Counting Lattice Paths with Privileged Access using Sheffer Sequences

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Congressus Numerantium **143** (2000) *141 – 160*

Abstract

A lattice path has privileged access to the line $y = a(x - \ell)$, $a, \ell \in \mathbb{N}_1$ if there is an additional set of (privileged) step vectors P such that the lattice point $(n, a(n-\ell))$ for $n \geq \ell$ can only be reached by steps from P. If P is empty, then the lattice paths are bounded by a line; a classical problem in path enumeration. Different from bounded lattice path problems, privileged access problems give these boundaries new path entry and exiting characteristics. We find closed form solutions for some specific privileged access problems using linear operators and functionals with Sheffer sequences.

Keywords: Umbral Calculus, Sheffer sequences, privileged access
lattice path counting
AMS Classification: 05140, 05A15

1 Introduction

Denote by $D(n, m)$ the number of lattice paths from the origin to (n, m). The *standard* lattice path has *step set* $\mathfrak{E} = \{\rightarrow, \uparrow\} = \{(1,0),(0,1)\}$; a (c,γ)-*path* has *step set* $\mathfrak{E} = \{\rightarrow, \uparrow, (c,\gamma)\}$ where $c \in \mathbb{N}_1, \gamma \in \mathbb{Z}$. The paths stay in the first quadrant above a line $y = a(x-\ell)$, $a, \ell \in \mathbb{N}_1$. The integer lattice $\{(n,m) \mid n, m \in \mathbb{Z}\}$ is a subset of the (x, y) plane; although we refer to the line $y = a(x - \ell)$ and to the *restricted (half)line* $y = a(x-\ell)$ for $x \geq \ell$, we are mainly interested in the *restricted points* $\{(n, a(n-\ell)) \mid n \geq \ell\}$. Introduced to us in [1] by Merlini, Rogers, Sprugnoli, and Verri, a lattice path has *privileged access* to the restricted line if there is a (privileged) *access step set* $\mathcal{P}$ such that the restricted points $(n, a(n-\ell))$ can only be reached by access steps from $\mathcal{P}$. Figure 1 shows a sample boundary. The solution method in this paper does not require the first piece of the boundary to be horizontal nor must the boundary have only two pieces [4]; however, we limit the scope of this paper to a two piece boundary (3) like that shown in Figure 1.

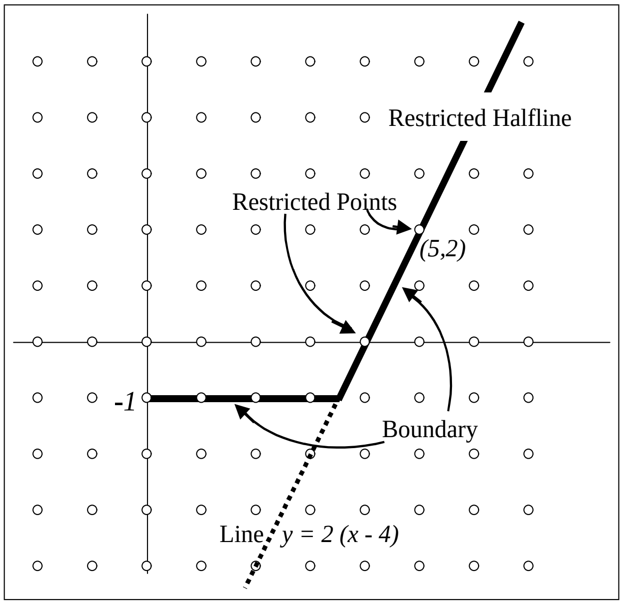

Figure 1: Restricted Line

If the path can exit the restricted line via the step set to lattice points above the boundary then this defines a lattice path problem with recursive initial conditions at $(n, a(n-\ell))$. In 1999, we [5] gave explicit solutions to Sheffer sequences with initial conditions determined by functionals on the sequence. In this paper, we translate the privileged access condition to a functional that evaluates to zero on the restricted line. We combine the earlier results of path counts strictly above a line in [4] and this newer technique of specifying initial conditions to give "closed form" solutions to privileged access lattice path counting problems. If the path is not allowed to exit the boundary, the number of paths reaching the boundary is found in section 4.1.

In Section 2 we give the necessary theory of linear operators and functionals on the polynomial ring $k[x]$ which can be found in [3], [6], and [7]. In section 3 the results from our prior work, [4] and [5], are specialized to this new problem. We establish sufficient conditions that guarantee an explicit solution for counting lattice paths with privileged access to the restricted line $y = a(x-\ell)$, $a, \ell \in \mathbb{N}_1$. We derive a general solution and then demonstrate its use by some specific examples.

We first exemplify the goal. In figures 2 and 3 we show a sample path and the values $D(n,m)$ for a privileged access problem of Type S in Table 1 with line $y = x - 2$, step set $\mathfrak{E} = \{\rightarrow, \uparrow\}$, and access step set $\mathcal{P} = \{(3,0)\}$. The path may exit the boundary to points above it with a step from $\mathfrak{E}$. Notice the path counts on the boundary line at $(n, n-2)$ are equal to the counts at $(n-3, n-2)$.

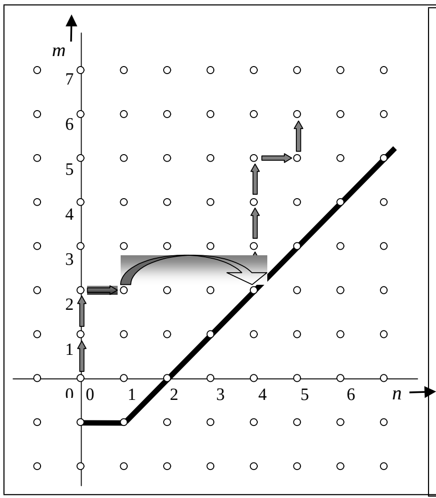

Figure 2: Sample Path to (5,6)

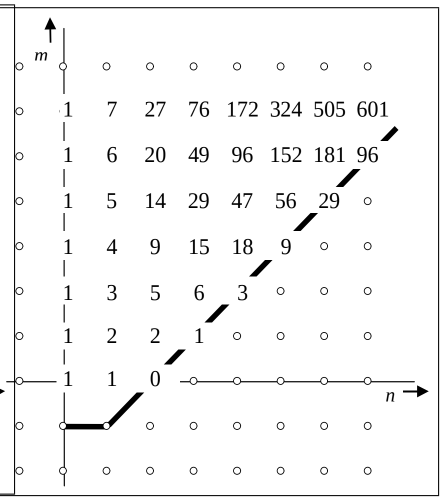

Figure 3: Path Counts

In Figure 3, we can view each column n as the values of a polynomial $d_n(x)$ evaluated at $x = m$, the height of the path[1]. For our example, the polynomials are

$$d_n(x) = \sum_{j=0}^{\lfloor \frac{n}{3} \rfloor} \frac{x - n + 2 + 3j}{x+2} \binom{x + 1 + n - 3j}{n - 3j}.$$

The first few polynomials in the sequence are $d_0(x) = 1$, $d_1(x) = x + 1$, $d_2(x) = \frac{1}{2}x^2 + \frac{3}{2}x$, and $d_3(x) = \frac{1}{6}x^3 + x^2 + \frac{5}{6}x - 1$. For example, the number of paths from $(0,0)$ to $(3,4)$ is $D(3,4) = d_3(4) = 29$. We show how to derive this answer in section 4.2.3.

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[1] These polynomials, written in the generic independent variable x, are evaluated at integer values m along the y axis; n runs along the x axis.

2 Sheffer Sequences

The necessary theory of linear operators on the polynomial ring $k[x]$ where k is a field of characteristic zero can be found in [3], which gives the underlying theory of the Umbral Calculus, and in [7] which introduces the Binomial Theorem of Sheffer sequences. The theory of linear functionals on $k[x]$ and how it is connected with the linear operator theory can be found in [6].

2.1 Operators

Let $k\left[x\right]$ be the algebra of all polynomials over a field k of characteristic 0. A *polynomial sequence* in $k\left[x\right]$ is denoted by $(p_n(x))$ where p_n has degree n. We set $p_n(x)=0$ for all x iff $n<0$. A *Sheffer sequence* $(s_n(x))$ is a polynomial sequence with generating function

$$\sum_{n\geq 0} s_n(x)t^n = \rho(t)e^{x\beta(t)}$$

where $\rho(t)$ and $\beta(t)$ are power series of order 0 and 1 respectively [7].

Let $Hom_k(k[x],k[x])$ be the linear operators on $k[x]$. Let $Q(t)$ be the formal inverse (compositional) of $\beta(t)$ and D_x be the derivative operator on $k\left[x\right]$. The linear operator $Q:=Q(D_x)$ is the *delta operator* associated with $(s_n(x))$. Delta operators and their Sheffer sequences are related by the "backwards recursion" [2]

$$Qs_n = s_{n-1} \text{ for } n \geq 0.$$

Every delta operator is *shift-invariant*; it commutes with all shift operators E^a where $E^a p(x) = p(x+a)$ for every $a\in k$ and $p(x)\in k\left[x\right]$.

The *basic sequence* $(q_n(x))$ is the unique Q- Sheffer sequence with initial values $q_n(0)=\delta_{0,n}$. The sequence $\left(\left(n+1\right)q_{n+1}(x)/x\right)_{n=0,1,\ldots}$ is a Q-Sheffer sequence [7, p. 702] and the linear combination of Sheffer sequences

$$s_n(x) := \frac{x-an-b}{x-b}q_n(x-b)$$

is again a Sheffer sequence.

Given that (s_n) is a Sheffer sequence and (q_n) is the basic sequence for delta operator Q, the *Binomial Theorem* for Sheffer sequences [7] states

$$s_n(x+y) = \sum_{i=0}^{n} s_i(y)q_{n-i}(x). \tag{1}$$

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[2] Divide the Sheffer sequences in [7] by $n!$ to agree with our definition.

2.2 Functionals

Let $k^*[x]$ be the vector space of linear functionals on $k[x]$. We denote the action of a linear functional L on a polynomial $p(x)$ by

$$\langle L \mid p(x) \rangle .$$

The *umbral algebra* [6, p. 101] of linear functionals defines the product of two functionals L and M as

$$\left\langle LM \mid \frac{x^n}{n!} \right\rangle = \sum_{i=0}^{n} \left\langle L \mid \frac{x^i}{i!} \right\rangle \left\langle M \mid \frac{x^{n-i}}{(n-i)!} \right\rangle .$$

The product can be computed using any basic sequence instead of $\left(\frac{x^n}{n!}\right)$.

The functional ε_a is defined as *evaluation at a,*

$$\langle \varepsilon_a \mid p(x) \rangle = p(a).$$

The identity of the product is evaluation at 0, called *augmentation* and denoted by ε.

For all linear functionals $L \in k^*[x]$ define $\mu : k^*[x] \rightarrow Hom_k(k[x], k[x])$ by

$$\mu(L) = \sum_{n \geq 0} \left\langle L \mid \frac{x^n}{n!} \right\rangle D_x^n. \tag{2}$$

The mapping μ is a one-to-one homomorphism onto the set of shift invariant operators such that

$$\langle L \mid p(x) \rangle = \langle \varepsilon \mid \mu(L) p(x) \rangle .$$

Example 1 $\mu(\varepsilon_a) = E^a$.

With respect to the above product, a linear functional L is invertible iff $\langle L \mid 1 \rangle \neq 0$ and therefore a shift invariant operator $\mu(L)$ is invertible iff $\mu(L)1 \neq 0$.

3 Prior Results

3.1 Paths Strictly above a Boundary

The following results for paths staying strictly above the boundary are needed for our method to solve privileged access problems. All results for (c, γ) paths in this section are derived in [4].

From the Binomial Theorem (1), we see that given the basic sequence a Sheffer sequence is determined if one value of $d_n(x)$ is known for each n. We call these values the initial conditions of the sequence and usually choose them on the boundary. Let $\nu_n : \mathbb{N}_0 \to \mathbb{Z}$ be a specific boundary function

$$\nu_n = \begin{cases} -1 & \text{if} \quad 0 \le n < \ell \\ a(n-\ell) & \text{if} \quad \ell \le n \end{cases} \tag{3}$$

with $a, \ell \in \mathbb{N}_1$. For clarity we designate the lattice points (n, m) where $m > \nu_n$ and $n \ge 0$ as *upper points* and points where $D(n, m) = d_n(m)$ as *congruous points*. For $n < 0$ there is no boundary defined and no path reaches these points; by convention $d_n(x) = 0$, and so these points are also congruous points. Exceptional congruous points are points (n, m) weakly below the boundary where m is a root of the polynomial $d_n(x)$ for $n > 0$. A lattice path problem utilizing exceptional congruous points is considered in the Appendix, but in the main part of the paper, the *congruous region* and "congruous points" will refer to the nonexceptional congruous points. Figure 4 shows upper points and congruous points for a boundary function with $a = 2$ and $\ell = 4$. For all $n \ge 0$ the points (n, ν_n) are the path boundary. We talk about values of the polynomials (and of lattice path counts when $m > \nu_n$) in the *rectangles* when $0 \le n < \ell$ and in the *triangles* when $n \ge \ell$.

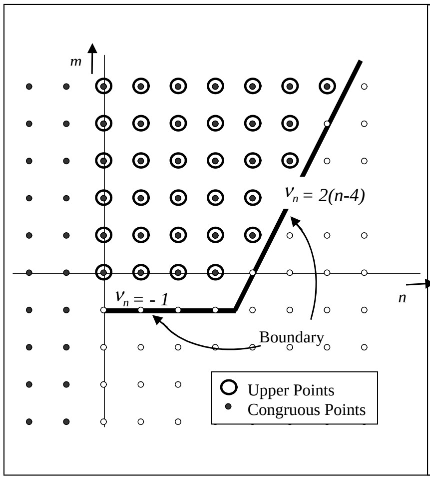

Figure 4: Boundary and Points

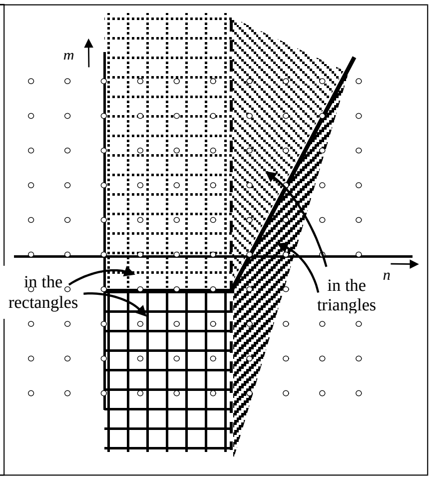

Figure 5: Rectangles and Triangles

We call a lattice path problem *admissible* if, given a step set and the recursion on $D(n, m)$, a constraint on the boundary function (e.g. a choice for a and ℓ) guarantees that only one initial value $D(n, m_n)$, say, for each n is needed to obtain the path counts. The counts of (c, γ)-paths strictly above a boundary will be values of Sheffer polynomials for all upper points if the problem is admissible. Admissibility, however, does not imply that the Sheffer polynomials can be found explicitly. We remark that there are lattice path counting problems that are not admissible in the strict sense as defined here, but that have *piecewise* Sheffer sequence solutions. In that case there is a second boundary $\nu_n^{'} \geq \nu_n$, where $d_n(m) = D(n,m)$ for all $m > \nu_n^{'}$ for some Sheffer sequence $(d_n(x))$.

For standard and for (c,γ) paths we state the delta operators with their associated basic sequences. We find the conditions on the step set and on a and ℓ for admissible problems (in the strict sense) along with appropriate initial values. For both step sets, we state the explicit solutions for paths staying strictly above $y=-1$ and for the paths strictly above the boundary (3).

3.1.1 Basic Sequences for Delta Operators

To use the Binomial Theorem we must know the basic sequence associated with a delta operator. For the standard lattice path [3, p. 186] the delta operator is the backwards difference operator ∇

$$\nabla d_n(x) = d_n(x) - d_n(x-1) = d_{n-1}(x)$$

and its basic polynomials are $\binom{n+x-1}{n}$.

A (c,γ) path [4, p. 760] has delta operator W and associated basic sequence (w_n) where

$$W d_n(x) = d_n(x) - d_n(x-1) - d_{n-c}(x-\gamma) = d_{n-1}(x) \quad \text{and}$$

$$w_n(x) = x \sum_{k \geq 0} \binom{n+k(1-c)}{k} \frac{1}{n+k(1-c)} \binom{x+n+k(1-c-\gamma)-1}{n+k(1-c)-1}.$$

This simplifies for $\gamma = 1$ to

$$w_n(x) = \sum_{k=0}^{\lfloor n/c \rfloor} \binom{x}{k} \binom{x+n-1-kc}{n-ck}.$$

3.1.2 Admissibility and Initial Values

Admissibility for (c,γ) paths with a boundary function

$$\nu_n = \begin{cases} -1 & \text{if} \quad 0 \leq n < \ell \\ a(n-\ell) & \text{if} \quad \ell \leq n \end{cases}$$

means those (c,γ)- lattice path problems for which there is a Sheffer sequence $(d_n(x))$ such that $d_n(m) = D(n,m)$ for all $m > \nu_n$.

Any positive integers a and ℓ are admissible for a standard lattice path. For a (c,γ)-path step set (which assumes a positive c), admissible values for γ, a and ℓ derived from [4] are listed in Table 1 below. The table also shows the initial values $d_n(-1)$ for $0 \leq n < \ell$ and $d_n(a(n-\ell))$ for $n \geq \ell$

Type	a	ℓ	γ	$d_n(-1)$	$d_n(a(n-\ell))$
S	$a \geq 0$	$\ell \geq 1$	standard path	$\delta_{n,0}$	0
A	$a \geq 0$	$\ell \geq 1$	$\gamma \leq 0$	$\delta_{n,0}$	0
B	$a \geq 0$	$\ell > c$	$\gamma = 1$	$(-1)^{n \div c}$	0
C	$a \geq 1$	$1 < \ell = c$	$\gamma = 2$	$\delta_{n,0}$	$-\delta_{c,n}$
D	$a = 1$	$1 = \ell \leq c$	$\gamma = c+1$	$\delta_{n,0}$	$(-1)^{n \div c}$
E	$a > 1$	$1 = \ell < c$	$\gamma = a(c-1)+2$	$\delta_{n,0}$	$-\delta_{c,n}$
F	$a \geq 1$	$1 < \ell < c$	$\gamma = a(c-\ell)+2$	$\delta_{n,0}$	$-\delta_{c,n}$
G	$a \geq 1$	$1 \leq \ell \leq c$	$\gamma < a(c-\ell)+2$	$\delta_{n,0}$	0

Table 1: Admissibility and Initial Conditions

where $(-1)^{n \div c} := (-1)^{\frac{n}{c}}$ if c divides n and zero otherwise.

3.1.3 Explicit Solutions

The path counts in the upper rectangle have initial conditions along the constant $m = -1$. The expansion of a Q-Sheffer sequence $(d_n(x))$ with basic sequence $(q_n(x))$ along this horizontal boundary is given by the Binomial Theorem (1) with $y = -1$,

$$d_n(x) = \sum_{i=0}^{n} d_i(-1) q_{n-i}(x+1).$$

The initial conditions $d_i(-1)$ are found in Table 1 and the appropriate basic polynomial from section 3.1.1 is substituted for q_{n-i}. If $\gamma > 0$, then by Type B in Table 1, $\gamma = 1$ if $c < \ell$. If $c \geq \ell$ then in the upper rectangle the path is equivalent to a standard lattice path. The path counts in the upper rectangle and the values of the Sheffer polynomials in both rectangles are

- for $\mathfrak{E} = \{\rightarrow, \uparrow\}$ (Type S) and $\mathfrak{E} = \{\rightarrow, \uparrow, (c,\gamma)\}$ where $c \geq \ell$ (Types C - G)

$$d_n(m) = \binom{n+m}{n}, \tag{4}$$

- for $\mathfrak{E} = \{\rightarrow, \uparrow, (c,\gamma)\}$ where $\gamma \leq 0$ (Type A)

$$\begin{aligned} d_n(m) \quad = \quad & \sum_{i=0}^{\lfloor n/c \rfloor} (-1)^i \sum_{k=0}^{\left\lfloor \frac{n}{c} \right\rfloor - i} \left(\binom{m-k(\gamma-1)+1}{k} \times \right. \\ & \left. \frac{m+1}{m-k(\gamma-1)+1} \binom{m+n-c(i+k+\gamma)+k}{n-c(i+k)} \right), \end{aligned} \tag{5}$$

- and for the special case $\mathfrak{E} = \{\rightarrow, \uparrow, (c,1)\}$, $c \in \mathbb{N}_1$ (Type B)

$$d_n(m) = \sum_{j=0}^{\lfloor n/c \rfloor} \binom{m+n-cj}{n-cj}\binom{m}{j}. \tag{6}$$

To enumerate paths strictly above a boundary like (3) we use the following variation of the Binomial Theorem which lends itself to initial conditions at points $(n, a(n-\ell))$.

If (q_n) is the basic sequence of a delta operator Q, then with [3, p. 194]

$$q_n(x \mid a) := \frac{x}{x+an} q_n(x+an)$$

is the basic sequence for $E^{-a}Q$, the *Abelization of* Q [7, p. 711]. If $(d_n(x))$ is Sheffer for Q, then $(d_n(x+an))$ is Sheffer for $E^{-a}Q$. Apply the Binomial Theorem (1) to the basic polynomials $q_n(x \mid a)$ and the Sheffer polynomials $d_n(x+an)$ to derive the 'extended' Binomial Theorem:

$$d_n(x+y+an) = \sum_{i=0}^{n} d_i(y+ai)\left(\frac{x}{x+a(n-i)} q_{n-i}(x+a(n-i))\right). \tag{7}$$

With the appropriate substitutions the explicit solution for paths staying strictly above the boundary ν_n is therefore

$$d_n(x) = \sum_{i=0}^{n} d_i(a\,(i-\ell))\left(\frac{x-a\,(n-\ell)}{x-a\,(i-\ell)} q_{n-i}(x-a\,(i-\ell)\right). \tag{8}$$

For $i = 0, ..., \ell - 1$, the coefficients $d_i(a\,(i-\ell))$ are values in the rectangles as computed in (4), (5), or (6). For $i \geq \ell$ the values $d_i(a\,(i-\ell))$ are the initial conditions for path counts in the triangles as given in section 3.1.2. The appropriate basic polynomial from section 3.1.1 must be substituted for q_{n-i}.

3.2 Defining Initial Conditions with a Functional

In a more narrow treatment than given in [5], we state a 'generalized' Binomial Theorem for Sheffer sequences.

Theorem 2 *Suppose L is an invertible linear functional and (b_n) the basic sequence and (s_n) a Sheffer sequence for the delta operator B, then*

$$s_n(x) = \sum_{i=0}^{n} \langle L \mid s_i \rangle \, \mu(L)^{-1} b_{n-i}(x) \tag{9}$$

where $\mu(L)^{-1} = \mu(L^{-1})$.

We note the similarity of this theorem and Theorem 11 in [6]. Theorem 2 has a computational advantage calculating $s_n(x)$ when initial conditions of the Sheffer sequence $\langle L \mid s_i(x)\rangle = 0$ are given for $i \geq \ell$.

Example 3 *For a trivial example we derive the Binomial Theorem (1) from Theorem 2. Let L be evaluation at* $-y$. *We have* $\mu(L)^{-1} = \mu\left(\varepsilon_{-y}\right)^{-1} = E^y$ *per (2) and hence*

$$\begin{aligned} s_n(x) &= \sum_{i=0}^{n} \langle \varepsilon_{-y} \mid s_i \rangle\, E^y b_{n-i}(x) \\ &= \sum_{i=0}^{n} s_i(-y) b_{n-i}(x+y). \end{aligned}$$

4 Privileged Access

A lattice path has *privileged access* to the restricted line $y = a\,(x-\ell)$ for $x \geq \ell$ if there is a step $(p,q) \in \mathbb{Z}^2$ in the access step set $\mathcal{P}$ by which the *restricted point* $(n, a\,(n-\ell))$ can be reached from an upper point $(n-p, a(n-\ell)-q)$ in the path. An *access point* is any point $(n-p, a(n-\ell)-q)$ where $(p,q) \in \mathcal{P}$.

Sheffer sequences are sought to give explicit solutions for counting privileged access paths. Recall that *admissibility* guarantees that the lattice path counts are values of Sheffer polynomials. In addition to admissibility of the boundary and step (c,γ) as described in section 3.1.2, admissibility must also be characterized for the access step set $\mathcal{P}$. Foremost, since no value of a polynomial is infinite, the number of access points with nonzero values $d_{n-p}(a\,(n-\ell)) - q)$ for each restricted point must be finite; we call this the *finiteness condition.* To further describe admissibility and problem solutions, we divide privileged access lattice path counting problems into two types: paths that can not exit the boundary via a step vector and paths that can.

When paths can not exit to the upper points, the boundary ν_n acts like an absorbing boundary; the lattice paths that reach an upper point must have stayed strictly above the boundary ν_n. Therefore, besides conditions on $\mathcal{P}$, the admissibility for these lattice path problems and the Sheffer sequence solutions for the counts at the upper points are the same as in section 3.1.3. The number of paths S_n via privileged access to the restricted point $(n, a(n-\ell)$ is a function of these Sheffer sequence solutions and is not part of the Sheffer sequence itself.

When the lattice path can exit to the upper points, the path counts in the upper triangle are different from those with paths strictly above the boundary. This defines a new lattice path problem. Admissibility differs from the no exit problems. The points on the restricted line become part of the Sheffer sequence; the privileged path counts at the restricted points become the new initial conditions. The initial conditions are expressed by an invertible functional L defined so that $\langle L | d_n(an - x) \rangle = 0$ for $n \geq \ell$. With this functional and the generalized Binomial Theorem 2 some privileged access problems can be solved explicitly.

4.1 No Exit from the Boundary

If the path is not allowed to exit the boundary, then S_n, the number of paths to the restricted point $(n, a(n - \ell))$, is the sum of paths to its corresponding access points. The access set $\mathcal{P}$ must ensure that for all $(p, q) \in \mathcal{P}$, an access point $(n - p, a(n - \ell) - q)$ is in the congruous region. Therefore, $\mathcal{P}$ is admissible if it satisfies the finiteness condition and if $\mathcal{P}$ is a subset of

$$\{(p, q) \mid p \leq 0, q < ap\} \cup \{(p, q) \mid p > 0, q \leq \max(a(p - \ell), 0)\}. \tag{10}$$

(See the Appendix for more details.)

We address two cases. If the paths must end when they reach a restricted point, then

$$S_n = \sum_{(p,q) \in \mathcal{P}} D(n - p, a(n - \ell) - q) = \sum_{(p,q) \in \mathcal{P}} d_{n-p}(a(n - \ell) - q)$$

is a mere sum of explicitly known path counts. If the path can continue along the boundary, then the path counts S_n are recursive. An example of this is the *Fast Collector*, where paths can not exit but can continue up the restricted line in arbitrarily large steps $\{(j, aj) \mid j > 0\}$. Let $(d_n(x))$ be the Sheffer sequence such that $D(n, m) = d_n(m)$ for all upper points (for $m > \max\{-1, a(n - \ell)\}$). We let $D_n := \sum_{(p,q) \in \mathcal{P}} d_{n-p}(a(n - \ell) - q)$. The numbers S_n follow the recursion

$$S_n = D_n + \sum_{k=\ell}^{n-1} S_k = 2S_{n-1} + D_n - D_{n-1}$$

for $n \geq \ell$. By induction, for $n > \ell$,

$$S_n = 2^{n-\ell-1} D_\ell + D_n + \sum_{k=1}^{n-\ell-1} 2^{k-1} D_{n-k}. \tag{11}$$

Example 4

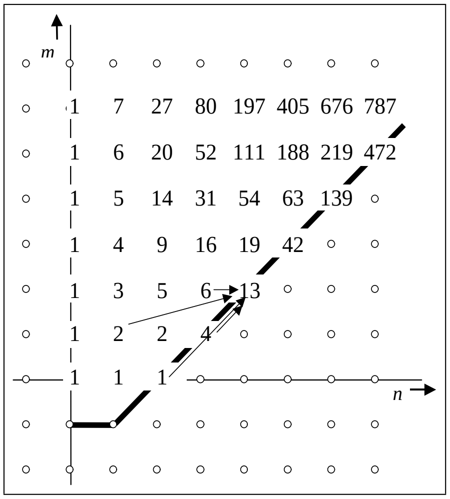

Figure 7: Path Counts

Suppose $(3,1)$*-paths* ($\mathfrak{S} = \{\rightarrow, \uparrow, (3,1)\}$) *stay weakly above the subdiagonal* $y = a\,(x-\ell)$. *Let the restricted line* $y = a\,(x-\ell)$ *be a Fast Collector with access step set* $\mathcal{P} = \{\rightarrow, (3,1)\}$. *How many paths reach the restricted point* $(n, a\,(n-\ell))$ *for* $n \geq \ell$? *Figure 6 shows a sample path to the restricted point* $(7,5)$ *on the restricted line* $y = x-2$ *for* $x \geq 2$. *Figure 7 gives the path counts from the origin to* (n,m) *and shows the specific components for point* $(4,2)$. *Given the step and access step sets, this problem is admissible by condition (10) for* $\ell \leq 2$ *where* $a, \ell \in \mathbb{N}_1$. *We let* $a = 1$ *and* $\ell = 2$ *as in figures 6 and 7 (Type G).*

$$D_n = \sum_{(p,q)\in\mathcal{P}} d_{n-p}(n-2-q) = d_{n-1}(n-2) + d_{n-3}(n-3)$$

and

$$\begin{aligned} S_n &= 2^{n-3} + d_{n-1}(n-2) + d_{n-3}(n-3) \\ &\quad + \sum_{k=1}^{n-3} 2^{k-1}\left(d_{n-k-1}(n-k-2) + d_{n-k-3}(n-k-3)\right) \end{aligned}$$

for $n > 2$ *and* $S_2 = D_2 = d_1(0) + d_{-1}(-1) = 1$. *For* $d_n(x)$, *substitute the explicit solution (8) to path counts strictly above the boundary.*

4.2 Exit to the Upper Points

Allow a path that reaches the restricted line to exit to the upper points with a step vector; in the case of our boundary (3) this means with a vertical step up ↑. The exiting paths add to those paths that stay strictly above the boundary. In this section, "privileged access problems" means problems where the paths can exit from the restricted line to upper points.

4.2.1 Admissibility

Admissibility guarantees that the Sheffer polynomial recursion can replace the path recursion. For a (c,γ)-path this means

$$d_n(x) = d_n(x-1) + d_{n-1}(x) + d_{n-c}(x-\gamma)$$

can replace

$$D(n,m) = D(n,m-1) + D(n-1,m) + D(n-c,m-\gamma).$$

At $x = \nu_n + 1$, we have

$$d_n(\nu_n + 1) = d_n(\nu_n) + d_{n-1}(\nu_n + 1) + d_{n-c}(\nu_n + 1 - \gamma). \tag{12}$$

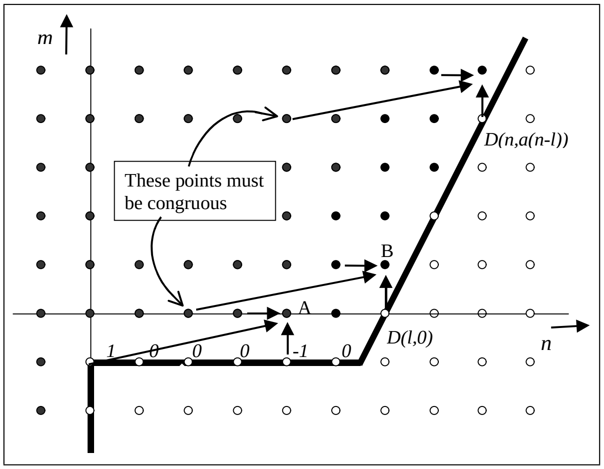

Figure 8: Access Points and Initial Values

In a privileged access problem, the numbers of paths that reach the restricted line at n

$$D\left(n, a\left(n-\ell\right)\right) = \sum_{(p,q)\in\mathcal{P}} D\left(n-p, a\left(n-\ell\right)-q\right) \tag{13}$$

are the initial values $d_n(a\,(n-\ell) = d_n\,(\nu_n)$ for the Sheffer polynomials in the triangles. For $n \geq \ell$ the number of paths that reach $(n, \nu_n + 1)$ (point B in Figure 8) without going through the restricted point $(n, a\,(n-\ell))$ must be the sum of $d_{n-1}(\nu_n+1)$ and $d_{n-c}(\nu_n+1-\gamma)$ by (12). As (n, ν_n+1) and $(n-1, \nu_n+1)$ are both congruous points and $d_n(\nu_n)$ is already given by (13) for $n \geq \ell$, the access point $(n-c, \nu_n+1-\gamma)$ must also be a congruous point for $n \geq \ell$. Notice, $(n-c, \nu_n+1-\gamma)$ when $n < \ell$ need not be congruous (see access points for A in Figure 8) because we are free to define $d_n(\nu_n)$ appropriately ($d_4(-1) = -1$ in Figure 8). With $c, a \in \mathbb{N}_1$ the point $(n-c, a(n-\ell)+1-\gamma)$ is congruous when $\gamma - 1 \leq \max(a(c-\ell), 0)$. Admissibility types and corresponding initial values are listed in Table 2.

Type	a	ℓ	γ	$d_n(-1)$
S	$a \geq 0$	$\ell \geq 1$	standard path	$\delta_{n,0}$
A	$a \geq 0$	$\ell \geq 1$	$\gamma \leq 0$	$\delta_{n,0}$
B	$a \geq 0$	$\ell > c$	$\gamma = 1$	$(-1)^{n \div c}$
G	$a \geq 1$	$1 \leq \ell \leq c$	$\gamma < a(c-\ell)+2$	$\delta_{n,0}$

Table 2: Privileged Access Admissibility and Initial Values

The admissibility defined here could be relaxed to include, for example, all problem types given in Table 1. In that case, the Sheffer sequence solutions need correcting at some noncongruous, restricted points.

It follows from the finiteness condition on the privileged access set $\mathcal{P}$ that the access steps can not create self intersecting paths. For any $q \in \mathbb{Z}$ this condition requires an access step (p, q) to have $p > 0$.

4.2.2 General Solution

If the access points are congruous points, then for $n \geq \ell$ the number of paths on the restricted line

$$D\,(n, a\,(n-\ell)) = \sum_{(p,q)\in\mathcal{P}} D\,(n-p, a\,(n-\ell)-q)$$

can be replaced by

$$d_n\,(a\,(n-\ell)) = \sum_{(p,q)\in\mathcal{P}} d_{n-p}\,(a\,(n-\ell)-q)\,,$$

and there is a solution in terms of Sheffer sequences.

Theorem 5 *Suppose $\gamma, a,$ and ℓ are admissible per Type S, A, B or G and the step sets are $\mathfrak{E} = \{\rightarrow, \uparrow\}$ or $\mathfrak{E} = \{\rightarrow, \uparrow, (c,\gamma) \mid c \in \mathbb{N}_1, \gamma \in \mathbb{Z}\}$. If the access step set $\mathcal{P}$ is contained in $\{(p,q) \mid p, q \in \mathbb{Z}, p > 0, q \leq \max\,(a(p-\ell), 0)\}$,* *then the number $D(n,m)$ of lattice paths above and with privileged access to the restricted line $y = a(x-\ell)$ for $x \geq \ell$ are values of Sheffer polynomials $d_n(x)$ such that $D(n,m) = d_n(m)$ for all $m \geq \max\{0, a(n-\ell)\}$, i.e., the points (n,m) are congruous.*

Because the initial values on the restricted line for privileged access problems are values of the same function for all n, they lend themselves to functional evaluation on the restricted line. Starting with the privileged access condition

$$d_n\left(a\left(n-\ell\right)\right) - \sum_{(p,q)\in\mathcal{P}} d_{n-p}\left(a\left(n-\ell\right)-q\right) = 0,$$

we define L so that

$$\langle L \mid d_n(an+x)\rangle = 0 \text{ for all } n \geq \ell.$$

In functional notation $\langle L \mid d_n\left(an+x\right)\rangle$

$$\begin{aligned} &= \langle \varepsilon_{-a\ell} \mid d_n\left(an+x\right)\rangle - \sum_{(p,q)\in\mathcal{P}} \left\langle \varepsilon_{a(p-\ell)-q} \mid d_{n-p}\left(a\left(n-p\right)+x\right)\right\rangle \\ &= \langle \varepsilon_{-a\ell} \mid d_n\left(an+x\right)\rangle - \sum_{(p,q)\in\mathcal{P}} \left\langle \varepsilon_{a(p-\ell)-q} \mid \left(E^{-a}Q\right)^p d_n\left(an+x\right)\right\rangle \end{aligned}$$

which gives

$$L = \varepsilon_{-a\ell} - \sum_{(p,q)\in\mathcal{P}} \varepsilon_{a(p-\ell)-q}\mu^{-1}\left(\left(E^{-a}Q\right)^p\right).$$

Because we have $p > 0$ for all $(p,q)\in\mathcal{P}$, it is easily seen that L is invertible.

$$\begin{aligned} \langle L \mid 1\rangle &= \left\langle \varepsilon_{-a\ell} - \sum_{(p,q)\in\mathcal{P}} \varepsilon_{a(p-\ell)-q}\mu^{-1}(E^{-ap}Q^p) \mid 1 \right\rangle \\ &= \langle \varepsilon_{-a\ell} \mid 1\rangle - \sum_{(p,q)\in\mathcal{P}} \langle \varepsilon_{-a\ell-q} \mid Q^p 1\rangle \\ &= 1 - 0. \end{aligned}$$

Given this functional L, we use the generalized Binomial Theorem 2

$$s_n(x) = \sum_{i=0}^{n} \langle L \mid s_i(x)\rangle\, \mu(L^{-1}) b_{n-i}(x)$$

where $s_n(x) = d_n(an+x)$ is a Sheffer sequence for $E^{-a}Q$ and by Abelization $b_n(x) = \frac{x}{x+an} q_n(x+an)$ is the basic sequence for $E^{-a}Q$. With

$\langle L \mid d_n(an+x)\rangle = 0$ for all $n \geq \ell$ we have the general solution

$$d_n(an+x) = \sum_{i=0}^{\ell-1} \langle L \mid d_i(ai+x)\rangle \, \mu(L^{-1}) \frac{x}{a\,(n-i)+x} q_{n-i}(a\,(n-i)+x). \tag{14}$$

For the functional L we get the unique operator

$$\begin{aligned} \mu(L) &= E^{-a\ell} - \sum_{(p,q)\in\mathcal{P}} E^{a(p-\ell)-q} \left(E^{-a}Q\right)^p \\ &= E^{-a\ell} \left(1 - \sum_{(p,q)\in\mathcal{P}} E^{-q}Q^p\right) \end{aligned}$$

and therefore

$$\mu(L)^{-1} = E^{a\ell} \left(1 - \sum_{(p,q)\in\mathcal{P}} E^{-q}Q^p\right)^{-1}.$$

Q and q_n are substituted with the appropriate delta operator and basic polynomial from section (3.1.1). $\langle L \mid d_i(ai+x)\rangle$ for $i = 0 \ldots \ell-1$ are values of Sheffer polynomials in the rectangles as computed in (4), (5), or (6).

4.2.3 Examples

The application of the general solution requires that closed forms can be found for $\langle L \mid d_n(an+x)\rangle$ in the rectangles, and that $\mu(L)^{-1}$ can be explicitly applied to the basic polynomial. We give two of the examples for which we found solutions, jump access and line access. Figure 9 shows a $(3,0)$ jump step and Figure 10 shows some access points and access steps for horizontal line access for the restricted point $(6, 4)$.

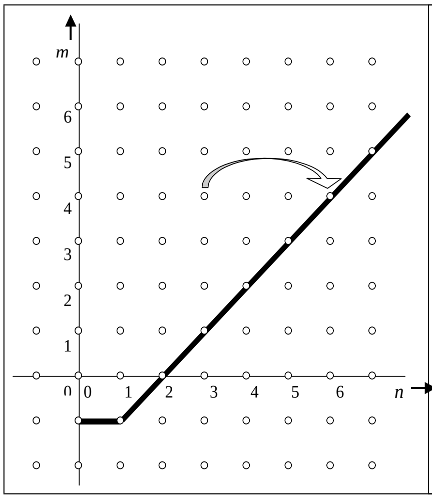

Figure 9: Jump Access

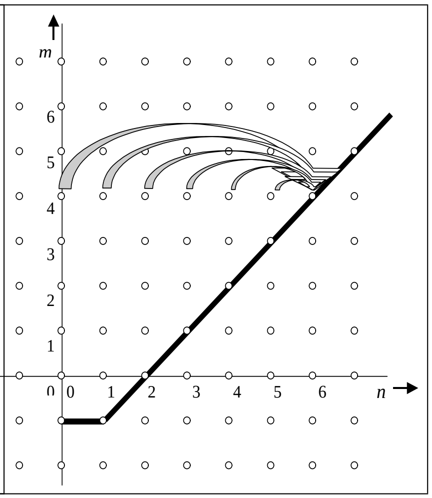

Figure 10: Line Access

If there is only one access step in $\mathcal{P}$ we call this type of privileged access problem *Jump Access.* With $a = \ell = 1$, jump access problems are solved with generating functions in [2]. Since $\mathcal{P}$ is finite, it clearly satisfies the finiteness condition. In this case

$$\mu(L)^{-1} = \frac{E^{a\ell}}{(1 - E^{-q}Q^p)}$$

is realized with a geometric sum. We show how $\mu(L)^{-1}$ is explicitly applied to the basic polynomial.

Because $b_n(x) := \frac{x}{x+an} q_n(x + an)$ is basic for $E^{-a}Q$,
$\mu(L)^{-1} b_{n-i}(a\,(n - i) + x)$

$$\begin{aligned}
&= E^{a\ell} \sum_{j \geq 0} \left(E^{-q}Q^p\right)^j b_{n-i}(a\,(n-i) + x) \\
&= E^{a\ell} \sum_{j \geq 0} E^{(ap-q)j} \left(E^{-a}Q\right)^{pj} b_{n-i}(a\,(n-i) + x)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^{\left\lfloor \frac{n-i}{p} \right\rfloor} b_{n-i-pj}(a\,(n + \ell - i) + x - qj) \\
&= \sum_{j=0}^{\left\lfloor \frac{n-i}{p} \right\rfloor} \frac{a\,(\ell + pj) + x - qj}{a\,(n + \ell - i) + x - qj} q_{n-i-pj}(a\,(n + \ell - i) + x - qj).
\end{aligned}$$

For $n < \ell$ the functional L subtracts two values of explicitly known Sheffer polynomials in the rectangles (see 3.1.3),

$$\langle L \mid d_n(an+x) \rangle = d_n(a(n-\ell)) - d_{n-p}(a(n-\ell)-q).$$

Example 6 *In the introduction we showed a jump access problem with* $\mathfrak{E} = \{\rightarrow, \uparrow\}$ *and* $\mathcal{P} = \{(3,0)\}$ *with the restricted line* $y = x - 2$ *(Type S). The enumeration problem is solved by*

$$d_n(n+x) = \sum_{i=0}^{1} \langle L \mid d_i \rangle \sum_{j=0}^{\left\lfloor \frac{n-2}{3} \right\rfloor} \frac{2+3j+x}{n+2-i+x} q_{n-i-3j}(n+2-i+x)$$

where

$$\begin{aligned} \langle L \mid d_i \rangle &= \binom{a(i-\ell)+i}{i} - \binom{a(i-\ell)+i-3}{i-3} \\ &= \binom{2i-2}{i} - \binom{2i-5}{i-3} \\ &= \delta_{i,0} \text{ for } i = 0,\ 1 \end{aligned}$$

and

$$q_n(x) = \binom{n+x-1}{n}.$$

When $m = an + x$ *the explicit solution is*

$$D(n,m) = \sum_{j=0}^{\left\lfloor \frac{n}{3} \right\rfloor} \frac{m-n+2+3j}{x+2} \binom{m+1+n-3j}{n-3j}.$$

When the access step set is $\mathcal{P} = \{(\alpha j, \beta j) \mid j > 0\}$ with $\alpha > 0$ we call this type of privileged access problem *Line Access.* The access set $\mathcal{P}$ satisfies the finiteness condition; as $d_n(x) = 0$ for all $n < 0$, the maximum number of access points $(n - \alpha j, a(n-\ell) - \beta j)$ for which $d_{n-\alpha j}(a(n-\ell) - \beta j) \neq 0$ for any $n \geq 0$ is $j = \left\lfloor \frac{n}{\alpha} \right\rfloor$.

In this case

$$\begin{aligned} E^{-a\ell} \mu(L)^{-1} &= \frac{1}{1 - \sum_{j \geq 1} \left(E^{-\beta} Q^{\alpha}\right)^j} = \frac{\left(1 - E^{-\beta} Q^{\alpha}\right)}{1 - 2E^{-\beta} Q^{\alpha}} \\ &= \sum_{j \geq 0} 2^j \left(E^{-\beta} Q^{\alpha}\right)^j - \sum_{j \geq 1} 2^{j+1} \left(E^{-\beta} Q^{\alpha}\right)^j \\ &= \left(1 + \sum_{j \geq 0} 2^j \left(E^{-\beta} Q^{\alpha}\right)^{j+1}\right) \end{aligned}$$

and

$$\langle L \mid d_n\left(an+x\right)\rangle = d_n\left(a\left(n-\ell\right)\right) - \sum_{j\geq 1} d_{n-\alpha j}\left(a\left(n-\ell\right)-\beta j\right).$$

If $\beta \leq \max(a\left(\alpha-\ell\right),0)$ then the access points are congruous points and the general solution (14) holds.

Example 7 *If we let $\alpha = 1$ and $\beta = 0$, then line access is horizontal as in Figure 10. Let $\mathfrak{E} = \{\rightarrow, \uparrow, (3,1)\}$ and $\mathcal{P} = \{(j,0) \mid j > 0\}$ with the restricted line $y = 2(x-3)$ for $x \geq 3$ (Type G). The enumeration problem is solved by*

$$D\left(n,m\right) = \sum_{i=0}^{1} \langle L \mid d_i\rangle \left(b_{n-i}(6-2i+m) + \sum_{k=0}^{n-i-1} 2^k b_{n-i-k-1}(6-2i+m)\right)$$

where

$$\langle L \mid d_i\rangle = \binom{2(i-3)+i}{i} \text{ for } i = 0,\ 1$$

and

$$b_n(x) = \frac{x-an}{x}\sum_{j=0}^{\lfloor n/3\rfloor}\binom{x}{j}\binom{x+n-1-3j}{n-3j}.$$

5 Appendix

5.1 Congruous Region

The region of solid points shown in the figures below is the congruous region. We show that the points $(n-p, a(n-\ell)-q)$ for $n \geq \ell$ are in the congruous region if (p,q) is contained in the set

$$\{(p,q) \mid p \leq 0,\ q < ap\} \cup \{(p,q) \mid p > 0, q \leq \max(a(p-\ell),0)\}.$$

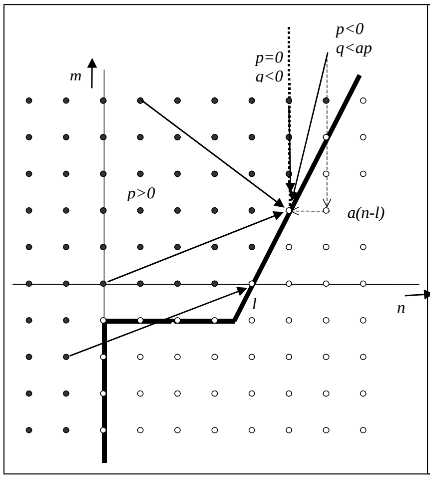

Figure A1: Congruous Region

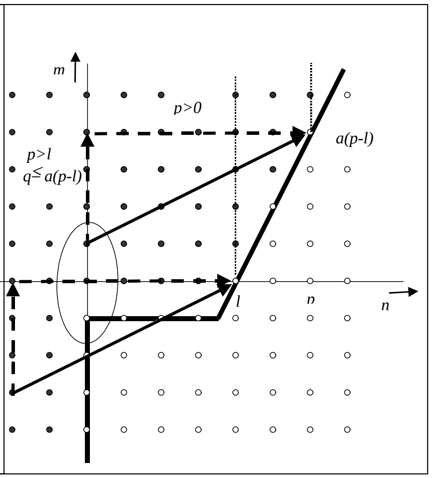

Figure A2: When $p > \ell$

- For $p < 0$, q is constrained by the boundary with slope a, so $q < ap$.
- For $p = 0$, paths can step down from above so $q < 0$.
- For $p > 0$, we look at cases where $0 < p \leq \ell$ and where $p > \ell$. We see from the restricted point $(\ell, 0)$ that if $p \leq \ell$ then q can not be positive, so $q \leq 0$. For $p > \ell$ refer to Figure A2. The point $(n-p, a(n-\ell)-q)$ will be in the negative n space for finitely many restricted points. At the critical point $n = p$, q can not be greater than $a(p-\ell)$, the vertical space available before the "corner point" $(0, -1)$ is erroneously included in the congruous points.

5.2 Applications to Privileged Access

No Exit from the Boundary In privileged access problems where the boundary absorbs the paths that reach it via an access step, all congruous points can be access points. In that case the access step set $\mathcal{P}$ is contained in the set

$$\{(p,q) \mid p \leq 0, q < ap\} \cup \{(p,q) \mid p > 0, q \leq \max(a(p-\ell), 0)\}.$$

Exit to the Upper Points For admissibility of the privileged access problem, we have the requirement that the point $(n-c, a(n-\ell)+1-\gamma)$ must be congruous for $n \geq \ell$. Since $c > 0$ in the definition of the (c,γ)-path step set, we know from above that for $p > 0$, $q \leq \max(a(p-\ell), 0)$ is the constraint on q for $(n-p, a(n-\ell)-q)$ to be a congruous point. We let $p=c$ and $q=\gamma-1$. Admissibility becomes the requirement that

$$\gamma-1 \leq \max(a(c-\ell), 0).$$

Because our general solution assumes all access points are congruous points and the finiteness condition requires that $p>0$, the access step set $\mathcal{P}$ must be contained in the set

$$\{(p, q) \mid p>0, q \leq \max(a(p-\ell), 0)\}.$$

Exceptional Congruous Points as Access Points Privileged access step vectors which utilize access points out of the congruous region may still be admissible if the access point is an exceptional congruous point, a point (n, m) where m is an integer root of $p_n(x)$ for $n>0$. For example, let $\mathfrak{E}=\{\rightarrow, \uparrow\}$ and access step set $\mathcal{P}=\{(2,1)\}$ with the restricted line $y=x-4$ for $x \geq 4$ (Type S). When $n=\ell=4$, the access point $(n-p, a(n-\ell)-q)=(4-2,0-1)$ is not in the congruous region, but it is a congruous point. We see that $d_2(x)=\binom{2+x}{2}=\frac{1}{2} x^2+\frac{3}{2} x+1$ evaluated at $x=-1$ gives $d_2(-1)=0=D(2,-1)$.

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