$$

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Restate the path S_1 as the path starting at $(t, 0)$, then following $(N^sE^t)^{\infty}$. In this case, the formula above counts paths strictly above S_1 from the origin to $(tn, sn + 1)$. This is the formulation we will use to prove the conjecture.

Reversing these paths, and considering the weak boundary above S_1, we have $|S_1|$ as the number of paths with steps W and S from the point $(tn, sn + 1)$ to the origin,staying weakly above $W\left(S^kW^k\right)^{n-1}S^{k+1}W^k$.

The number of (E, N) paths from the origin to the point $(sn + 1, tn)$ that avoid the path $E\left(E^bN^b\right)^{n-1}E^{b+1}N^b$ is counted by $\sum_{i=0}^{b-1} t_{bn}(bn - i)$, that is, the paths that stay weakly above the paths $N\left(E^bN^b\right)^{n-1}E^{b+1}N^b$. Replacing each N step with a W step and each E step with an S step, and interchanging x and y coordinates (as the horizontal and vertical steps have changed places), we see that this counts the number of paths from the point $(tn, sn + 1)$ to the origin,staying weakly above $W\left(S^kW^k\right)^{n-1}S^{k+1}W^k$, for which the formula, given in Chapman et al., is $b\binom{bn+bn}{bn} - b\binom{bn+bn}{bn-1} = b \cdot C_{bn}$.

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