

# Planar Walks with Recursive Initial Conditions

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## Abstract

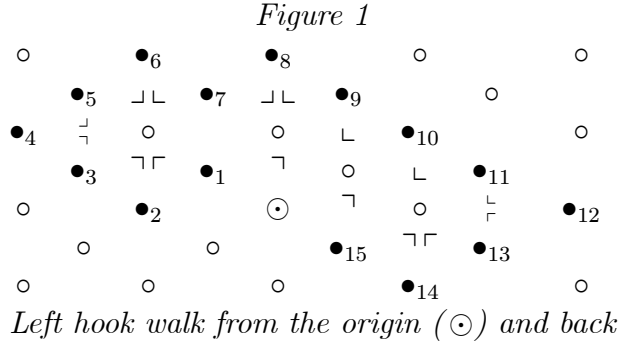
We enumerate two-dimensional walks inside a band parallel to an axis of symmetry, using an algebraic approach. The restrictions along the boundary lines can be more complicated than just initial values of zeroes; we consider *recursive initial values* prescribing which predecessors a position at the boundary can have. More general, the initial conditions are expressed by certain linear operators which vanish along the axis of symmetry.

## 1 Introduction

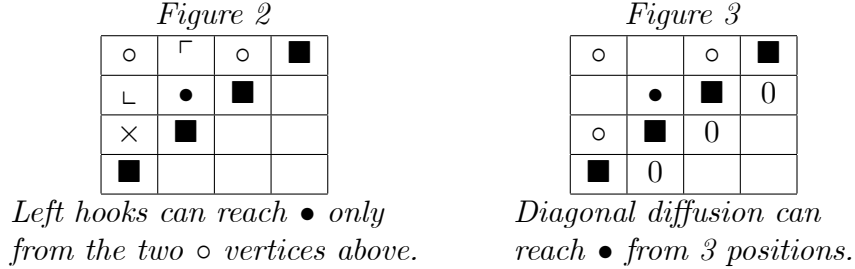
We like to think of a planar walk as a sequence of vertices connected by edges which must be selected from a prescribed set of step vectors. All paths start at the origin, and they are allowed to intersect with themselves. Enumerating such walks means finding the number of paths that reach a certain vertex in  $k$  steps, say. We are interested in *explicit enumeration*, finding “closed forms” for such numbers; however, we also use spread sheets (not shown in this paper!), one for each  $k$ , for *recursive enumeration* of the walks, counting the paths that arrive at a certain vertex in  $k$  steps by adding what we counted at the neighboring vertices in  $k - 1$  steps, in the previous spread sheet. Recursive enumeration validates the derived explicit formulas, and is of course the only efficient counting method when no closed forms can be derived from the recurrence equations and initial conditions.

There may be more meaning attached to a step vector than just being a directed edge in a graph. The following example will show this. Consider a *diagonal diffusion walk* in  $\mathbb{Z}^2$ , a random walk with steps  $\nearrow, \swarrow, \searrow$  and  $\nearrow$ . Suppose the walk has to move along the grid lines; diagonal steps are not physically possible, i.e., every diagonal step is a *hook*, a horizontal and a vertical step combined into one. If all four diagonal steps are left hooks,  $\nwarrow = \lrcorner$ ,  $\swarrow = \ulcorner$ ,  $\searrow = \llcorner$

and  $\nearrow = \lrcorner$ , we call the resulting path a *left hook walk*.



The above example shows 16 vertices, sequentially numbered, and visited by a left hook walk. Because there are no further restrictions, the corresponding example of a diagonal diffusion walk would look exactly the same. However, the difference gets noticeable if certain kinds of boundaries are present.



In the above example a diagonal boundary is blocking the walks. The left hooks are “more severely restricted” by such a boundary than the diagonal diffusion. Let  $H_k[m, n; \nearrow_r]$  be the number of left hook walks starting at the origin, and reaching  $(n, m)$  in  $k$  moves while staying strictly above  $y = x - r$ . The recurrence relation for  $H_k[m, n; \nearrow_r]$  is obvious, and Fig. 2 shows that we must apply the initial values

$$H_k[n - r + 1, n; \nearrow_r] = H_{k-1}[n - r + 2, n + 1; \nearrow_r] + H_{k-1}[n - r + 2, n - 1; \nearrow_r],$$

a prototype of *recursive initial values*. If we extend the recurrence to lattice points on or below the boundary, the above condition will hold if we require that

$$H_{k-1}[n - r, n - 1; \nearrow_r] + H_{k-1}[n - r, n + 1; \nearrow_r] = 0.$$

This extension of the recurrence creates infinitely many new entries in each of the matrices  $H_k[m, n; \nearrow_r]$ , but fortunately at lattice points that cannot be reached by any of the restricted walks.

Enumeration of the bounded diagonal diffusion is much easier. All it requires are plain boundary values of zeroes as indicated in Fig. 3; closed forms can be obtained from the *reflection principle* [4]. In this paper we surround the reflection principle by some linear algebra to find an explicit formula (closed form) for  $H_k[m, n; \nearrow_r]$ , and the number of paths in other recursive initial value problems. With an explicit solution we mean an expansion in terms of the unrestricted counts,  $H_k[m, n]$ . Both expansions, Theorems 23 and 29, for

one- and for two-sided restrictions, can only be applied to boundaries parallel to an axis of symmetry  $py = qx$ , say, of the unrestricted walk ( $\gcd(p, q) = 1$ ). A shift by one “basis” step vector  $(p, q)$  in the positive direction of the selected axis is denoted by  $\mathcal{X}$ ; the shift  $\mathcal{Y}$  moves a basis step  $(-q, p)$  perpendicular to  $\mathcal{X}$ . For example, the above boundary for the left hooks is parallel to  $y = x$ . Hence  $\mathcal{X}H_k[m, n; \nearrow_r] = H_k[m + 1, n + 1; \nearrow_r]$ , and  $\mathcal{Y}H_k[m, n; \nearrow_r] = H_k[m + 1, n - 1; \nearrow_r]$ . Recursive initial conditions can be translated into a linear operator  $\mathcal{L}$ , which is a linear combination (Laurent series) of powers in  $\mathcal{X}$  and  $\mathcal{Y}$ , evaluating to 0 along the axis of symmetry. In our example we saw that for the left hooks holds  $(\mathcal{X}^{-1} + \mathcal{Y}^{-1})H_k[n - r + 1, n; \nearrow_r] = 0$ , and therefore  $\mathcal{L} = (\mathcal{X}^{-1} + \mathcal{Y}^{-1})(\mathcal{X}\mathcal{Y})^{(1-r)/2}$ . Theorem 23 solves such recursive initial value problem in general, and says that

$$H_k[m, n; \nearrow_r] = H_k[m, n] - (\mathcal{L}^T / \mathcal{L})H_k[m, n]$$

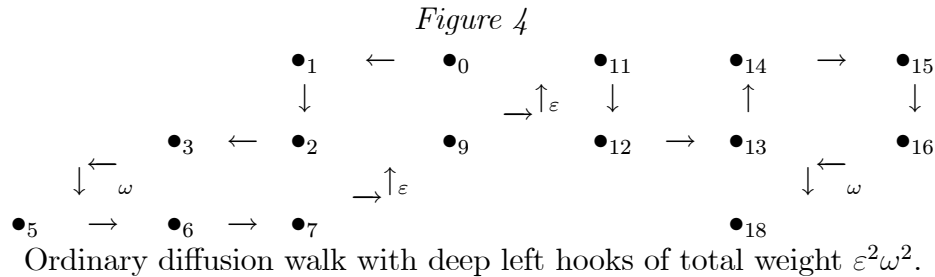
where  $\mathcal{L}^T$  is the conjugate of  $\mathcal{L}$  with respect to transposition (see Section 2.1). In particular,

$$H_k[m, n; \nearrow_r] = \binom{k}{\frac{k+m}{2}} \binom{k}{\frac{k+n}{2}} - \binom{k+1}{\frac{k+m+r+1}{2}} \binom{k-1}{\frac{k+n-r-1}{2}}$$

(all binomial coefficients  $\binom{u}{v}$  are assumed to be zero if  $u$  or  $v$  are negative integers). The algebraic approach via Laurent series tends to overshadow the simple beauty of the ‘reflection principle’: The theorem is based on the observation that matrices like  $\mathcal{L}H_k[m, n; \nearrow_r]$  vanish along the axis of symmetry, because they are the difference between some matrix and its reflection.

Formulas for counting walks inside a band are more complicated, of course. In Section 4 we prove the main result of this paper, Theorem 29, which allows us to expand the expression for the number of symmetric two-dimensional walks with recursive initial values along two boundaries parallel to an axis of symmetry. Both expansion theorems simplify to the well known reflection principle when the linear operators  $\mathcal{L}$  and  $\mathcal{J}$  are monomials in  $\mathcal{Y}$ .

The left hook walks are a prime example for the theory developed in this paper. In Sections 3.1 and 4.1 they can be counted by one-line formulas without being trivial. In general, the formulas will involve multiple summations; we show only one such example, the ordinary diffusion walk with weighted E-N and S-W *deep* left hooks, in Section 3.2. A deep left hook (Fig. 4) looks the same as an ordinary hook, but counts as two separate moves.



If the weights  $\varepsilon$  and  $\omega$  in the generating function for deep left hooks are replaced by  $\mu_{\downarrow} - 1$  and  $\mu_{\rightarrow} - 1$ , then we are enumerating the ordinary diffusion walk by weighted East-North



**page walks** with step vectors  $\pm(2, 1)$  and  $\pm(-1, 2)$ . Unrestricted:  $P_k[m, n]$  (Example 15).

As in [4] we need the concept of reflectable points.

**Definition 1 (Symmetry)** *Suppose  $q$  and  $p$  are relative prime integers, and  $p \geq 0$  (that assumption will be made throughout this paper). A lattice point  $(ps + qt, qs - pt) \in \mathbb{Z}^2$  is called  $q/p$ -reflectable. Its reflection at the line  $py = qx$  is the point  $(ps - qt, qs + pt)$ . The set of  $q/p$ -reflectable points is called the  $q/p$ -grid. A board  $U$  is symmetric about the line  $py = qx$ , or  $q/p$ -symmetric, iff*

$$U[qs - pt, ps + qt] = U[qs + pt, ps - qt] \quad (1)$$

for all  $q/p$ -reflectable lattice points. In this case the line  $py = qx$  is the axis of symmetry. A sequence of boards is  $q/p$ -symmetric iff every board in the sequence is  $q/p$ -symmetric.

The cases  $p = 0, q = 1$  and  $p = 1, q = 0$  correspond to symmetry about the  $y$ -axis and  $x$ -axis, respectively. The  $0/1$ -,  $1/0$ -,  $1/1$ -, and  $-1/1$ -grids are all equal to each other (and to  $\mathbb{Z}^2$ ). This situation is typical (see [4]): A point which is reflectable at a certain line ( $py = qx$ ) is also reflectable at the line perpendicular to it through the origin ( $-qy = px$ ), and at the two bisectors  $((p+q)y = (q-p)x$  and  $(p-q)y = (p+q)x$ ). The  $q/p$ -,  $-p/q$ -,  $(q-p)/(q+p)$ - and  $(q+p)/(p-q)$ -grid are the same. Note that we are abusing the notation: We should write  $\frac{q+p}{2}/\frac{p-q}{2}$ -grid if  $p$  and  $q$  are both odd.

**Remark 2** *If  $p + q$  is odd, then all  $q/p$ -reflectable points are of the form  $(ps - qt, qs + pt)$  where  $s$  and  $t$  are integers. The perpendicular vectors  $(p, q)$  and  $(-q, p)$  are a basis for the whole  $q/p$ -grid, which is a subspace of  $\mathbb{Z}^2$  seen as a vector space over the integers. We call them the basis step vectors.*

*If  $q+p$  is even, then all  $q/p$ -reflectable points are of the form  $\frac{1}{2}(ps - qt, qs + pt)$  where  $s$  and  $t$  are integers of the same parity. The perpendicular vectors  $(p, q)$  and  $(-q, p)$  only span a subspace of the  $q/p$ -grid; the whole grid is spanned by the bisectors  $\{\frac{1}{2}(p + q, q - p), \frac{1}{2}(p - q, p + q)\}$ .*

The following extension of  $\mathbb{Z}^2$  will simplify the notation. Let

$$\mathbb{Z}_{q/p} = \begin{cases} \mathbb{Z}^2 & \text{if } p + q \text{ is odd,} \\ \{(i, j) \mid (2i, 2j) \in \mathbb{Z}^2 \text{ and } i + j \in \mathbb{Z}\} & \text{if } q + p \text{ is even.} \end{cases}$$

We can view  $\mathbb{Z}_{q/p}$  as the vector space over  $\mathbb{Z}$  spanned by  $\{(1, 0), (0, 1)\}$  if  $p + q$  is odd, and spanned by  $\{(1/2, 0), (0, 1/2)\}$  if  $p + q$  is even. All  $q/p$ -reflectable points are of the form

$$(s, t) \begin{pmatrix} p & q \\ -q & p \end{pmatrix} = (ps - qt, qs + pt)$$

where  $(s, t) \in \mathbb{Z}_{q/p}$ . This notation has the advantage that we can write all  $q/p$ -grid points in the form  $(ps - qt, qs + pt)$ ,  $(s, t) \in \mathbb{Z}_{q/p}$ , without referring to the parity of  $p + q$ .

**Example 3** Taking an ordinary diffusion walk means that we can select among the step vectors  $(1, 0)$ ,  $(-1, 0)$ ,  $(0, 1)$ , and  $(0, -1)$ . Let  $D_k[m, n]$  be the number of unrestricted paths from the origin to  $(n, m)$  in  $k$  moves. It is well known that

$$D_k[m, n] = \binom{k}{(k+m+n)/2} \binom{k}{(k-m+n)/2} \quad (2)$$

(see also Lemma 17).

Figure 6																	
<b>2</b>	0	0	0	0	0	<b>2</b>	0	0	0	0	0	<b>2</b>	0	0	1	0	0
<b>1</b>	0	0	0	0	0	<b>1</b>	0	0	1	0	0	<b>1</b>	0	2	0	2	0
<b>0</b>	0	0	1	0	0	<b>0</b>	0	1	0	1	0	<b>0</b>	1	0	4	0	1
<b>-1</b>	0	0	0	0	0	<b>-1</b>	0	0	1	0	0	<b>-1</b>	0	2	0	2	0
<b>-2</b>	0	0	0	0	0	<b>-2</b>	0	0	0	0	0	<b>-2</b>	0	0	1	0	0
$m \uparrow / n \rightarrow$	<b>-2</b>	<b>-1</b>	<b>0</b>	<b>1</b>	<b>2</b>	$m \uparrow / n \rightarrow$	<b>-2</b>	<b>-1</b>	<b>0</b>	<b>1</b>	<b>2</b>	$m \uparrow / n \rightarrow$	<b>-2</b>	<b>-1</b>	<b>0</b>	<b>1</b>	<b>2</b>
$D_0[m, n]$					$D_1[m, n]$					$D_2[m, n]$							

The sequence of boards  $(D_k)$  is 1/1-symmetric and also 0/1-symmetric. The basis step vectors  $(1, 0)$  and  $(0, 1)$  span the whole  $\mathbb{Z}^2$ ; the bisectors  $(1, 1)$  and  $(-1, 1)$  span the subgrid which only contains lattice points with an even sum of components. Because of the step vectors,  $D_k[m, n]$  follows the recursion

$$D_k[m, n] = D_{k-1}[m, n+1] + D_{k-1}[m, n-1] + D_{k-1}[m+1, n] + D_{k-1}[m-1, n],$$

with initial values  $D_0[m, n] = \delta_{(m,n),(0,0)}$ .

## 2.1 Transposition

In order to explicitly enumerate restricted walks with recursive initial values, we need a small amount of linear algebra.

**Definition 4** We denote by  $\mathfrak{B}_{q/p}$  the vector space (over  $\mathbb{R}$ , or some ring of generating functions) of all boards  $B$  which vanish outside the  $q/p$ -grid.

Because the grids are the same, it follows that  $\mathfrak{B}_{q/p} = \mathfrak{B}_{-p/q} = \mathfrak{B}_{(q-p)/(q+p)} = \mathfrak{B}_{(q+p)/(p-q)}$ .

**Definition 5 (Transposition of boards)** The linear operator  $\mathcal{T}$  on  $\mathfrak{B}_{q/p}$  that reflects any board  $B \in \mathfrak{B}_{q/p}$  along  $py = qx$  in the sense of Definition 1 is the transposition operator,

$$(\mathcal{T}B)[qs + pt, ps - qt] = B[qs - pt, ps + qt] \quad (3)$$

for all  $(s, t) \in \mathbb{Z}_{q/p}$ . We also write  $B^\top$  for  $\mathcal{T}B$ , and call  $B^\top$  the  $q/p$ -transpose of  $B$ .

Note that the notation  $\mathcal{T}$  and  $B^\top$  does not indicate the dependence on  $q/p$ , which will be obvious from the context. The  $q/p$ -transposition is involutory,  $\mathcal{T}^2 = 1$ . Ordinary matrix transposition equals 1/1-transposition.

**Definition 6 (Transposition of operators)** The transpose  $\mathcal{L}^T$  of any linear operator  $\mathcal{L}$  on  $\mathfrak{B}_{q/p}$  is the conjugate of  $\mathcal{L}$  with respect to  $\mathcal{T}$ ,

$$\mathcal{L}^T = \mathcal{T}\mathcal{L}\mathcal{T}$$

$\mathcal{L}$  is symmetric iff  $\mathcal{L} = \mathcal{L}^T$ .

For all boards  $B \in \mathfrak{B}_{q/p}$  holds

$$(\mathcal{L}^T B)[qs - pt, ps + qt] = (\mathcal{L} B^\top)[qs + pt, ps - qt]$$

in general, and

$$(\mathcal{L} B)[qs - pt, ps + qt] = (\mathcal{L} B^\top)[qs + pt, ps - qt]$$

if  $\mathcal{L}$  is symmetric. In the following we will omit the parenthesis in an expression like  $(\mathcal{L} B^\top)[m, n]$ ; the cell  $[m, n]$  will always refer to the image board at the left,  $(\mathcal{L} B^\top)[m, n] = \mathcal{L} B^\top[m, n]$ .

**Definition 7 (Basic boards)** A symmetric linear operator  $\mathcal{R}$  on  $\mathfrak{B}_{q/p}$  is the recursion operator of a sequence of boards  $(B_k)$  in  $\mathfrak{B}_{q/p}$  iff  $B_k = \mathcal{R}B_{k-1}$  for all  $k > 0$ . We call  $(B_k)$  an  $\mathcal{R}$ -recursive sequence. The sequence is basic (for  $\mathcal{R}$ ) iff  $B_0[m, n] = \delta_{(n,m),(0,0)}$ .

If  $(B_k)$  is an  $\mathcal{R}$ -recursive sequence, then  $B_k$  is  $q/p$ -symmetric, as can be shown by induction:

$$B_k = \mathcal{R}B_{k-1} = \mathcal{R}B_{k-1}^\top = (\mathcal{R}^T B_{k-1})^\top = (\mathcal{R}B_{k-1})^\top = B_k^\top.$$

All the linear operators on  $\mathfrak{B}_{q/p}$  in this paper will be composed of shift operators, which are defined as follows.

**Definition 8 (Shift operators)** Let  $(v, u)$  and  $(n, m)$  be  $q/p$ -grid points and  $B \in \mathfrak{B}_{q/p}$ .

$$\mathcal{E}^{u,v} : B[m, n] \rightarrow B[m + u, n + v]$$

is the shift operator  $\mathcal{E}^{u,v}$  on  $\mathfrak{B}_{q/p}$ . We write  $\mathcal{X}$  for  $\mathcal{E}^{q,p}$ , and  $\mathcal{Y}$  for  $\mathcal{E}^{p,-q}$ ;  $\mathcal{X}$  and  $\mathcal{Y}$  are the shifts by basis step vectors.

The dependence of  $\mathcal{X}$  and  $\mathcal{Y}$  on  $q$  and  $p$  will follow from the context and is not indicated by the notation. Any finite linear combination of  $\mathcal{X}^i \mathcal{Y}^j$ ,  $(i, j) \in \mathbb{Z}_{q/p}$ , maps  $\mathfrak{B}_{q/p}$  to itself.

**Example 9 (Ordinary diffusion continued)** Suppose we focus on the  $0/1$ -symmetry of the ordinary diffusion walk (Example 3). Then  $\mathcal{X} = \mathcal{E}^{0,1}$  and  $\mathcal{Y} = \mathcal{E}^{1,0}$ . Note that  $0/1$ -transposition means reflection along the  $x$ -axis,  $(\mathcal{T}U)[t, s] = U[-t, s]$  for any board  $U$  in  $\mathfrak{B}_{0/1}$ , and therefore  $\mathcal{X}^T = \mathcal{X}$  and  $\mathcal{Y}^T = \mathcal{E}^{-1,0} = \mathcal{Y}^{-1}$  (see also (4)). The recursion in Example 3 can be written as  $D_0[m, n] = \delta_{(n,m),(0,0)}$  and

$$D_k[m, n] = \mathcal{X}D_{k-1}[m, n] + \mathcal{X}^{-1}D_{k-1}[m, n] + \mathcal{Y}D_{k-1}[m, n] + \mathcal{Y}^{-1}D_{k-1}[m, n] = \mathcal{R}D_{k-1}[m, n],$$

where  $\mathcal{R} = \mathcal{X} + \mathcal{X}^{-1} + \mathcal{Y} + \mathcal{Y}^{-1}$  is the recursion operator for ordinary diffusion walks, and  $(D_k)$  is its basic sequence. It is easy to check that this recursion operator  $\mathcal{R}$  is symmetric,  $\mathcal{R}^T = \mathcal{X} + \mathcal{X}^{-1} + \mathcal{Y}^{-1} + \mathcal{Y} = \mathcal{R}$ .

With the help of  $\mathcal{X}$  and  $\mathcal{Y}$  we can express symmetry of a board  $B$  (see (3)) as

$$\mathcal{T}\mathcal{X}^s\mathcal{Y}^t B[0,0] = \mathcal{X}^s\mathcal{Y}^{-t} B[0,0]$$

for all  $(s,t) \in \mathbb{Z}_{q/p}$ . It follows immediately that

$$(\mathcal{X}^s\mathcal{Y}^t)^T = \mathcal{X}^s\mathcal{Y}^{-t} \quad (4)$$

for all  $(s,t) \in \mathbb{Z}_{q/p}$ . Hence  $\mathcal{X}^s\mathcal{Y}^t (\mathcal{X}^s\mathcal{Y}^t)^T = \mathcal{X}^{2s}$  is symmetric. This is just an example of more general results about transposes, which we list without proof.

**Lemma 10** *The linear operator  $\mathcal{L}$  commutes with its transpose  $\mathcal{L}^T$  iff  $\mathcal{L}\mathcal{L}^T$  is symmetric.*

**Lemma 11** *If  $\mathcal{R}$  is a symmetric linear operator which commutes with some other linear operator  $\mathcal{L}$ , then  $\mathcal{R}$  also commutes with  $\mathcal{L}^T$ .*

## 2.2 Laurent Series of Shifts

Commutativity is an important aspect of the algebraic manipulations we are planning to use. Instead of phrasing the expansion theorems in their most general form, we will enforce commutativity of the linear operators by restricting them to suitable rings of Laurent series in  $\mathcal{X}$  and  $\mathcal{Y}$ . The choice of Laurent series, and the subspaces of boards they act on, may look rather arbitrary at first reading, but are actually tailored to the type of walks we will discuss later. We postpone the motivation for the following definitions until Remark 26, after we have seen an application.

**Definition 12** *For given  $q$  and  $p$  let  $\mathcal{X} = \mathcal{E}^{q,p}$  and  $\mathcal{Y} = \mathcal{E}^{p,-q}$  as before. For any set of (real) coefficients  $a_{i,j}$  we call  $\sum_{(i,j) \in \mathbb{Z}_{q/p}} a_{i,j} \mathcal{X}^i \mathcal{Y}^j$  a series in  $\mathcal{X}$  and  $\mathcal{Y}$ . The set of series*

$$\left\{ \sum_{(i,j) \in \mathbb{Z}_{q/p}, j \geq \alpha_1, iq+jp \geq \alpha_2} a_{i,j} \mathcal{X}^i \mathcal{Y}^j \mid a_{i,j} \in \mathbb{R}, \text{ and } (\alpha_1, \alpha_2) \in \mathbb{Z}_{q/p} \right\}$$

*is denoted by  $\mathfrak{L}_{q/p}^{\geq}$ , and the set of their transposes by  $\mathfrak{L}_{q/p}^{\leq}$ . The elements of both sets are called Laurent series. The support of a series  $\mathcal{S}$  is the index set*

$$\text{Supp}(\mathcal{S}) := \{(i,j) \in \mathbb{Z}_{q/p} \mid a_{i,j} \neq 0\}.$$

*The intersection of both types of Laurent series will be denoted by  $\mathfrak{L}_{q/p}^{\cap} := \mathfrak{L}_{q/p}^{\geq} \cap \mathfrak{L}_{q/p}^{\leq}$ .*

*The subspace of boards  $U$  from  $\mathfrak{B}_{q/p}$  with the property*

*$U[aq+bp, ap-bq] \neq 0$  implies  $b \leq \beta_1$  and  $aq+bp \leq \beta_2$ , for some  $(\beta_1, \beta_2) \in \mathbb{Z}_{q/p}$  (dependent on  $U$ ), is denoted by  $\mathfrak{B}_{q/p}^{\leq}$ , and the subspace of their transposes by  $\mathfrak{B}_{q/p}^{\geq}$ . The intersection of both subspaces will be denoted by  $\mathfrak{B}_{q/p}^{\cap}$ .*

For simplicity we will assume in this section that the coefficients  $a_{i,j}$  of a Laurent series are real. In weighted enumeration, the coefficients may be power series in the weights.

A series  $\mathcal{L}$  is in  $\mathfrak{L}_{q/p}^{\geq}$  iff  $\min\{j \mid (i,j) \in \text{Supp}(\mathcal{L})\}$  and  $\min\{iq + jp \mid (i,j) \in \text{Supp}(\mathcal{L})\}$  both exist. Transposition shows that the Laurent series in  $\mathfrak{L}_{q/p}^{\leq}$  are of the form

$$\sum_{(i,j) \in \mathbb{Z}_{q/p}, j \leq \alpha_1, iq - jp \geq \alpha_2} a_{i,j} \mathcal{X}^i \mathcal{Y}^j.$$

The support of Laurent series in  $\mathfrak{L}_{q/p}^{\cap}$  is contained in  $\{(i,j) \in \mathbb{Z}_{q/p} \mid \beta \geq j \geq \alpha_1, iq \geq \alpha_2\}$  for some  $(\alpha_1, \alpha_2), (\beta, \alpha_2) \in \mathbb{Z}_{q/p}$ . The support of boards in  $\mathfrak{B}_{q/p}^{\cap}$  is contained in  $\{(ap - bq, aq + bp) \mid (a,b) \in \mathbb{Z}_{q/p} \text{ and } \beta_1 \geq b \geq \alpha, \beta_2 \geq aq \text{ for some } (\beta_1, \beta_2), (\alpha, \beta_2) \in \mathbb{Z}_{q/p}\}$ .

**Lemma 13** *The Laurent series  $\mathfrak{L}_{q/p}^{\geq}$  and  $\mathfrak{L}_{q/p}^{\leq}$  are rings (over  $\mathbb{R}$ ) of linear operators on  $\mathfrak{B}_{q/p}^{\leq}$  and  $\mathfrak{B}_{q/p}^{\geq}$ , respectively.*

**Proof.** It is straightforward to show that  $\mathfrak{L}_{q/p}^{\geq}$  and  $\mathfrak{L}_{q/p}^{\leq}$  are both closed under addition; showing that the Cauchy product of  $\mathcal{L} := \sum_{j \geq \alpha_1, iq + jp \geq \alpha_2} a_{i,j} \mathcal{X}^i \mathcal{Y}^j$  and  $\mathcal{J} := \sum_{l \geq \beta_1, kq + lp \geq \beta_2} b_{k,l} \mathcal{X}^k \mathcal{Y}^l$  is again in  $\mathfrak{L}_{q/p}^{\geq}$  requires verifying that

$$\mathcal{L}\mathcal{J} = \sum_{v \geq \alpha_1 + \beta_1, uq + vp \geq \alpha_2 + \beta_2} \mathcal{X}^u \mathcal{Y}^v \sum_{j = \alpha_1}^{v - \beta_1} \sum_{\alpha_2 - jp \leq iq \leq uq + (v - j)p - \beta_2} a_{i,j} b_{u-i, v-j}.$$

The product is well defined, because the inner double sum is finite. The form of the outer sum shows that  $\mathcal{L}\mathcal{J}$  is in  $\mathfrak{L}_{q/p}^{\geq}$ . Hence,  $\mathfrak{L}_{q/p}^{\geq}$  is a ring, and so is  $\mathfrak{L}_{q/p}^{\leq}$ .

Let  $B \in \mathfrak{B}_{q/p}^{\leq}$  with  $B[aq + bp, ap - bq] = 0$  where  $b > \beta_1$  or  $aq + bp > \beta_2$ . It can be shown that  $\mathcal{L}B[qs + pt, ps - qt]$

$$= \sum_{\beta_1 - t \geq j \geq \alpha_1, \beta_2 - qs - p(j+t) \geq iq \geq \alpha_2 - jp} a_{i,j} B[q(i+s) + p(j+t), p(i+s) - q(j+t)].$$

Hence  $\mathcal{L}B[qs + pt, ps - qt]$  is well defined for all  $(s, t) \in \mathbb{Z}_{q/p}$ . The summation range gets empty if  $\beta_1 - \alpha_1 < t$  or if  $\beta_2 - \alpha_2 < qs + pt$ . Therefore  $\mathcal{L}B$  is again in  $\mathfrak{B}_{q/p}^{\leq}$ .

Suppose  $\mathcal{J} \in \mathfrak{L}_{q/p}^{\leq}$ , and  $C \in \mathfrak{B}_{q/p}^{\geq}$ . We saw above that  $\mathcal{J}^T C^T \in \mathfrak{B}_{q/p}^{\leq}$ , and from  $\mathcal{J}^T C^T = (\mathcal{J}C)^T$  follows  $\mathcal{J}C = (\mathcal{J}^T C^T)^T \in \mathfrak{B}_{q/p}^{\geq}$ . ■

Algebraically, the Laurent series  $\mathfrak{L}_{q/p}^{\leq}$  and the boards  $\mathfrak{B}_{q/p}^{\geq}$  are just matrices, and there is no need to distinguish between them. The distinction is made for combinatorial reasons only.

**Lemma 14** *Every  $q/p$ -symmetric operator  $\mathcal{R} \in \mathfrak{L}_{q/p}^{\cap}$  can be expanded as*

$$\sum_{(i,j) \in \mathbb{Z}_{q/p}, \beta_1 \geq j \geq 0, iq \geq \beta_2} b_{i,j} \mathcal{X}^i (\mathcal{Y}^j + \mathcal{Y}^{-j}).$$

**Proof.** Straightforward verification. ■

**Example 15** The page walk takes steps  $\pm(2, 1)$  and  $\pm(-1, 2)$ . The number  $P_k[m, n]$  of page walks from the origin to  $(n, m)$  in  $k$  steps follows the recursion

$$P_k[m, n] = P_{k-1}[m+1, n+2] + P_{k-1}[m+2, n-1] + P_{k-1}[m-2, n+1] + P_{k-1}[m-1, n-2].$$

Hence,  $P_k[m, n] = \mathcal{R}P_{k-1}[m, n]$  if we define the recursion operator  $\mathcal{R} := \mathcal{E}^{1,2} + \mathcal{E}^{2,-1} + \mathcal{E}^{-2,1} + \mathcal{E}^{-1,-2}$ . The recursion is 1/2-symmetric; with  $\mathcal{X} := \mathcal{E}^{1,2}$  and  $\mathcal{Y} := \mathcal{E}^{2,-1}$  we can write  $\mathcal{R} = \frac{1}{2}(\mathcal{X}\mathcal{Y}^0 + \mathcal{X}\mathcal{Y}^0) + (\mathcal{X}^0\mathcal{Y}^1 + \mathcal{X}^0\mathcal{Y}^{-1}) + \frac{1}{2}(\mathcal{X}^{-1}\mathcal{Y}^0 + \mathcal{X}^{-1}\mathcal{Y}^0)$ . The recursion is also -2/1-symmetric; with  $\mathcal{X} := \mathcal{E}^{-2,1}$  and  $\mathcal{Y} := \mathcal{E}^{1,2}$  we can write  $\mathcal{R}$  as  $\mathcal{Y}^1 + \mathcal{Y}^{-1} + \mathcal{X} + \mathcal{X}^{-1}$ . Some results about restricted page walks can be found in [4]. The numbers  $P_k[m, n]$  can be easily found from Lemma 17.

**Example 16 (Ordinary diffusion continued)** The operator  $\mathcal{R} = \mathcal{X} + \mathcal{X}^{-1} + \mathcal{Y} + \mathcal{Y}^{-1}$  is also the recursion for ordinary diffusion (Example 9), but with different  $p = 1$  and  $q = 0$  then in the example above. We can calculate the numbers  $D_k[m, n]$  in equation (2) from  $D_k[m, n] = \mathcal{R}D_{k-1}[m, n] = \dots = \mathcal{R}^k D_0[m, n]$  by a multinomial expansion of  $\mathcal{R}$ , using that  $D_0[m, n] = \delta_{(n,m),(0,0)}$ . That expansion, followed by an application of the Chu-Vandermonde convolution formula, nothing but mirrors the elementary combinatorics approach to finding  $D_k[m, n]$ . A faster road to formula (2) is based on the observation that

$$\mathcal{R} = (1 + \mathcal{X}\mathcal{Y})(\mathcal{X}^{-1} + \mathcal{Y}^{-1}). \quad (5)$$

We saw that this recursion operator plays a role in many different settings; hence we present the expansion as a lemma.

**Lemma 17** Let  $(B_k)$  be the basic board sequence in  $\mathfrak{B}_{q/p}$  for the recursion operator  $\mathcal{R} = \mathcal{X} + \mathcal{X}^{-1} + \mathcal{Y} + \mathcal{Y}^{-1}$ , i.e.,

$$B_k[m, n] = B_{k-1}[m+q, n+p] + B_{k-1}[m-q, n-p] + B_{k-1}[m+p, n-q] + B_{k-1}[m-p, n+q],$$

and  $B_0[m, n] = \delta_{(n,m),(0,0)}$ . Then

$$B_k[m, n] = \binom{k}{\frac{q(n-m)-p(m+n)}{2(q^2+p^2)} + \frac{k}{2}} \binom{k}{\frac{p(n-m)+q(m+n)}{2(q^2+p^2)} + \frac{k}{2}}$$

for all  $q/p$ -grid points  $(n, m)$ .

**Proof.** Note that  $\mathcal{R} = (1 + \mathcal{X}\mathcal{Y})(\mathcal{X}^{-1} + \mathcal{Y}^{-1})$ , and therefore

$$D_k[m, n] = \mathcal{R}^k D_0[m, n] = \sum_{i,j} \binom{k}{i} \binom{k}{j} \mathcal{X}^{i-j} \mathcal{Y}^{i+j-k} D_0[m, n].$$

Solving the equations

$$n + p(i-j) - q(i+j-k) = 0 \quad \text{and} \quad m + q(i-j) + p(i+j-k) = 0$$

for  $i$  and  $j$  proves the lemma. ■

The introduction of the algebra  $\mathfrak{L}_{q/p}^{\geq}$  was necessary because we will need inverses of operators in our Expansion Theorems 23 and 29. Not every element from  $\mathfrak{L}_{q/p}^{\geq}$  has an inverse in  $\mathfrak{L}_{q/p}^{\geq}$ . For example,  $1 - \mathcal{X}^{-2}\mathcal{Y}$  has an inverse in  $\mathfrak{L}_{1/1}^{\leq}$ , but not in  $\mathfrak{L}_{1/1}^{\geq}$ .

**Lemma 18** Suppose  $\mathcal{L}$  is a Laurent series in  $\mathfrak{L}_{q/p}^{\geq}$ , with coefficients  $z_{a,b}$ ,  $(a,b) \in S_{\mathcal{L}} := \text{Supp}(\mathcal{L})$ . Let  $b' := \min\{b \mid (a,b) \in S_{\mathcal{L}}\}$ . We can invert  $\mathcal{L}$  in  $\mathfrak{L}_{q/p}^{\geq}$  if there is an index  $a'$  such that  $(a',b') \in S_{\mathcal{L}}$  and

$$aq + bp \geq a'q + b'p \text{ for all } (a,b) \in S_{\mathcal{L}}. \quad (6)$$

**Proof.** Let  $\hat{z}_{a,b} := -z_{a,b}/z_{a',b'}$  if  $(a,b) \neq (a',b')$ , and  $\hat{z}_{a',b'} := 0$ . Then  $\mathcal{L} = z_{a',b'} \mathcal{X}^{a'} \mathcal{Y}^{b'} \mathcal{J}$ , where  $\mathcal{J} = 1 - \sum_{(a,b) \in S_{\mathcal{L}}} \hat{z}_{a,b} \mathcal{X}^{a-a'} \mathcal{Y}^{b-b'}$ . It is enough to find the inverse of  $\mathcal{J}$ ,

$$\begin{aligned} \mathcal{J}^{-1} &= \sum_{k \geq 0} \left( \sum_{(a,b) \in S_{\mathcal{L}}} \hat{z}_{a,b} \mathcal{X}^{a-a'} \mathcal{Y}^{b-b'} \right)^k \\ &= \sum_{k \geq 0} \sum_{\substack{k_{a,b} > 0, k = \sum_{(a,b) \in S_{\mathcal{L}}} k_{a,b}}} \binom{k}{\dots, k_{a,b}, \dots} \prod_{(a,b) \in S_{\mathcal{L}}} \left( \hat{z}_{a,b} \mathcal{X}^{a-a'} \mathcal{Y}^{b-b'} \right)^{k_{a,b}} \\ &= \sum_{k \geq 0} \sum_{k_{a,b} > 0, k = \sum_{(a,b) \in S_{\mathcal{L}}} k_{a,b}} k! \left( \prod_{(a,b) \in S_{\mathcal{L}}} \hat{z}_{a,b}^{k_{a,b}} / k_{a,b}! \right) \mathcal{X}^{\sum k_{a,b}(a-a')} \mathcal{Y}^{\sum k_{a,b}(b-b')} \\ &= \sum_{(i,j) \in \mathbb{Z}_{q/p}} \mathcal{X}^i \mathcal{Y}^j \sum_{\substack{k_{a,b} > 0, i = \sum k_{a,b}(a-a'), j = \sum k_{a,b}(b-b')}} \left( \sum_{(a,b) \in S_{\mathcal{L}}} k_{a,b} \right)! \prod_{(a,b) \in S_{\mathcal{L}}} \hat{z}_{a,b}^{k_{a,b}} / k_{a,b}!. \end{aligned}$$

Because of the assumption (6) we know that  $\sum k_{a,b}((a-a')q + (b-b')p)$  is a partition of  $iq + jp$  into nonnegative terms. For given  $i$  and  $j$  there are only finitely many  $k_{a,b} > 0$  in this sum. Hence the inner sum is well defined. This series is in  $\mathfrak{L}_{q/p}^{\geq}$  because  $j$  and  $iq + jp$  are nonnegative. Also note that  $(a,b) \in \mathbb{Z}_{q/p}$  implies  $\sum k_{ab}(a,b) \in \mathbb{Z}_{q/p}$ . ■

**Example 19** The 2-term series  $\mathcal{X}^{a'} \mathcal{Y}^{b'} - z \mathcal{X}^a \mathcal{Y}^b$  is invertible if  $b \geq b'$  and  $ap + bq \geq a'p + b'q$ . In this case

$$\frac{1}{\mathcal{X}^{a'} \mathcal{Y}^{b'} - z \mathcal{X}^a \mathcal{Y}^b} = \frac{\mathcal{X}^{-a'} \mathcal{Y}^{-b'}}{1 - z \mathcal{X}^{a-a'} \mathcal{Y}^{b-b'}} = \mathcal{X}^{-a'} \mathcal{Y}^{-b'} \sum_{k \geq 0} z^k \mathcal{X}^{k(a-a')} \mathcal{Y}^{k(b-b')}.$$

**Remark 20** If we write  $1/\mathcal{L}$  we always mean  $\mathcal{L}^{-1}$  which is in  $\mathfrak{L}_{q/p}^{\geq}$ , and we call  $1/\mathcal{L}$  the  $\mathfrak{L}_{q/p}^{\geq}$ -inverse of  $\mathcal{L}$ . The inverse of  $\mathcal{K} \in \mathfrak{L}_{q/p}^{\leq}$  will be denoted by  $\mathcal{K}^{-1}$  which is in  $\mathfrak{L}_{q/p}^{\leq}$ . In other words, if  $\mathcal{L} \in \mathfrak{L}_{q/p}^{\geq}$  then  $\mathcal{L}^T (\mathcal{L}^{-1})^T = (\mathcal{L} \mathcal{L}^{-1})^T = 1$ , and therefore

$$(\mathcal{L}^T)^{-1} = (\mathcal{L}^{-1})^T,$$

provided one of the two inverses exists. If  $\mathcal{R}$  is a symmetric and invertible Laurent series then  $(\mathcal{R}^{-1})^T = \mathcal{R}^{-1}$ , but  $\mathcal{R}^{-1}$  is not symmetric in general,  $\mathcal{R}^{-1} \neq \mathcal{R}^{-1}$ . A simple counter example is  $\mathcal{Y} + \mathcal{Y}^{-1}$ . However, if we know that  $U$  and  $\mathcal{R}^{-1}U$  are symmetric boards, then

$$\mathcal{R}^{-1}U = (\mathcal{R}^{-1})^T U^T = (\mathcal{R}^{-1}U)^T = \mathcal{R}^{-1}U. \quad (7)$$

**Lemma 21** A Laurent series  $\mathcal{K} \in \mathfrak{L}_{q/p}^{\geq}$  satisfies the condition  $\mathcal{K}^T = \mathcal{K}^{-1}$  iff  $\mathcal{K} = \mathcal{Y}^w$  for some integer  $w$ .

**Proof.** Because  $\mathcal{K}$  is  $\mathfrak{L}_{q/p}^{\geq}$ -invertible, we must be able to write  $\mathcal{K}$  as  $z\mathcal{X}^u\mathcal{Y}^w(1+\mathcal{N})$ , where all powers of  $\mathcal{Y}$  in  $\mathcal{N}$  must be positive. It is obvious that  $z = 1$ ,  $u = 0$ , and it can be shown that  $\mathcal{N} = 0$ . ■

### 2.3 Left hooks

The number  $H_k[m, n]$  of unrestricted diagonal diffusion walks from the origin to  $(n, m)$  in  $k$  moves follows the recursion

$$\begin{aligned} H_k[m, n] &= H_{k-1}[m+1, n+1] + H_{k-1}[m-1, n-1] + H_{k-1}[m+1, n-1] + H_{k-1}[m-1, n+1] \\ &= (\mathcal{E}^{1,1} + \mathcal{E}^{-1,-1} + \mathcal{E}^{1,-1} + \mathcal{E}^{-1,1}) H_k[m, n]. \end{aligned}$$

The recursion is obviously 0/1-symmetric, but also 1/1-symmetric. In the latter case we get  $\mathcal{X} = \mathcal{E}^{1,1}$ ,  $\mathcal{Y} = \mathcal{E}^{1,-1}$ , and we can write the recursion operator as  $\mathcal{R} = \mathcal{X} + \mathcal{X}^{-1} + \mathcal{Y} + \mathcal{Y}^{-1}$ . In the Introduction we saw why a diagonal step, like  $\nearrow$ , in some models must be interpreted as the result of a single “left hook step”  $\lrcorner$ , first going right and then up in one step. There are four different left hooks; EN, NW, WS, and SE. The left hook walk follows the same recursion  $\mathcal{R}$  as the diagonal diffusion; without restrictions, left hook walks are indistinguishable from diagonal diffusion walks, and it is well known that

$$H_k[m, n] = \binom{k}{(k+m)/2} \binom{k}{(k+n)/2} \quad (8)$$

(see [1] for references, or apply Lemma 17). Vertical and horizontal boundaries for the left hooks do not require any considerations different from ordinary diagonal diffusion walks. Now suppose that we restrict the walks by a lower boundary parallel to the 1/1-axis, the main diagonal. Hook steps cannot slide directly parallel to a diagonal barrier of slope 1 (see Figures 2 and 3): If  $y = x - r$  is the lower barrier, the diagonal step  $\nearrow$  can go from  $(n, n - r + 1)$  to  $(n + 1, n - r + 2)$ , but the hook step  $\lrcorner$  can't, because it would hit the barrier when stepping from  $(n, n - r + 1)$  to  $(n + 1, n - r + 1)$ .

The physical description of the left hook  $\lrcorner$  resembles a left turn, a combination of the two steps  $\rightarrow\uparrow$ . Suppose we want to enumerate ordinary diffusion walks (with steps  $\rightarrow, \uparrow, \leftarrow, \downarrow$ , as in Examples 3 and 9), sorted by their number of East-North and West-South left turns. Corresponding to the step vectors we have the shift operators  $\mathcal{E}^{0,1} = \mathcal{X}^{1/2}\mathcal{Y}^{-1/2}$ ,  $\mathcal{E}^{1,0} = \mathcal{X}^{1/2}\mathcal{Y}^{1/2}$ ,  $\mathcal{E}^{0,-1} = \mathcal{X}^{-1/2}\mathcal{Y}^{1/2}$ , and  $\mathcal{E}^{-1,0} = \mathcal{X}^{-1/2}\mathcal{Y}^{-1/2}$  (note that  $y = x$  is the chosen axis of symmetry, hence  $q = p = 1$ ). We give the weight  $\mu_{\lrcorner}$  to the E-N left turns, and  $\mu_{\leftarrow}$  to the W-S left turns. Denote the weighted number (generating function) of such walks by  $T_k[m, n]$ ; the coefficient of  $\mu_{\lrcorner}^i \mu_{\leftarrow}^j$  in  $T_k[m, n]$  is the number of ordinary diffusion walks from the origin to  $(n, m)$  in  $k$  steps with  $i$  E-N and  $j$  W-S left turns. This sequence of basic boards follows the recursion

$$\begin{aligned} T_k[m, n] &= T_{k-1}[m, n+1] + T_{k-1}[m+1, n] + T_{k-1}[m, n-1] + T_{k-1}[m-1, n] \\ &\quad + (\mu_{\leftarrow} - 1)T_{k-2}[m+1, n+1] + (\mu_{\lrcorner} - 1)T_{k-2}[m-1, n-1] \\ &= (\mathcal{X}^{1/2} + \mathcal{X}^{-1/2})(\mathcal{Y}^{1/2} + \mathcal{Y}^{-1/2})T_{k-1}[m, n] + ((\mu_{\leftarrow} - 1)\mathcal{X} + (\mu_{\lrcorner} - 1)\mathcal{X}^{-1})T_{k-2}[m, n]. \end{aligned}$$

We found it easier to think of that walk as a diffusion walk with the additional *deep hooks* (double steps) E-N ( $=\xrightarrow{\quad}\uparrow$ ) and W-S ( $=\downarrow\overleftarrow{\quad}$ ). At every move we can choose among the step vectors

$$\{(0, 1), (0, -1), (1, 0), (-1, 0), (1, 1), (-1, -1)\}$$

where the deep hooks  $(1, 1) = \xrightarrow{\quad}\uparrow$  and  $(-1, -1) = \downarrow\overleftarrow{\quad}$  each increase the length of the path by two. The weight  $\varepsilon$  is assigned to the “ $\varepsilon$ -N” deep hooks, and the weight  $\omega$  to the “ $\omega$ -S” deep hooks. Let  $D_k[m, n; \varepsilon, \omega]$  denote the weighted number (generating function) of diffusion walks with weighted deep hooks from  $(0, 0)$  to  $(n, m)$  in  $k$  steps; the sequence is 1/1 symmetric and follows the recursion

$$D_k[m, n; \varepsilon, \omega] = (\mathcal{X}^{1/2} + \mathcal{X}^{-1/2}) (\mathcal{Y}^{1/2} + \mathcal{Y}^{-1/2}) D_{k-1}[m, n; \varepsilon, \omega] + (\omega\mathcal{X} + \varepsilon\mathcal{X}^{-1}) D_{k-2}[m, n; \varepsilon, \omega]. \quad (9)$$

A comparison of the recursions shows that  $D_k[m, n; \mu_{\downarrow} - 1, \mu_{\uparrow} - 1] = T_k[m, n]$ . Suppose the walk takes  $u$  E-N deep hooks of weight  $\varepsilon$ , and  $d$  W-S deep hooks of weight  $\omega$ . There are  $\binom{k-u-d}{u+d} \binom{u+d}{u}$  positions for the deep hooks among the  $k$  moves. Hence

$$\begin{aligned} D_k[m, n; \varepsilon, \omega] &= \sum_{\substack{b-f+u-d=m, a-c+u-d=n \\ a+b+c+2d+f+2u=k}} \varepsilon^u \omega^d \binom{k-u-d}{u+d} \binom{u+d}{u} \binom{k-2u-2d}{a, b, c, f} \\ &= \sum_{u, d \geq 0} \varepsilon^u \omega^d \binom{k-u-d}{u, d, \frac{k+m+n}{2} - 2u, \frac{k-m-n}{2} - 2d} \binom{k-2u-2d}{\frac{k-m+n}{2} - u - d}. \end{aligned} \quad (10)$$

We will refer to this walk as *diffusion with deep hooks*.

### 3 Initial conditions along a line

**Example 22 (Reflected paths)** *Suppose an ordinary diffusion walk (with steps  $\rightarrow, \uparrow, \leftarrow, \downarrow$ ) is reflected at the line  $y = -r$ , where  $r$  is a positive integer. This means that whenever the path hits the mirror  $y = -r$  (in a downward step  $\downarrow$ ), it must leave the mirror in the next move taking an upward step  $\uparrow$ . For short, we will call such walks reflected paths, and we denote the number of reflected paths from  $(0, 0)$  to  $(n, m)$  in  $k$  moves by  $D_k[m, n; \bar{\tau}]$ . Because there are no horizontal steps along the mirror, we get the recursive initial condition*

$$D_k[-r, n; \bar{\tau}] = D_{k-1}[1 - r, n; \bar{\tau}].$$

*In operator terminology,*

$$\mathcal{Y}^{-r} D_k[0, n; \bar{\tau}] = \mathcal{Y}^1 \mathcal{Y}^{-r} D_{k-1}[0, n; \bar{\tau}] \quad (11)$$

*(note that  $q = 0$  and  $p = 1$ , because  $y = 0$  is the relevant axis of symmetry). Replacing  $D_k[0, n; \bar{\tau}]$  by  $\mathcal{R} D_{k-1}[0, n; \bar{\tau}]$ , with  $\mathcal{R} = (1 + \mathcal{X}\mathcal{Y})(\mathcal{X}^{-1} + \mathcal{Y}^{-1})$  (see Example 16) we can write (11) as*

$$(\mathcal{Y}^{-r} \mathcal{R} - \mathcal{Y}^1 \mathcal{Y}^{-r}) D_k[0, n; \bar{\tau}] = 0.$$

*Hence this recursive initial condition (reflection) is expressed by the operator  $\mathcal{L} := (\mathcal{R} - \mathcal{Y}) \mathcal{Y}^{-r}$  in this example.*

We will now discuss the above construction in a more general situation. For given  $t$  we can define a line parallel to the symmetry axis  $py = qx$  in the parametric form  $a(p, q) + t(q, -p)$ ,  $(a, t) \in \mathbb{Z}_{q/p}$ . The parametric line  $(pa + qt, qa - pt)$  can also be written as  $py = qx - t(q^2 + p^2)$  in the  $x$ - $y$ -plane. We want to find a sequence of  $\mathcal{R}$ -recursive boards  $(U_k[m, n])_{k \geq 0}$  with (recursively) given initial values  $U_k[qa - pt, pa + qt]$  along that line, i.e., with conditions of the form

$$U_k[qa - pt, pa + qt] = \mathcal{K}U_{k-1}[qa - pt, pa + qt] \quad (12)$$

or equivalently  $0 = (\mathcal{K}\mathcal{Y}^{-t} - \mathcal{Y}^{-t}\mathcal{R})U_{k-1}[qa, pa]$ , for some linear operator  $\mathcal{K}$ . The following Theorem shows how to expand  $U_k[m, n]$ .

**Theorem 23** *Suppose  $(C_k)_{k \geq 0}$  is a sequence of boards in  $\mathfrak{B}_{q/p}^\cap$  following a  $q/p$ -symmetric recursion  $\mathcal{R} \in \mathfrak{B}_{q/p}^\cap$ ,*

$$C_k = \mathcal{R}C_{k-1}$$

*for all  $k \geq 1$ . Let  $\mathcal{L} \in \mathfrak{L}_{q/p}^\cap$  be invertible in  $\mathfrak{L}_{q/p}^\geq$ . Define a sequence  $(U_k)_{k \geq 0}$  of boards in  $\mathfrak{B}_{q/p}^\leq$  by the relation*

$$U_k[m, n] = C_k[m, n] - (\mathcal{L}^T / \mathcal{L})C_k^\top[m, n] \quad (13)$$

*for all points  $(n, m)$  in the  $q/p$ -grid. Then  $(U_k)_{k \geq 0}$  follows the same recursion  $\mathcal{R}$ , and has, for all  $q/p$ -grid points  $(pa, qa)$  on the symmetry axis, the recursive initial values*

$$\mathcal{L}U_k[qa, pa] = 0. \quad (14)$$

**Remark 24** *The board sequence  $(C_k)$  can be chosen from  $\mathfrak{B}_{q/p}^\geq$ , but then the solution  $(U_k)_{k \geq 0}$  will be in  $\mathfrak{B}_{q/p}^\geq - \mathfrak{B}_{q/p}^\leq$ . If  $(C_k) \subseteq \mathfrak{B}_{q/p}^\cap$ , and a solution  $(U_k)_{k \geq 0}$  of boards in  $\mathfrak{B}_{q/p}^\geq$  is desired, let  $U_k[m, n] = C_k[m, n] - (\mathcal{L}^T \mathcal{L}^{-1})C_k^\top[m, n]$  (if  $\mathcal{L}^{-1}$  exists).*

**Proof.** It is straight forward to verify that the above expansion follows the right recursion. We check the initial values:

$$\mathcal{L}U_k = \mathcal{L}C_k - \mathcal{L} \left( \frac{\mathcal{L}^T}{\mathcal{L}} \right) C_k^\top = \mathcal{L}C_k - \mathcal{L}^T C_k^\top = \mathcal{L}C_k - (\mathcal{L}C_k)^\top.$$

From  $V[qa, pa] = V^\top[qa, pa]$  for any board  $V$  follows  $\mathcal{L}U_k[qa, pa] = 0$ . ■

**Example 25 (Reflected paths continued)** *A path too short for being reflected by the mirror will behave like an unrestricted ordinary diffusion,  $D_k[m, n; \bar{r}] = D_k[m, n]$  for  $k \leq r$ . The  $\mathcal{R}$ -recursive sequence  $(C_k)$  in the Theorem therefore equals  $(D_k)$ , and formula (13) gives with  $\mathcal{L} = (\mathcal{R} - \mathcal{Y})\mathcal{Y}^{-r}$*

$$\begin{aligned} D_k[m, n; \bar{r}] &= D_k[m, n] - \frac{((\mathcal{R} - \mathcal{Y})\mathcal{Y}^{-r})^T}{(\mathcal{R} - \mathcal{Y})\mathcal{Y}^{-r}} D_k^\top[m, n] \\ &= D_k[m, n] - \frac{\mathcal{R} - \mathcal{Y}^{-1}}{\mathcal{R} - \mathcal{Y}} \mathcal{Y}^{2r} D_k[m, n]. \end{aligned} \quad (15)$$

All what remains to do is expanding

$$\begin{aligned} (\mathcal{R} - \mathcal{Y})^{-1} &= (\mathcal{X} + \mathcal{X}^{-1} + \mathcal{Y}^{-1})^{-1} = \mathcal{Y} (\mathcal{Y} (\mathcal{X} + \mathcal{X}^{-1}) + 1)^{-1} \\ &= \sum_{j \geq 0} (-1)^j \mathcal{Y}^{j+1} \sum_{i=0}^j \binom{j}{i} \mathcal{X}^{2i-j}. \end{aligned}$$

Hence, with  $D_k[m, n] = \binom{k}{s} \binom{k}{s-m}$  where  $s := \frac{k+m+n}{2}$  (see (2)), we get the expansion  $D_k[m, n; \bar{r}]$

$$\begin{aligned} &= D_k[m, n] - \sum_{j \geq 0} (-1)^j \sum_{i=0}^j \binom{j}{i} \times \\ &\left[ \binom{k+1}{s+1+r+i} \binom{k+1}{s-m-r+i-j} - \binom{k}{s+r+i} \binom{k}{s-m-r+i-j} \right] \quad (16) \\ &= D_k[m, n] - \sum_{i=0}^{k-s-r} (-1)^i \left[ \binom{k+1}{s+1+r+i} \binom{k-i}{s-m-r} - \binom{k}{s+r+i} \binom{k-i-1}{s-m-r} \right] \end{aligned}$$

As in the unrestricted case, all we really need to know is  $D_0[m, n; \bar{r}]$ ; the boards  $D_k[m, n; \bar{r}] = \mathcal{R}^k D_0[m, n; \bar{r}]$  can be recursively constructed. In this example,  $D_0[m, m; \bar{r}] = 0$  for  $m \geq -2r$  except  $D_0[0, 0; \bar{r}]$ . Fig. 7 shows the first six rows of  $D_0[m, n; \bar{r}]$  for  $m < -2r$ , the correction terms, which could be called a generalized signed Pascal's triangle.

Figure 7														
$-2r$	...	0	0	0	0	0	0	0	0	0	0	0	0	...
$-2r - 1$	...	0	0	0	0	0	-1	0	-1	0	0	0	0	...
$-2r - 2$	...	0	0	0	0	1	0	1	0	1	0	0	0	...
$-2r - 3$	...	0	0	0	-1	0	-2	0	-2	0	-1	0	0	...
$-2r - 4$	...	0	0	1	0	3	0	4	0	3	0	1	0	...
$-2r - 5$	...	0	-1	0	-4	0	-7	0	-7	0	-4	0	-1	...
$-2r - 6$	...	1	0	5	0	11	0	14	0	11	0	5	0	...
$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$
$m \uparrow / n \rightarrow$	...	<b>-6</b>	<b>-5</b>	<b>-4</b>	<b>-3</b>	<b>-2</b>	<b>-1</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b> ...
$D_0[m, n; \bar{r}]$														

The next three boards, Fig. 8, demonstrate how the correction terms achieve the reflection (at  $y = -1$ ). The boards are still symmetric around the  $y$ -axis; hence only entries with  $n \geq 0$

are displayed.

Figure 8																	
<b>1</b>	0	0	0	0	0	<b>1</b>	1	0	0	0	0	<b>1</b>	0	2	0	0	
<b>0</b>	1	0	0	0	0	<b>0</b>	0	1	0	0	0	<b>0</b>	4	0	0	0	
<b>-1</b>	<span style="border: 1px solid black; padding: 1px;">0</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	<b>-1</b>	<span style="border: 1px solid black; padding: 1px;">1</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	<b>-1</b>	<span style="border: 1px solid black; padding: 1px;">0</span>	<span style="border: 1px solid black; padding: 1px;">1</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	<span style="border: 1px solid black; padding: 1px;">0</span>	
<b>-2</b>	0	0	0	0	0	<b>-2</b>	0	-1	0	0	0	<b>-2</b>	-2	0	-1	0	
<b>-3</b>	0	-1	0	0	0	<b>-3</b>	-1	0	0	0	0	<b>-3</b>	0	-3	0	0	
<b>-4</b>	1	0	1	0	0	<b>-4</b>	0	-1	0	0	0	<b>-4</b>	-2	0	0	0	
<b>-5</b>	0	-2	0	-1	0	<b>-5</b>	1	0	1	0	0	<b>-5</b>	0	-1	0	0	
<b>-6</b>	4	0	3	0	1	<b>-6</b>	0	-2	0	-1	0	<b>-6</b>	1	0	1	0	
$m \uparrow / n \rightarrow$	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	$m \uparrow / n \rightarrow$	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	$m \uparrow / n \rightarrow$	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	
$D_0[m, n; \bar{1}]$						$D_1[m, n; \bar{1}]$						$D_2[m, n; \bar{1}]$					

**Remark 26** The above boards can help to explain the definitions of  $\mathfrak{B}_{q/p}^{\leq}$  and  $\mathfrak{L}_{q/p}^{\geq}$ . We enumerate walks which stay on one side (the interior) of a boundary line parallel to the axis of symmetry  $py = qx$ . For a given vector  $t(q, -p)$  on that boundary, we write the line in parametric form as

$$a(p, q) + t(q, -p) \text{ where } (a, t) \in \mathbb{Z}_{q/p}, t > 0 \text{ fixed.}$$

The line divides the grid into two sides, the interior where we find the number of walks under investigation, and the exterior containing the positions of the correction terms. Given  $t$ , the interior is the set

$$\{(pa + qb, qa - pb) \mid (a, b) \in \mathbb{Z}_{q/p}, b \leq t\},$$

and the exterior is the set

$$\{(pa + qb, qa - pb) \mid (a, b) \in \mathbb{Z}_{q/p}, b \geq t\}.$$

They have the boundary line in common. The exterior of a board may contain infinitely many nonzero correction term. The entries in the interior have finite support. For every board we can therefore find another parallel to the chosen symmetry axis such that the support of the board is completely “below” that parallel. That is the meaning of the condition “ $U[aq + bp, ap - bq] \neq 0$  implies  $b \leq \beta_1$ ” in the definition of  $\mathfrak{B}_{q/p}^{\leq}$ . The second restriction on the support of boards in  $\mathfrak{B}_{q/p}^{\leq}$  in Definition 12, “ $U[aq + bp, ap - bq] \neq 0$  implies  $aq + bp \leq \beta_2$ ”, means that the support of any given board can also be bounded by a line parallel to the  $y$ -axis; depending on the number of moves there is a highest point beyond no path will go, nor any of the correction terms. In the above example the two conditions coincide.

Suppose that  $(B_k)$  is a basic sequence of boards for the recursion operator  $\mathcal{R}$ . The Laurent series  $\mathfrak{L}_{q/p}^{\geq}$  have been defined such that they can create the correction terms from  $B_0[m, n]$ . Let  $(i, j) \in \mathbb{Z}_{q/p}$ ,  $j \geq t$ , and  $iq + jp \geq M$  for some lower bound  $M$ . The board  $(1 - a_{i,j} \mathcal{X}^i \mathcal{Y}^j) B_0[m, n] = B_0[m, n] - a_{i,j} B_0[m + iq + jp, ip - jq]$  can only contain two nonzero values, a 1 at  $(0, 0)$ , and the correction term  $-a_{i,j}$  at  $(-ip + jq, -iq - jp)$ , in the exterior and below a certain height  $-M$ . Additional terms (possibly infinitely many) of the form

$a_{i,j}\mathcal{X}^i\mathcal{Y}^j$  introduce additional correction terms in the exterior. As in the Reflection Principle, the correction terms start random walks “behind the mirror”, i.e., in the exterior. If they are correctly positioned, they do not interfere with the random walk in the interior except for forcing the required recursive initial values on the boundary. If the walks are restricted by an upper boundary, the solutions will be boards in  $\mathfrak{B}_{q/p}^{\geq}$ . Because of symmetry, they are transposes of the corresponding lower boundary problem.

Two features of the above example are typical for lattice path applications. First, the beginning of the  $\mathcal{R}$ -recursive sequence  $(C_k)$  will be unrestricted, and therefore equal to the (symmetric) sequence  $(B_k)$  of basic boards for  $\mathcal{R}$ . Note that the proof does not use any symmetry of  $C_k$ , only the symmetry of the recursion  $\mathcal{R}$ . This will be more important in Theorem 29, which is usually applied to unsymmetrical boards. Next, as already announced in (12), the recursive initial conditions are of the form

$$\mathcal{Y}^{-t}U_k[qa, pa] = \mathcal{K}\mathcal{Y}^{-t}U_{k-1}[qa, pa]$$

for all  $k = 1, 2, \dots$ . Theorem 23 applies to this situation if  $\mathcal{K} \in \mathfrak{L}_{q/p}^{\cap}$  and  $\mathcal{L} := \mathcal{Y}^{-t}(\mathcal{R} - \mathcal{K})$ :

$$U_k = B_k - \frac{\mathcal{R} - \mathcal{K}^T}{\mathcal{R} - \mathcal{K}}\mathcal{Y}^{2t}B_k. \quad (17)$$

If  $\mathcal{R}$  is invertible, it can be an advantage to expand  $U_k$  in the form

$$U_k = B_k - \frac{1 - \mathcal{K}^T\mathcal{R}^{-1}}{1 - \mathcal{K}\mathcal{R}^{-1}}\mathcal{Y}^{2t}B_k = B_k - (1 - \mathcal{K}^T\mathcal{R}^{-1})\mathcal{Y}^{2t}\sum_{j \geq 0} \mathcal{K}^j B_{k-j} \quad (18)$$

Note that the negative powers of  $\mathcal{R}$  introduce basic boards with negative indices,  $B_{-k} := \mathcal{R}^{-k}B_0$  for  $k \geq 0$ . However, in applications to planar walks at most a finite number of those boards will contribute to the counting results (the interior).

**Example 27 (Reflected paths continued)** We saw in (15) that  $D_k[m, n; \bar{r}] = D_k[m, n] - \frac{\mathcal{R} - \mathcal{Y}^{-1}}{\mathcal{R} - \mathcal{Y}}\mathcal{Y}^{2r}D_k[m, n]$ . Applying (18) we find the expansion

$$D_k[m, n; \bar{r}] = D_k[m, n] - (1 - \mathcal{Y}^{-1}\mathcal{R}^{-1})\sum_{j \geq 0} D_{k-j}[m + j + 2r, n].$$

It is easy to verify that for all integers  $k$  holds  $D_k[m, n] = 0$  if  $m > k$ . Hence  $j \leq \frac{k-m}{2} - r$  in the above sum. Again, let  $s = \frac{k+m+n}{2}$ . We find

$$\begin{aligned} D_k[m, n; \bar{r}] &= \binom{k}{s} \binom{k}{s-m} \\ &\quad - \sum_{j = \lfloor \frac{k+m}{2} \rfloor + r - 1}^{k-1} \left( \binom{j+1}{s+r} \binom{j+1}{s-n+r} - \binom{j}{s+r-1} \binom{j}{s-n+r-1} \right) \end{aligned}$$

a nonalternating expansion (compare to (16)).

### 3.1 Left hooks above a diagonal barrier

The left hook walks only occupy positions where both coordinates are of equal parity. Depending on the parity of  $r$  the path approaches the lower right boundary  $y = x - r$  in two different ways, as shown in Fig. 9.

Figure 9

The positions  $\bullet$  of a left hook path at the boundary  $y = x - r$

0		○	┌	○	■
-1		└	●	■	
-2		×	■		
	-1	0	1	2	3

If  $r$  is odd,  $\bullet$  can only be reached from two predecessors.

0		○	┌	○	
-1		└	●		■
-2		○	└	■	
	-1	0	1	2	3

If  $r$  is even,  $\bullet$  can be reached from three predecessors.

For even  $r$  the boundary can be represented by (nonrecursive) initial values of zeroes along the line  $y = x - r$ . The classical reflection principle applies in this case. We will therefore assume that  $r$  is odd for left hook walks. The number  $H_k[m, n; \nearrow_r]$  of left hook walks from the origin to  $(n, m)$ , in  $k$  steps strictly above  $y = x - r$ , is uniquely determined by the recursion  $\mathcal{R} = \mathcal{Y} + \mathcal{Y}^{-1} + \mathcal{X} + \mathcal{X}^{-1}$  where  $\mathcal{X} = \mathcal{E}^{1,1}$  and  $\mathcal{Y} = \mathcal{E}^{1,-1}$ , and the recursive initial condition

$$\begin{aligned} H_k[n - r + 1, n; \nearrow_r] &= H_{k-1}[n - r + 2, n + 1; \nearrow_r] + H_{k-1}[n - r + 2, n - 1; \nearrow_r] \\ &= (\mathcal{X} + \mathcal{Y}) H_{k-1}[n - r + 1, n; \nearrow_r]. \end{aligned}$$

Hence

$$\mathcal{Y}^{(1-r)/2} \mathcal{R} H_k[n, n; \nearrow_r] = \mathcal{Y}^{(1-r)/2} (\mathcal{X} + \mathcal{Y}) H_k[n, n; \nearrow_r]$$

We can apply Theorem 23 in the form of formula (17) with  $p = q = 1$ ,  $t = (r - 1) / 2$

$$\begin{aligned} H_k[m, n; \nearrow_r] &= H_k[m, n] - \frac{\mathcal{R} - (\mathcal{X} + \mathcal{Y})^T}{\mathcal{R} - (\mathcal{X} + \mathcal{Y})} \mathcal{Y}^{r-1} H_k^\top[m, n] \\ &= H_k[m, n] - \frac{\mathcal{Y} + \mathcal{X}^{-1}}{\mathcal{X}^{-1} + \mathcal{Y}^{-1}} H_k[m + r - 1, n - r + 1] \end{aligned} \quad (19)$$

where we use the symmetry of  $H_k[m, n]$ . For the actual expansion of  $(\mathcal{X}^{-1} + \mathcal{Y}^{-1})^{-1}$  we could apply Example 19 with  $z = -1$ ,  $a' = 0$ , and  $b' = -1$ ,

$$\frac{1}{\mathcal{X}^{-1} + \mathcal{Y}^{-1}} = \frac{\mathcal{Y}}{1 + \mathcal{X}^{-1}\mathcal{Y}} = \mathcal{Y} \sum_{j \geq 0} (-1)^j \mathcal{X}^{-j} \mathcal{Y}^j.$$

However, this example is very special, and there is a more elegant way to expand  $H_k[m, n; \nearrow_r]$ . We used already in Lemma 17 that this recurrence operator  $\mathcal{R}$  can be factored,  $\mathcal{R} = (1 + \mathcal{X}\mathcal{Y})(\mathcal{Y}^{-1} + \mathcal{X}^{-1})$ , and therefore

$$\begin{aligned} \frac{\mathcal{Y} + \mathcal{X}^{-1}}{\mathcal{Y}^{-1} + \mathcal{X}^{-1}} H_k[m + r - 1, n - r + 1] &= \frac{(\mathcal{Y} + \mathcal{X}^{-1})(1 + \mathcal{X}\mathcal{Y})}{\mathcal{R}} \mathcal{Y}^{-1} H_k[m + r, n - r] \\ &= (1 + \mathcal{X}^{-1}\mathcal{Y}^{-1})(1 + \mathcal{X}\mathcal{Y}) H_{k-1}[m + r, n - r] = (2 + \mathcal{X}^{-1}\mathcal{Y}^{-1} + \mathcal{X}\mathcal{Y}) H_{k-1}[m + r, n - r]. \end{aligned}$$

Hence  $H_k[m, n; \nearrow_r] =$

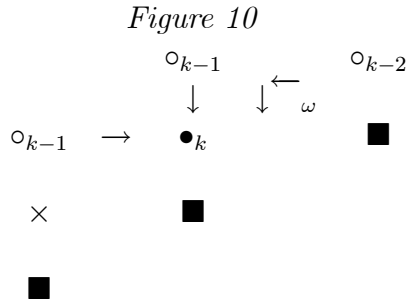
$$\begin{aligned}
&= \binom{k}{\frac{k+m}{2}} \binom{k}{\frac{k+n}{2}} - 2 \binom{k-1}{\frac{k+m+r-1}{2}} \binom{k-1}{\frac{k+n-r-1}{2}} - \binom{k-1}{\frac{k+m+r-1}{2} - 1} \binom{k-1}{\frac{k+n-r-1}{2}} \\
&\quad - \binom{k-1}{\frac{k+m+r-1}{2} + 1} \binom{k-1}{\frac{k+n-r-1}{2}} \\
&= \binom{k}{\frac{k+m}{2}} \binom{k}{\frac{k+n}{2}} - \binom{k+1}{\frac{k+m+r-1}{2} + 1} \binom{k-1}{\frac{k+n-r-1}{2}}. \tag{20}
\end{aligned}$$

### 3.2 Deep hooks above a diagonal barrier

We saw in (9) that the recursion  $\mathcal{R}$  for deep hooks is defined by the equation

$$\mathcal{R}^2 = (\mathcal{X}^{1/2} + \mathcal{X}^{-1/2}) (\mathcal{Y}^{1/2} + \mathcal{Y}^{-1/2}) \mathcal{R} + \omega \mathcal{X} + \varepsilon \mathcal{X}^{-1}.$$

Solving for  $\mathcal{R}$  shows that this recursion operator is symmetric and an element of  $\mathfrak{L}_{q/p}^{\geq}$ .



*The point  $\bullet$  at the border in the  $k$ -th move  
can only be reached by three predecessors.*

Deep hooks will stay above the barrier  $y = x - r$  iff we require the recursive initial values (see Fig. 10)

$$\begin{aligned}
D_k[n-r+1, n; \varepsilon, \omega; \nearrow_r] &= D_{k-1}[n-r+2, n; \varepsilon, \omega; \nearrow_r] + D_{k-1}[n-r+1, n-1; \varepsilon, \omega; \nearrow_r] \\
&\quad + \omega D_{k-2}[n-r+2, n+1; \varepsilon, \omega; \nearrow_r] \\
&= ((\mathcal{X}^{1/2} + \mathcal{X}^{-1/2}) \mathcal{Y}^{1/2} \mathcal{R} + \omega \mathcal{X}) D_{k-2}[n-r+1, n; \varepsilon, \omega; \nearrow_r]
\end{aligned}$$

We can apply Theorem 23 in a form similar to formula (18) (with  $p = q = 1$ ,  $t = (r-1)/2$ ,  $\mathcal{K} = (\mathcal{X}^{1/2} + \mathcal{X}^{-1/2}) \mathcal{Y}^{1/2} \mathcal{R} + \omega \mathcal{X}$ , and  $\mathcal{R}^2$  instead of  $\mathcal{R}$ ) to expand  $D_k[m, n; \varepsilon, \omega; \nearrow_r]$  in terms of  $D_k[m, n; \varepsilon, \omega]$  (see (10)):

$$\begin{aligned}
D_k[m, n; \varepsilon, \omega; \nearrow_r] &= D_k[m, n; \varepsilon, \omega] - \frac{\mathcal{R}^2 - \mathcal{K}^T}{\mathcal{R}^2 - \mathcal{K}} \mathcal{Y}^{r-1} D_k[m, n; \varepsilon, \omega] \\
&= D_k[m, n; \varepsilon, \omega] - \frac{\mathcal{Y} + (\mathcal{X}^{1/2} + \mathcal{X}^{-1/2})^{-1} \mathcal{Y}^{1/2} \mathcal{R}^{-1} \varepsilon \mathcal{X}^{-1}}{1 + (\mathcal{X}^{1/2} + \mathcal{X}^{-1/2})^{-1} \mathcal{Y}^{1/2} \mathcal{R}^{-1} \varepsilon \mathcal{X}^{-1}} \mathcal{Y}^{r-1} D_k[m, n; \varepsilon, \omega].
\end{aligned}$$

Expanding the fraction gives the explicit formula

$$\begin{aligned}
& D_k[m, n; \varepsilon, \omega; \nearrow_r] \\
&= D_k[m, n; \varepsilon, \omega] - D_k[m + r, n - r; \varepsilon, \omega] - \sum_{j=1}^k \varepsilon^j \sum_{i \geq 0}^{(k-m-n)/2} \binom{i+j-1}{i} (-1)^{i+j} \\
&\times (D_{k-j}[m + i + r, n + i - j - r; \varepsilon, \omega] - D_{k-j}[m - 1 + i + r, n + 1 + i - j - r; \varepsilon, \omega]).
\end{aligned}$$

## 4 Recursive initial conditions along two lines

We now extend the scope of counting with recursive initial values, to problems where such values are prescribed along two lines parallel to an axis of symmetry of the recursion  $\mathcal{R}$ .

**Example 28 (Paths between two mirrors)** *Suppose an ordinary diffusion walk (with steps  $\rightarrow, \uparrow, \leftarrow, \downarrow$ ) is reflected at the bottom mirror  $y = -r$ , and also at the top mirror  $y = l$ , where  $r$  and  $l$  are positive integers. We denote the number of reflected paths from  $(0, 0)$  to  $(n, m)$  in  $k$  moves between two mirrors by  $D_k[m, n; \frac{l}{r}]$ . Because there are no horizontal moves along the mirrors, reflection is equivalent to the recursive initial condition*

$$D_k[-r, n; \frac{l}{r}] = D_{k-1}[1 - r, n; \frac{l}{r}] \text{ and } D_k[l, n; \frac{l}{r}] = D_{k-1}[l - 1, n; \frac{l}{r}].$$

In operator terminology,

$$(\mathcal{R} - \mathcal{Y}) \mathcal{Y}^{-r} D_k[0, n; \frac{l}{r}] = 0 \text{ and } (\mathcal{R} - \mathcal{Y}^{-1}) \mathcal{Y}^l D_k[0, n; \frac{l}{r}] = 0.$$

**Theorem 29** *Let  $(U_k)_{k \geq 0}$  be a sequence of boards in  $\mathfrak{B}_{q/p}^{\leq}$  following the  $q/p$ -symmetric recursion  $\mathcal{R} \in \mathfrak{B}_{q/p}^{\cap}$ , and let  $\mathcal{L}$  and  $\mathcal{J}$  be  $\mathfrak{L}_{q/p}^{\geq}$ -invertible linear operators from  $\mathfrak{L}_{q/p}^{\cap}$ . Define  $\tilde{\mathcal{L}} := \mathcal{L}^T / \mathcal{L}$  and  $\tilde{\mathcal{J}} := \mathcal{J}^T / \mathcal{J}$ , and the sequence  $(V_k)_{k \geq 0}$  by*

$$V_k = \frac{1}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}} U_k - \left( \frac{\tilde{\mathcal{J}}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}} U_k \right)^{\top} = \sum_{i \geq 0} (\tilde{\mathcal{J}}\tilde{\mathcal{L}})^i U_k - \sum_{j \geq 0} (\tilde{\mathcal{J}}^j \tilde{\mathcal{L}}^j)^T \tilde{\mathcal{J}}^T U_k^{\top}. \quad (21)$$

Then  $(V_k)$  follows the same recursion  $\mathcal{R}$ , and has the initial values

$$\mathcal{L}V_k[qa, pa] = \mathcal{L}U_k[qa, pa] \text{ and } \mathcal{J}^T V_k[qa, pa] = 0$$

for all  $q/p$ -grid points  $(pa, qa)$  on the symmetry axis.

**Remark 30** *In general, the sequence  $(V_k)$  is neither in  $\mathfrak{B}_{q/p}^{\leq}$  nor in  $\mathfrak{B}_{q/p}^{\geq}$ , because  $V_k$  is a difference of elements from both spaces. Only operators that act on both spaces can be applied to  $V_k$ .*

**Proof.** First we check the recursive initial condition  $\mathcal{J}^T V_k[qa, pa] = 0$ .

$$\mathcal{J}^T V_k = \frac{\mathcal{J}^T}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}} U_k - \mathcal{J}^T \left( \frac{\tilde{\mathcal{J}}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}} U_k \right)^{\top} = \frac{\mathcal{J}^T}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}} U_k - \left( \frac{\mathcal{J}^T}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}} U_k \right)^{\top}.$$

All terms in this sum equal zero along the axis of symmetry,  $py = qx$ .  
 Finally we check the initial condition  $\mathcal{L}V_k[qa, pa] = \mathcal{L}U_k[qa, qa]$ .

$$\begin{aligned}\mathcal{L}V_k &= \frac{\mathcal{L}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}}U_k - \left( \frac{\mathcal{L}^T \tilde{\mathcal{J}}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}}U_k \right)^\top = \frac{\mathcal{L} (1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}) + \mathcal{L}\tilde{\mathcal{J}}\tilde{\mathcal{L}}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}}U_k - \left( \frac{\mathcal{L}^T \tilde{\mathcal{J}}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}}U_k \right)^\top \\ &= \mathcal{L}U_k + \frac{\mathcal{L}^T \tilde{\mathcal{J}}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}}U_k - \left( \frac{\mathcal{L}^T \tilde{\mathcal{J}}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}}U_k \right)^\top.\end{aligned}$$

The difference of the last two terms equals 0 along the symmetry axis  $py = qx$ .

Expanding  $\frac{1}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}}U_k - \left( \frac{\tilde{\mathcal{J}}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}}U_k \right)^\top$  gives

$$\begin{aligned}& \sum_{i \geq 0} (\tilde{\mathcal{J}}\tilde{\mathcal{L}})^i U_k[qa + pb, pa - qb] - \left( \sum_{j \geq 0} (\tilde{\mathcal{J}}\tilde{\mathcal{L}})^j \tilde{\mathcal{J}}U_k[qa + pb, pa - qb] \right)^\top \\ &= \sum_{i \geq 0} (\tilde{\mathcal{J}}\tilde{\mathcal{L}})^i U_k[qa + pb, pa - qb] - \sum_{j \geq 0} (\tilde{\mathcal{J}}^j \tilde{\mathcal{L}}^j)^T \tilde{\mathcal{J}}^T U_k[qa - pb, pa + qb].\end{aligned}$$

■

**Example 31 (Paths between two mirrors continued)** We saw in Example 28 that the recursive initial conditions for an ordinary diffusion between two mirrors are given by the operators  $\mathcal{L} := (\mathcal{R} - \mathcal{Y})\mathcal{Y}^{-r}$  and  $\mathcal{J}^T := (\mathcal{R} - \mathcal{Y}^{-1})\mathcal{Y}^l$ . In Theorem 29 we must choose  $D_k[m, n; \frac{l}{r}]$  for  $U_k[m, n]$ , because we are looking for  $V_k[m, n] = D_k[m, n; \frac{l}{r}]$  with the properties  $\mathcal{L}D_k[0, n; \frac{l}{r}]$  ( $= \mathcal{L}V_k[0, n] = \mathcal{L}U_k[0, n]$ )  $= \mathcal{L}D_k[0, n; \frac{l}{r}] = 0$  and  $\mathcal{J}^T D_k[0, n; \frac{l}{r}] = 0$ . Substitute  $\tilde{\mathcal{L}} := \mathcal{L}^T / \mathcal{L} = \frac{\mathcal{R} - \mathcal{Y}^{-1}}{\mathcal{R} - \mathcal{Y}}\mathcal{Y}^{2r}$  and  $\tilde{\mathcal{J}} := \mathcal{J}^T / \mathcal{J} = \frac{\mathcal{R} - \mathcal{Y}^{-1}}{\mathcal{R} - \mathcal{Y}}\mathcal{Y}^{2l}$  into (21) and get

$$\begin{aligned}D_k[m, n; \frac{l}{r}] &= \sum_{i \geq 0} \left( \frac{\mathcal{R} - \mathcal{Y}^{-1}}{\mathcal{R} - \mathcal{Y}} \right)^{2i} D_k[m + 4i(r + l), n; \frac{l}{r}] \\ &- \sum_{j \geq 0} \left( \left( \frac{\mathcal{R} - \mathcal{Y}^{-1}}{\mathcal{R} - \mathcal{Y}} \right)^{2j+1} D_k[m + 4j(r + l) + 2l, n; \frac{l}{r}] \right)^\top,\end{aligned}$$

where  $D_k[m, n; \frac{l}{r}]$  has been expanded in Examples 25 and 27.

The above example is typical for planar walks applications where the boards  $U_k$  satisfy the condition  $\mathcal{L}U_k[qa, pa] = 0$ , with  $\mathcal{L} := \mathcal{Y}^{-t}(\mathcal{R} - \mathcal{K})$  (see (12)). In this case Theorem 23 tells us that

$$U_k[m, n] = (1 - \tilde{\mathcal{L}}T)C_k[m, n].$$

Usually  $(C_k)$  is the basic sequence  $(B_k)$  for the recursion  $\mathcal{R}$ ; hence  $U_k[m, n] = (1 - \tilde{\mathcal{L}})B_k[m, n]$ , because  $B_k$  is symmetric.

Adding a few more conditions, the typical sum over **all** integers occurs in the expansion of  $V_k$ , as we will see in the next corollary. This type of summation occurs in most known formulas for counting lattice paths between parallel boundaries. The additional conditions

are rather restrictive; we must require that  $(\tilde{\mathcal{J}}\tilde{\mathcal{L}})^T = (\tilde{\mathcal{J}}\tilde{\mathcal{L}})^{-1}$ , which implies  $\tilde{\mathcal{J}}\tilde{\mathcal{L}} = \mathcal{Y}^w$  for some integer  $w$  (by Lemma 21). However there are nontrivial applications, as shown in Section 4.1.

**Corollary 32** *Let  $(B_k)_{k \geq 0}$  be the basic sequence of boards for the  $q/p$ -symmetric recursion  $\mathcal{R} \in \mathfrak{L}_{q/p}^\cap$ , and let  $\mathcal{L}$  and  $\mathcal{J}$  be  $\mathfrak{L}_{q/p}^\geq$ -invertible linear operators from  $\mathfrak{L}_{q/p}^\cap$ . If  $(\mathcal{L}\mathcal{J})^T / (\mathcal{L}\mathcal{J}) = \mathcal{Y}^w$  for some positive integer  $w$ , and*

$$(\mathcal{L}^T)^{-1} B_k = (\mathcal{L}^{-1})^T B_k \text{ for all } k = 0, 1, \dots$$

then the sequence  $(V_k)_{k \geq 0}$  defined by

$$V_k = \sum_{j \in \mathbb{Z}} \mathcal{Y}^{jw} (1 - \mathcal{L}^T / \mathcal{L}) B_k$$

follows the same recursion  $\mathcal{R}$ , and has the recursive initial values

$$\mathcal{L}V_k[qa, pa] = 0 \text{ and } \mathcal{J}^T V_k[qa, pa] = 0$$

for all  $q/p$ -grid points  $(pa, qa)$  on the symmetry axis.

**Proof.** In Theorem 29 choose  $U_k[m, n] = (1 - \tilde{\mathcal{L}})B_k[m, n]$ . We know from Theorem 23 that  $\mathcal{L}U_k[qa, pa] = 0$  (the boards  $(B_k)$  are symmetric). Hence

$$\begin{aligned} V_k &= \frac{U_k}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}} - \left( \frac{\tilde{\mathcal{J}}}{1 - \tilde{\mathcal{J}}\tilde{\mathcal{L}}} U_k \right)^\top = \frac{(1 - \tilde{\mathcal{L}})B_k}{1 - \mathcal{Y}^w} - \left( \frac{\tilde{\mathcal{J}}}{1 - \mathcal{Y}^w} (1 - \tilde{\mathcal{L}})B_k \right)^\top \\ &= \frac{(1 - \tilde{\mathcal{L}})B_k}{1 - \mathcal{Y}^w} - \left( \frac{\tilde{\mathcal{J}}\tilde{\mathcal{L}}}{1 - \mathcal{Y}^w} (\tilde{\mathcal{L}}^{-1} - 1)B_k \right)^\top = \frac{(1 - \tilde{\mathcal{L}})B_k}{1 - \mathcal{Y}^w} + \mathcal{Y}^{-w} \sum_{k \geq 0} \mathcal{Y}^{-kw} \left( 1 - (\tilde{\mathcal{L}}^{-1})^T \right) B_k, \end{aligned}$$

where we used that for positive  $w$  we get  $((1 - \mathcal{Y}^w)^{-1})^T = \sum_{k \geq 0} \mathcal{Y}^{-kw}$ . Applying the assumption  $(\mathcal{L}^T)^{-1} B_k = (\mathcal{L}^{-1})^T B_k$  we find

$$(\tilde{\mathcal{L}}^{-1})^T B_k = \left( (\mathcal{L}^T \mathcal{L}^{-1})^{-1} \right)^T B_k = \left( \mathcal{L} (\mathcal{L}^T)^{-1} \right)^T B_k = \mathcal{L}^T \left( (\mathcal{L}^{-1})^T \right)^T B_k = \tilde{\mathcal{L}} B_k.$$

Hence

$$V_k = \left( \frac{1}{1 - \mathcal{Y}^w} + \sum_{k \geq 1} \mathcal{Y}^{-kw} \right) (1 - \tilde{\mathcal{L}}) B_k.$$

■

In applications to random walks the recursive initial conditions are usually of the form

$$\mathcal{Y}^{-t} U_k[qa, pa] = \mathcal{K} \mathcal{Y}^{-t} U_{k-1}[qa, pa] \text{ and } \mathcal{Y}^s U_k[qa, pa] = \mathcal{H} \mathcal{Y}^s U_{k-1}[qa, pa] \quad (22)$$

for all  $k = 1, 2, \dots$ , with positive integers  $t$  and  $s$ , and  $\mathcal{H}, \mathcal{K} \in \mathfrak{L}_{q/p}^\cap$ . Corollary 32 applies to this situation with  $\mathcal{L} := \mathcal{Y}^{-t} (\mathcal{R} - \mathcal{K})$  and  $\mathcal{J}^T := \mathcal{Y}^s (\mathcal{R} - \mathcal{H})$  if  $(\mathcal{R} - \mathcal{K}^T)^{-1} B_k =$

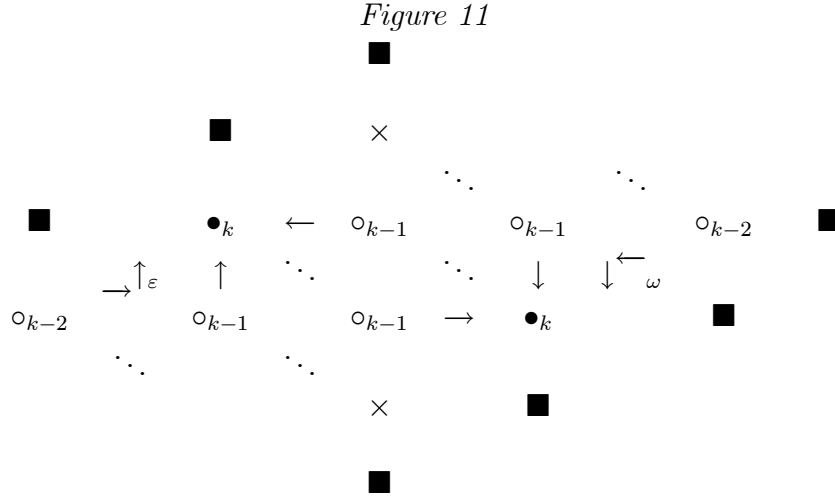
$((\mathcal{R} - \mathcal{K})^{-1})^T B_k$  and  $\frac{\mathcal{R} - \mathcal{K}^T}{\mathcal{R} - \mathcal{K}} = \frac{\mathcal{R} - \mathcal{H}^T}{\mathcal{R} - \mathcal{H}} \mathcal{Y}^v$  for some integer  $v > -2(s + t)$ . The first condition ensures that  $(\mathcal{L}^T)^{-1} B_k = (\mathcal{L}^{-1})^T B_k$ ; from the second condition follows  $(\mathcal{L}\mathcal{J})^T / (\mathcal{L}\mathcal{J}) = \mathcal{Y}^{2(s+t)+v}$ .

Hence

$$V_k = \sum_{j \in \mathbb{Z}} \mathcal{Y}^{2j(s+t)+jv} \left( 1 - \frac{\mathcal{R} - \mathcal{K}^T}{\mathcal{R} - \mathcal{K}} \mathcal{Y}^{2t} \right) B_k. \quad (23)$$

## 4.1 Left hooks inside a band

Fig. 11 shows how the left hook walks are restricted to a diagonal band.



*Points ● at the boundaries have only three predecessors ○*

The number  $H_k[m, n; \nearrow_r]$  of left hook walks from the origin to  $(n, m)$ , in  $k$  steps strictly above  $y = x - r$  and below  $y = x + l$ , is uniquely determined by the recursion  $\mathcal{R} = (1 + \mathcal{X}\mathcal{Y})(\mathcal{Y}^{-1} + \mathcal{X}^{-1})$  where  $\mathcal{X} = \mathcal{E}^{1,1}$  and  $\mathcal{Y} = \mathcal{E}^{1,-1}$ , and the recursive boundary conditions

$$\begin{aligned} H_k[n - r + 1, n; \nearrow_r] &= H_{k-1}[n - r + 2, n + 1; \nearrow_r] + H_{k-1}[n - r + 2, n - 1; \nearrow_r] \\ &= (\mathcal{X} + \mathcal{Y}) H_{k-1}[n - r + 1, n; \nearrow_r] \\ H_k[n + l - 1, n; \nearrow_r] &= H_{k-1}[n + l - 2, n - 1; \nearrow_r] + H_{k-1}[n + l - 2, n + 1; \nearrow_r] \\ &= (\mathcal{X}^{-1} + \mathcal{Y}^{-1}) H_{k-1}[n + l - 1, n; \nearrow_r] \end{aligned}$$

where  $r$  and  $l$  are positive integers. We apply formula (23) with  $p = q = 1$ ,  $t = (r - 1) / 2$ ,  $s = (l - 1) / 2$ . We will get a rather simple formula for  $H_k[m, n; \nearrow_r]$ , because the operators  $\mathcal{K} = \mathcal{X} + \mathcal{Y}$  and  $\mathcal{H} = \mathcal{X}^{-1} + \mathcal{Y}^{-1}$  have the special properties which make formula (23) applicable. First we verify that

$$\begin{aligned} (\mathcal{R} - \mathcal{K}^T)^{-1} &= (\mathcal{X}^{-1} + \mathcal{Y})^{-1} = \mathcal{X} (1 + \mathcal{X}\mathcal{Y})^{-1} = \mathcal{X} (\mathcal{Y}^{-1} + \mathcal{X}^{-1}) \mathcal{R}^{-1} \quad \text{and} \\ ((\mathcal{R} - \mathcal{K})^{-1})^T &= \left( (\mathcal{X}^{-1} + \mathcal{Y}^{-1})^{-1} \right)^T = (1 + \mathcal{X}\mathcal{Y})^T (\mathcal{R}^{-1})^T = (1 + \mathcal{X}\mathcal{Y}^{-1}) (\mathcal{R}^{-1})^T. \end{aligned}$$

We saw in (7) that  $\mathcal{R}^{-1}B_k = (\mathcal{R}^{-1})^T B_k$ ; hence  $(\mathcal{R} - \mathcal{K}^T)^{-1} B_k = ((\mathcal{R} - \mathcal{K})^{-1})^T B_k$ , as required for formula (23). Also,

$$\frac{\mathcal{R} - \mathcal{K}^T}{\mathcal{R} - \mathcal{K}} = \frac{\mathcal{X}^{-1} + \mathcal{Y}}{\mathcal{X}^{-1} + \mathcal{Y}^{-1}} = \mathcal{Y} \frac{\mathcal{X}^{-1} + \mathcal{Y}}{1 + \mathcal{X}^{-1}\mathcal{Y}} \quad \text{and} \quad \frac{\mathcal{R} - \mathcal{H}^T}{\mathcal{R} - \mathcal{H}} = \frac{\mathcal{X} + \mathcal{Y}^{-1}}{\mathcal{X} + \mathcal{Y}} = \mathcal{Y}^{-2} \frac{\mathcal{R} - \mathcal{K}^T}{\mathcal{R} - \mathcal{K}}.$$

With  $H_k[m, n; \nearrow_r] = \left(1 - \frac{\mathcal{R} - \mathcal{K}^T}{\mathcal{R} - \mathcal{K}} \mathcal{Y}^{r-1}\right) H_k[m, n]$ , we get

$$\begin{aligned} H_k[m, n; \nearrow_r] &= \sum_{i \in \mathbb{Z}} H_k[m + j(r+l), n - j(r+l); \nearrow_r] \\ &= \sum_{i \in \mathbb{Z}} \left( \binom{k}{\frac{k+m+j(r+l)}{2}} \binom{k}{\frac{k+n-j(r+l)}{2}} - \binom{k+1}{\frac{k+m+j(r+l)+r+1}{2}} \binom{k-1}{\frac{k+n-r-1-j(r+l)}{2}} \right). \end{aligned} \tag{24}$$

## References

- [1] Csáki, E. (1997). Some results for two-dimensional random walk, *Advances in Combinatorial Methods and Applications to Probability and Statistics* (Ed.: N. Balakrishnan), 115-124, Boston, Birkhäuser.
- [2] Guy, R.K., Krattenthaler, C. and Sagan, B.E. (1992). Lattice paths, reflections, and dimension changing bijections, *Ars Combinatoria*, **34**, 3-15.
- [3] Krattenthaler, C. (1997) The enumeration of lattice paths with respect to their number of turns, *Advances in Combinatorial Methods and Applications to Probability and Statistics* (Ed.: N. Balakrishnan), 19-58, Boston, Birkhäuser.
- [4] Niederhausen, H. (1998) Planar random walks inside a rectangle, *Congr. Numerantium*, **132**, 125-144.