

# Lagrange Inversion via Transforms

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## 1 Introduction

In [3] we described a technique for solving certain linear operator equations by studying the operator power series defined by the system. Essential for obtaining explicit solutions is a Lagrange inversion formula for power series with coefficients in an integral domain  $K$ . Such a formula can be found in “Recursive Matrices and Umbral Calculus” by Barnabei, Brini and Nicoletti [1]. J. F. Freeman’s [2] development of a theory of transforms of linear operators on generating functions provides us with a new interpretation of what inversion could mean in general (Theorem 1). We show how special choices of operators and generating functions then produce the desired formula.

Lagrange inversion requires imbedding of power series into Laurent series. Therefore, we have to investigate transforms in a slightly more general situation than it was done in [2]. The generalization carefully preserves all the important properties of transforms. These properties are listed in Section 4, omitting most of the straightforward proofs.

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## 2 Definitions

Let  $K$  be a commutative ring, and  $\mathbb{Z}$  the set of all integers. We denote the set of mappings  $\phi: \mathbb{Z} \rightarrow K$  by  $K^{\mathbb{Z}}$ . A Laurent series  $\phi = (k_j)_{j \in \mathbb{Z}}$  is an element in  $K^{\mathbb{Z}}$  iff there exist an  $n \in \mathbb{Z}$  such that  $k_n \neq 0$  and  $k_j = 0$  for all  $j < n$ . This  $n$  is called the *order* of  $\phi$ ,  $n = \text{ord}(\phi)$ , and the set of all Laurent series is denoted by  $K[t]$ . As indicated by the terminology, we view  $\phi$  also as a series,

$$\phi(t) = \sum_{j \geq n} k_j t^j,$$

for some formal parameter  $t$ . Thus, multiplication in  $K[t]$  is defined as multiplication of series. A multiplicative inverse (reciprocal) exists in  $K[t] \subseteq K[t]$ , the set of all Laurent series where the first non-zero term is a unit in  $K$ , i.e. has a multiplicative inverse. The terms  $k_j, j \in \mathbb{Z}$ , of a Laurent series  $\phi$  will be rarely mentioned in the next chapters (because we omit most proofs). If it becomes necessary we use the rather clumsy notation  $k_j = [\phi]_j$  for all  $j \in \mathbb{Z}$ . Replacing  $t$  by  $t^{-1}$  produces a *reverse* Laurent series. The collection of reverse Laurent series is denoted by  $K[x]$ . The degree of  $p(x) \in K[x]$  is the order of  $p(1/x)$ .  $K[x]$  is defined analogously. A sequence  $(\phi_n)_{n \in \mathbb{Z}}$  of Laurent series converges to zero,  $\phi_n \rightarrow 0$ , iff  $\text{ord}(\phi_n) \rightarrow 0$  as  $n \rightarrow \infty$ . Analogously,  $(p_m) \subseteq K[x]$  converges to zero iff  $\text{deg}(p_m) \rightarrow -\infty$  as  $m \rightarrow -\infty$ . In this way,  $K[x]$  and  $K[t]$  become homeomorphic topological  $K$ -algebras. We call a sequence  $(\phi_n) \subseteq K[t]$  *divergent*, iff  $\text{ord}(\phi_n) \rightarrow -\infty$  as  $n \rightarrow \infty$ . A sequence  $(p_m) \subseteq K[x]$  diverges iff  $\text{deg}(p_m) \rightarrow \infty$  as  $m \rightarrow -\infty$ .

### 3 Double series

Denote by  $K^{\mathbb{Z} \times \mathbb{Z}}$  the set of all double sequences  $(k_{i,j})_{i,j \in \mathbb{Z}}$  in  $K$ . Sequences of Laurent series can be viewed as elements of  $K^{\mathbb{Z} \times \mathbb{Z}}$ . If  $\phi_n \rightarrow 0$  in  $K[t]$ , a reverse Laurent series  $(p_m)$  is obtained by setting

$$[p_m]_n = [\phi_n]_m. \quad (1)$$

Now it is easy to show the following

**Lemma 1** (1) defines a bijection between those sequences of Laurent series which converge to zero, and the sequences of reverse Laurent series without divergent subsequences.

In the light of (1) we can pick elements  $f$  of  $K^{\mathbb{Z} \times \mathbb{Z}}$  which converge to zero as a sequence  $(\phi_n)$  of Laurent series as well as a sequence  $(p_m)$  of reverse Laurent series. Such an  $f$  we call a double series

$$f(x, t) = \sum_n \phi_n(t)x^n = \sum_m p_m(x)t^m.$$

The set of all such double series is denoted by  $K[x, t]$ . The lemma shows that  $K[x, t]$  contains those  $f \in K^{\mathbb{Z} \times \mathbb{Z}}$  which have no divergent subsequences, neither as Laurent nor as reverse Laurent series.

**Remark 1** Linear operators on  $K[t]$  ( and  $K[t]$  ) have their matrix representation in  $K^{\mathbb{Z} \times \mathbb{Z}}$ . The subset  $K[x, t]$  can be identified with those operators which are continuous on  $K[t]$  and  $K[t]^*$ . The latter is the space of all continuous linear functionals on  $K[t]$ , which in turn is homeomorphic to  $K[t]$ .

**Lemma 2 (Separation of Variables)** Suppose  $(q_m)$  is a sequence in  $K[x]^*$  such that  $\deg(q_m) = m$  for all  $m$ . Then for every

$$f(x, t) = \sum_n \phi_n(t)x^n$$

in  $K[x, t]$  there exists a unique sequence  $(\psi_n)$  in  $K[t]$ ,  $\psi_n \rightarrow 0$ , such that

$$f(x, t) = \sum_k q_k(x)\psi_k(t). \quad (2)$$

Moreover, if  $\text{ord}(\phi_n)$  is a strictly increasing sequence, then  $\text{ord}(\phi_n) = \text{ord}(\psi_n)$  for all  $n$ .

An analogous lemma holds if one starts with some sequence  $(\psi_n) \subseteq K[t]^*$ ,  $\text{ord}(\psi_n) = n$ , and verifies the existence of a unique  $(q_m) \subseteq K[x]$  such that (2) holds.

### 4 Transforms

A  $t$ -operator  $T$  acts on  $f(x, t) = \sum q_k(x)\psi_k(t) \in K[x, t]$  by

$$(Tf)(x, t) = \sum_k q_k(x)(T\psi_k)(t),$$

where  $T$  has to be a continuous linear operator on  $K[t]$ . Furthermore, we require that  $T$  also acts continuously on  $K[t]^*$ , where

$$(T\tilde{q})\psi = \tilde{q}(T\psi) \quad (3)$$

for all  $\tilde{q} \in K[t]^*$  and  $\psi \in K[t]$ . The set of  $t$ -operators is denoted by  $L_t$ . Analogously,  $L_x$  is defined.  $x$ - and  $t$ -operators commute.

A double series  $e(x, t) = \sum p_k(x)\phi_k(t) \in K[x, t]$  is called “full”, if  $\deg(p_k) = k = \text{ord}(\phi_k)$  for all  $k \in \mathbf{Z}$ .

**Proposition 1** Suppose  $e$  is a full double series. For every  $f \in K[x, t]$  there exists a unique  $x$ -operator  $X$  and a unique  $t$ -operator  $T$  such that

$$Xe = f = Te.$$

The  $t$ -operator in the above proposition is called the *transform* of  $X$ ,  $T = \hat{X}$ , with respect to  $e$ . Vice versa, we call  $X$  the transform of  $T$ , denoted by  $X = T^\smile$ .

Suppose  $\xi(t) \in K[t]$ . By  $M(\xi(t))$  we mean the  $t$ -operator which acts on  $t^n$  by multiplication  $M(\xi(t)) = \xi(t)t^n$  for all  $n \in \mathbf{Z}$ . We say that a  $t$ -operator  $T \neq 0$  belongs to the class  $\mathbf{MC} \subseteq L_t$ , iff

$$TM(t) = M(\gamma(t))T$$

for some  $\gamma(t) \in K[t]$  of positive order. It can easily be verified that  $T$  has to be of the form  $T = M(\xi)C(\gamma)$  for some  $\xi \in K[t]$  in order to be in  $\mathbf{MC}$ , where  $C$  is the composition  $t$ -operator

$$C(\gamma)t^n = \gamma(t)^n.$$

Let  $T_1 = M(\xi_1)C(\gamma_1)$  and  $T_2 = M(\xi_2)C(\gamma_2)$  be operators of the class  $\mathbf{MC}$ . Some straightforward calculations show that

$$T_1T_2 = M(\xi_1(t)\xi_2(\gamma_1(t)))C(\gamma_2(\gamma_1(t))). \quad (4)$$

Hence,  $\mathbf{MC}$  is a semigroup, called the *umbral semigroup* by Barnabei, Brini and Nicoletti [1, p. 554]. The umbral group is a subgroup of the umbral semigroup, containing only  $t$ -operators of the form  $T = M(\xi)C(\gamma)$ , where  $\xi$  and  $\gamma$  are from  $K[t]'$  and  $ord(\gamma) = 1$ .

All the definitions and statements we have made so far are straightforward generalizations of corresponding results in J. F. Freeman's paper [2] on *Transforms of Operators on  $K[x][[t]]$* . More such generalizations could be made, but for the purpose of a general Lagrange inversion formula we only have to verify proposition (4.9) of [2].

**Theorem 1 (Inversion Formula)** *Suppose  $e$  is a full double series and  $U = M(\phi)C(\gamma)$  an operator in the umbral group. If  $\Delta = M(t)^\smile$  w.r.t.  $e$ , then*

$$\eta(\Delta) = M(t) \text{ w.r.t. } Ue$$

where  $\eta(\gamma(t)) = t$ .

**Proof.**

$$\begin{aligned} \eta(\Delta)Ue &= U\eta(\Delta)e = UM(\eta)e = M(\phi(t)\eta(\gamma(t)))C(\gamma)e \text{ (product rule (4))} \\ &= M(t)Ue. \end{aligned}$$

■

## 5 Lagrange Inversion

In order to arrive at Lagrange inversion in its customary form we have to make some special choices in the general inversion formula above. Throughout this chapter, let

$$\begin{aligned} e &= \sum p_k(x)\phi_k(t) \text{ be a full double series,} \\ \beta &\in K[t], \xi, \gamma, \eta \in K[t]', \\ ord(\gamma) &= 1 = ord(\eta), \\ \eta(\gamma(t)) &= t, \\ U &= M(\xi)C(\gamma) \text{ a member of the umbral group.} \end{aligned}$$

As a direct consequence of the inversion formula we get

**Corollary 1**  $M(\beta)^\smile = \beta(\eta(\Delta))$  w.r.t.  $Ue$ .

By definition of transforms we have

**Corollary 2**

$$\beta(\eta(\Delta)) \sum_k x^k [\psi_k(t)]_n = \sum_k p_k(x) [\beta(t)\xi(t)\phi_k(\gamma(t))]_n$$

for all  $n \in \mathbf{Z}$ , where  $(\psi_k) \subseteq K[t]$  is defined by  $(Ue)(x, t) = \sum_k x^k \psi_k(t)$ .

Next, we make a very special choice for  $e$ ,

$$e(x, t) = \sum_k x^k t^k.$$

Noting that now  $\Delta x^n = x^{n-1}$ , we write  $\eta(1/x)$  for  $\eta(\Delta)$ .

**Corollary 3**

$$\beta(\eta(1/x)) \sum_k x^k [\xi(t)\gamma(t)^k]_n = \sum_k x^k [\beta(t)\xi(t)\gamma(t)^k]_n$$

for all  $n \in \mathbf{Z}$ .

Finally, we let  $n = -1$  and  $\xi(t) = D\gamma(t)$ , the formal derivative of  $\gamma$ .

**Corollary 4 (Lagrange Inversion Formula)**

$$\beta(\eta(1/x)) = \sum_k x^{1-k} \text{Res}[\beta(t)\gamma(t)^{-k} D\gamma(t)].$$

## References

- [1] Barnabei, M., Brini, A. and Nicoletti, G. (1982). Recursive matrices and Umbral Calculus. *J. Alg.* **75**, 546-573
- [2] Freeman, J. M. (1985). Transforms of operators on  $K[x][t]$ . *Congr. Numerantium* **48**, 115-132.
- [3] Niederhausen, H. (1985). A formula for explicit solutions of certain linear recursions on polynomial sequences. *Congr. Numerantium* **49**, 87-98.