Counting intersecting weighted pairs of lattice paths using transforms of operators

Heinrich Niederhausen
Florida Atlantic University, Boca Raton, FL 33431

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Abstract

Transforms of linear operators on bivariate generating functions can be used for constructing explicte solutions to certain generalized $q-$difference equations. The method is applied to counting intersecting pairs of lattice paths with weighted turns, a refinement of the q-Narayana numbers.

1 Introduction

The system of difference equations

$$h_a(m) - h_a(m-1) = h_{a-1}(m)$$

has a famous solution among the polynomial sequences $\{h_n(x) \mid h_n \in \mathbb{R}[x]\}$: The "basic sequence" $h_a(m) = \binom{a+m}{a}$. One way of visualizing the recurrence and this solution is by looking at the $\binom{a+m}{a}$ lattice paths taking horizontal and vertical unit steps (standard paths) from the origin to the lattice point (a, m). This may not be the most obvious interpretation of the binomial coefficient, but it generalizes easily to an interpretation of the q-binomial (Gaussian) coefficient $\begin{bmatrix} a+m \\ a \end{bmatrix}_q$, the equally famous solution to the q-difference equation

$$h_a(m) - h_a(m-1) = q^m h_{a-1}(m).$$

Here we think of $\begin{bmatrix} a+m \\ a \end{bmatrix}_q$ as the generating function in the formal variable q of the area (= total length of columns) underneath the paths. Of course, the a columns are exactly the parts ($\leq m$) in the partition of the area, giving the usual interpretation of $\begin{bmatrix} a+m \\ a \end{bmatrix}_q$ as in[1, Theorem 3.1], for example. The standard lattice path becomes a Ferrers graph in this case. Yet for the purpose of this paper we generate $\begin{bmatrix} a+m \\ a \end{bmatrix}_q$ in a different way by sorting the paths according to the sum of all their *left turn* coordinates. A standard path makes a left turn if a vertical step follows a horizontal step: $\underset{\rightarrow\circ}{\uparrow}$. There is a beautiful "Log Cabin" bijection proving the equality of both generating functions (Section 5).

Going one step further, we give to a path with l left turns the weight

$$\mu^l p^{\text{sum of all } l \text{ left turn } x\text{-coordinates}} q^{\text{sum of all } l \text{ left turn } y\text{-coordinates}}.$$

The sum of such weights satisfies the recurrence which will illustrate the usefulness of the transform methods applied in this paper:

$$h_a(m) - h_a(m-1) = h_{a-1}(m) + (\mu p^a q^m - 1)\, h_{a-1}(m-1).$$

Theorem 2 will show in a very general way how to find explicit solutions $\{h_n(x)\}$ for this and related recursions. Such solutions are polynomials in the variable q^x, where the coefficients are rational functions in p and q. We write $\mathbb{P}[q^x]$ for the algebra of these polynomials. The language of transforms of operators, developed by J. Freeman in [2] and Verde Star in [9], provides an extremely easy proof for Theorem 2.

In section 4 we investigate a bivariate path counting problem. The above recurrences become partial recurrence relations for the variables m and n. The "Narayana numbers"

$$\binom{a+m}{a}\binom{d+n}{n} - \binom{a+m}{a-1}\binom{d+n}{d-1}$$

are an example, counting the number all pairs of nonintersecting paths from $(0,1)$ to $(a, m+1)$, and from $(1,0)$ to $(n+1, d)$. If we weigh these nonintersecting paths by the sum of all left turn coordinates of the upper path plus the right turn coordinates of the lower path, we get the q-"Narayana numbers"

$$\begin{bmatrix} a+m \\ a \end{bmatrix}_q \begin{bmatrix} d+n \\ n \end{bmatrix}_q - q^2 \begin{bmatrix} a+m \\ a-1 \end{bmatrix}_q \begin{bmatrix} d+n \\ d-1 \end{bmatrix}_q .$$

The formulas are written in a way to explain why we are now switching from nonintersecting to intersecting pairs.

It is natural to ask how much the intersecting pairs with exactly l left turns in the upper and r right turns in the lower path contribute to the total. Such "Refinement of the Narayana numbers" were studied by R.A. Sulanke [8] for the case $q = 1$, using a bijection. In another approach [6], the generating function

$$\sum_l \mu^l \sum_r v^r \left|\{\text{intersecting pairs with } l \text{ left and } r \text{ right turns}\}\right|$$

has been derived. C. Krattenthaler and R. Sulanke solved the q-case in [4] by a rotation method. Analogously to the $q = 1$ case we want to show in this paper how the q-case can be approached via recurrence relations and initial conditions.

2 The difference equation for weighted turns

A path starting at $(0,1)$ is uniquely determined by its end point $(a, m+1)$, say, and its sequence $(\xi_1, \eta_1), ..., (\xi_l, \eta_l)$ of left turning points

$$1 \leq \xi_1 < ... < \xi_l \leq a, \quad 1 \leq \eta_1 < ... < \eta_l \leq m. \tag{1}$$

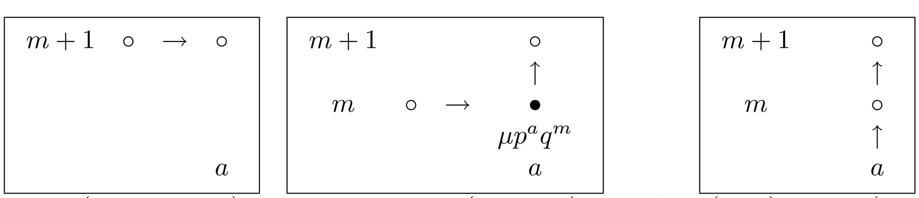
We give different weights, p and q, to the first and second component of the turning point: $D_{\mu pq}(a, m+1; l)$ denotes the total weight of all lattice paths that start at $(0,1)$, and reach the point $(a, m+1)$ with exactly l left turns, where a left turn at (ξ_i, η_i) gets the weight $\mu p^{\xi_i} q^{\eta_i}$. The total weight of a path taking left turns as specified in (1) is therefore $\mu^l p^{\xi_1+\dots+\xi_l} q^{\eta_1+\dots+\eta_l}$. This weight function is equivalent to weighing by the major and lesser index as described in [4] (choose $\mu = \tilde{\mu}/(pq)$). If $p = q = 1$ we get in the unrestricted case

$$D_{\mu,1,1}(a, m+1; l) = \binom{a}{l}\binom{m}{l}\mu^l.$$

$D_{\mu pq}(a, m+1) := \sum_{l \geq 0} D_{\mu pq}(a, m+1; l)$ is the number of paths with *μpq-weighted left turns* that start at $(0,1)$ and reach $(a, m+1)$.

The following recurrence holds for $D_{\mu pq}(a, m+1)$:

$D_{\mu pq}(a, m+1) =$



$D_{\mu pq}(a-1, m+1) \quad + \mu p^a q^m D_{\mu pq}(a-1, m) \quad + D_{\mu pq}(a, m) - D_{\mu pq}(a-1, m)$

Figure 1

In the forwards difference notation,

$$\Delta_m D_{\mu pq}(a,m) = (\mu p^a q^m - 1) D_{\mu pq}(a-1,m) + D_{\mu pq}(a-1,m+1). \tag{2}$$

It is important to realize that this recurrence allows us to find $D_{\mu pq}(a,m+1)$ even if some boundary values are given. For example, if the paths are restricted as in the figure at the right, the numbers $D_{\mu pq}(a,m+1)$ can still be recursively calculated from the above formula. Underneath a monotone increasing boundary, $D_{\mu pq}(a,m+1)$ takes the values of a polynomial $d_a(x;\mu,p,q)$ (in x if $q=1$, and in q^x if $q \neq 1$) with $\deg d_a = a$. However, if there are conditions that forbid reaching $(a,m+1)$ from below, the recurrence relation may break down.

m					0	•	•
6				0	•	•	•
5				0	•	•	•
4		0	0	•	•	•	•
3	0	•	•	•	•	•	•
2	0	•	•	•	•	•	•
1	•	•	•	•	•	•	•
$a:$	0	1	2	3	4	5	6

Figure 2

In the unrestricted case suitable initial values are

$$D_{\mu pq}(a,1) = 1 \text{ for all } a = 0,1,\dots \tag{3}$$

The polynomials $d_a(x)$ in q^x that represent the weighted counts, $d_a(m) = D_{\mu pq}(a,m+1)$ for all nonnegative integers m, have the initial values

$$d_a(0) = 1 \text{ for all } a = 0,1,\dots$$

and follow the recurrence

$$\Delta_m d_a(m-1) = (\mu p^a q^m - 1) d_{a-1}(m-1) + d_{a-1}(m). \tag{4}$$

3 pq-Sheffer sequences

Let $\{h_n(x)\}$ be any polynomial sequence in $\mathbb{P}[p^x]$ that follows the recurrence relation (4)

$$h_n(x) - h_n(x-1) = h_{n-1}(x) + (\mu p^n q^x - 1)\, h_{n-1}(x-1). \tag{5}$$

Remark 1 *Suppose $\{g_n\}$ is another polynomial sequence in $\mathbb{P}[p^x]$ that follows the same recurrence relation. If $g_n(x_n) = h_n(x_n)$ for some sequence $x_0, x_1, \dots$, then $g_n = h_n$ for all $n \in \mathbb{N}_0$.*

The generating function $h(x,t) := \sum_{n\geq 0} h_n(x) t^n$ of this sequence must satisfy the functional relation

$$(1-t)\,(h(x,t) - h(x-1,t)) = \mu tpq^x h(x-1,pt)$$

or

$$\Delta_x h(x,t) = q^{x+1} \tfrac{\mu tp}{1-t} h(x,pt), \tag{6}$$

where the forward difference operator Δ_x acts on the x-variable.

3.1 Some notation and an easy theorem about transforms

We study the relation (6) as an application of the concept of *transforms* (see [2]) or duals (see [9]) of operators. More details can be found in those references. Let $\{e_n(x)\}$ be a polynomial sequence (i.e. $\deg(e_n) = n$) such that $e_n(0) = \delta_{n,0}$ for all $n \in \mathbb{N}_0$. If $\{a_n(x)\}$ denotes an arbitrary polynomial sequence, then there exists a unique sequence $\{\alpha_n(t)\}$ of formal power series, $ord(\alpha_n) = n$, such that

$$\sum_{n\geq 0} a_n(x) t^n = \sum_{n\geq 0} e_n(x) \alpha_n(t).$$

We call $\{e_n(x)\}$ the *reference basis*. The sequences $\{\alpha_n(t)\}$ and $\{a_n(x)\}$ are *e-images* of each other. For example, $\{a_n(x)\}$ is a Sheffer sequence in the sense of the Umbral Calculus [7] iff $e_n(x) =$

$x^n/n!$ and the e-image of $\{a_n(x)\}$ is of the form $\{\rho(t)\beta(x)^n\}$, where $ord(\rho) = 0$ and $ord(\beta) = 1$. A *qp-Sheffer sequence* $\{a_n(x)\}$ *for* β *and* ρ is a polynomial sequence in $\mathbb{P}[q^x]$ with e-image $\left\{\beta_{[n]}(t)\rho(p^n t)\right\}$,

$$\sum_{n\geq 0} a_n(x)t^n = \sum_{n\geq 0} e_n(x)\beta_{[n]}(t)\rho(p^n t), \tag{7}$$

where

- $\rho(t)$ is a power series in t of order 0,
- $e_n(x)$ is a reference basis in $\mathbb{P}[q^x]$,
- $\beta_{[n]}(t) = \prod_{i=0}^{n-1} \beta(p^i t)$, where $\beta(t)$ is a power series of order 1, and $\beta_{[0]} = 1$.

Note that

$$\beta(t/p)\beta_{[n]}(t) = \prod_{i=-1}^{n-1} \beta(p^i t) = \prod_{i=0}^{n} \beta(p^i t/p) = \beta_{[n+1]}(t/p) \tag{8}$$

(see Garsia and Joni [3] for more details on $\beta_{[n]}(t)$). If necessary we indicate the dependence on parameters like p and q by writing $a_n(x; q, p)$, $e_n(x; q)$, and $\beta_{[n;p]}(t)$. We need the following t- and x-operators, given by their action on some basis.

- $M(\gamma) : t^n \mapsto \gamma(t)t^n$ is the *multiplication* t-operator for any power series $\gamma(t)$.
- $Xe_n(x) = e_{n-1}(x)$ for all nonnegative integers n. X is the e-dual (or x-transform with respect to e) of $M(t)$, because of

  $$Xe(x,t) = \sum_{n\geq 1} e_{n-1}(x)t^n = t\sum_{n\geq 0} e_n(x)t^n = M(t)e(x,t).$$

  We write

  $$X = \widehat{M(t)}^{e} \quad \text{and} \quad \widetilde{X}^{e} = M(t).$$

- $B : t^n \mapsto \beta_{[n]}(t)\rho(p^n t)$ for all nonnegative integers n is the t-operator that represents the e-image of the given polynomial sequence.
- $P : t^n \mapsto (pt)^n$ for all nonnegative integers n. It follows from (8) that

  $$\beta(t/p)\rho(p^n t)\beta_{[n]}(t) = P^{-1}\left(\rho(p^{n+1}t)\beta_{[n+1]}(t)\right),$$

  or

  $$M(P^{-1}\beta)B = P^{-1}BM(t). \tag{9}$$

  Obviously, $\widehat{P}^{a} a_n(x) = p^n a_n(x)$.

Theorem 2 *Let* $\{a_n(x)\}$ *be a polynomial sequence with generating function*

$$a(x,t) = \sum_{n\geq 0} a_n(x)t^n = \sum_{n\geq 0} e_n(x)\beta_{[n]}(t)\rho(p^n t) = Be(x,t)$$

and recurrence $A : a_n \mapsto a_{n-1}$ *for all nonnegative integers* n, *then, independent of* ρ,

$$M(\beta)P = \widetilde{X}^{a} \text{ and } Aa(x,t) = p\beta^{-1}(P^{-1}X)a(x,t)$$

Proof.

$$\begin{aligned} P^{-1}Xa(x,t) &= P^{-1}XBe(x,t) = P^{-1}BXe(x,t) \quad (x\text{- and } t\text{-transforms commute}) \\ &= P^{-1}B\,\widetilde{X}^{a} e(x,t) = P^{-1}BM(t)e(x,t) = M(P^{-1}\beta)Be(x,t) \\ &= M(P^{-1}\beta)a(x,t) \quad (\text{see } (9)). \end{aligned}$$

From $t/p = \beta^{-1}(P^{-1}\beta)$ follows $M(t/p)a(x,t) = \beta^{-1}(P^{-1}X)a(x,t)$. But

$$M(t)a(x,t) = \sum_{n\geq 0} a_n(x)t^{n+1} = \sum_{n\geq 1} a_{n-1}(x)t^n = Aa(x,t),$$

and the theorem follows. ∎

3.2 An application to weighted paths

We want to explore the class of polynomial sequences whose generating function $h(x,t)$ satisfies the relation $\Delta_x h(x,t) = q^{x+1}\frac{\mu t p}{1-t} P h(x,t)$ (see (6)). Theorem 2 tells us that $\beta(t)$ will be $\mu pt/(1-p)$ if we can find a reference sequence $\{e_n\}$ such that $X = q^{-x-1}\Delta_x$. We know for $q=1$ that $\Delta_x \binom{x}{n} = \binom{x}{n-1}$. Therefore we look for $\{e_n\}$ among the q-binomial coefficients

$$\begin{bmatrix} x \\ n \end{bmatrix}_q = \frac{(1-q^x)(1-q^{x-1})\cdot \ldots \cdot (1-q^{x-n+1})}{(1-q^n)(1-q^{n-1})\cdot \ldots \cdot (1-q)} = \frac{(q^{x-n+1};q)_n}{(q;q)_n}.$$

It is easy to check that $e_n(x) = q^{\binom{n+1}{2}} \begin{bmatrix} x \\ n \end{bmatrix}_q \in \mathbb{P}[q^x]$ is the right choice:

$$X e_n(x) = q^{-x-1+\binom{n+1}{2}} \left(\begin{bmatrix} x+1 \\ n \end{bmatrix}_q - \begin{bmatrix} x \\ n \end{bmatrix}_q \right) = q^{\binom{n+1}{2}-n} \begin{bmatrix} x \\ n-1 \end{bmatrix}_q = e_{n-1}(x).$$

The functional relation (6) for weighted paths can now be written as

$$X h(x,t) = M(\beta(t)) P h(x,t)$$

if we define $\beta(t) = \mu t p/(1-t)$. Thus

$$\beta_{[n]}(t) = \mu^n p^{\binom{n+1}{2}} t^n/(t;p)_n.$$

Theorem 2 also gives an 'explicit' expression for the recurrence $H^{\mu pq}$ of pq-Sheffer sequences $\{h_n\}$ satisfying (6). From

$$H^{\mu pq} = p\beta^{-1}(P^{-1}X) = p\sum_{m>0} (\mu p)^{-m} X^m \; {}^h\widehat{P^{-m}}$$

follows

$$h_{n-1}(x) = H^{\mu pq} h_n(x) = \sum_{m>0} \mu^{-m} p^{1-m(1+n)} \left(q^{-x-1}\Delta_x\right)^m h_n(x).$$

Remark 3 *Only the constants are in the kernel of $H^{\mu pq}$. Thus the initial value problem*

$$H^{\mu pq} h_n = h_{n-1},\; h_n(x_n) = y_n \textit{ for all } n \in \mathbb{N}_0$$

has a unique solution. This shows the same uniqueness that we mentioned in Remark 1.

The following family of initial values is well suited for $H^{\mu pq}$. Suppose, $\rho(t) = (t;p)_k^{-1}$, where $k = 0, 1, \ldots$ is a given parameter. The associated Sheffer sequence $\{h_n\}$ for $H^{\mu pq}$ has the generating function

$$\begin{aligned} h(x,t;\mu,p,q) &= \textstyle\sum_l h_l(x;\mu,p,q)t^l = \sum_{l\geq 0} (pq)^{\binom{l+1}{2}} \begin{bmatrix} x \\ l \end{bmatrix}_q \mu^l t^l \rho(p^l t)/(t;p)_l \\ &= \textstyle\sum_{l\geq 0} (pq)^{\binom{l+1}{2}} \begin{bmatrix} x \\ l \end{bmatrix}_q \mu^l t^l \frac{1}{(tp^l;p)_k}\frac{1}{(t;p)_l} = \sum_{l\geq 0} (pq)^{\binom{l+1}{2}} \begin{bmatrix} x \\ l \end{bmatrix}_q \mu^l t^l/(t;p)_{l+k} \\ &= \textstyle\sum_{l\geq 0} (qp)^{\binom{l+1}{2}} \begin{bmatrix} x \\ l \end{bmatrix}_q \mu^l \sum_{j\geq 0} \begin{bmatrix} l+j+k-1 \\ j \end{bmatrix}_p t^{l+j} = \sum_{n\geq 0} t^n \sum_{l=0}^n \mu^l (qp)^{\binom{l+1}{2}} \begin{bmatrix} x \\ l \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ l+k-1 \end{bmatrix}_p. \end{aligned}$$

So we found

$$h_n(x;\mu,p,q) = \textstyle\sum_{l=0}^n \mu^l (qp)^{\binom{l+1}{2}} \begin{bmatrix} x \\ l \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ l+k-1 \end{bmatrix}_p. \qquad (10)$$

3.2.1 $k = 0$: The basic sequence

We obtain the pq-basic sequence $\{b_n(x)\}$ for $H^{\mu pq}$ if we choose $\rho(t) = 1$, i.e. $k = 0$ in (10): $b_0(x) = 1$ and

$$b_n(x) = \sum_{l=1}^n \begin{bmatrix} x \\ l \end{bmatrix}_q \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_p (pq)^{\binom{l+1}{2}} \mu^l \text{ when } n \geq 1.$$

If in addition $q = p$ and $\mu = q^{-2}$, then $b(x,t) = \sum_{k\geq 0} \begin{bmatrix} x \\ k \end{bmatrix}_q \frac{t^k q^{(k-1)k}}{(t)_k} = 1/(t;q)_x$.

3.2.2 $k = 1$: Unrestricted weighted paths

In the unrestricted enumeration of weighted paths we determine $\rho(t)$ from the initial values

$$D_{\mu pq}(a,1) = d_a(0) = 1 \text{ for all } a = 0,1,\ldots$$

(see (3)). Hence

$$\tfrac{1}{1-t} = \sum_{n\geq 0} d_n(0)t^n = \sum_{n\geq 0} (pq)^{\binom{n+1}{2}} \left[\begin{smallmatrix}0\\n\end{smallmatrix}\right]_q \mu^n t^n \rho(p^n t)/(t;p)_n = \rho(t).$$

This is the case $k = 1$ in (10), giving

$$D_{\mu pq}(a, m+1) = d_a(m) = \sum_{l=0}^{a} \begin{bmatrix}m\\l\end{bmatrix}_q \begin{bmatrix}a\\l\end{bmatrix}_p (pq)^{\binom{l+1}{2}} \mu^l \quad \text{for all } a, m \geq 0, \tag{11}$$

a well-known result, because $q^{\binom{l+1}{2}} \left[\begin{smallmatrix}n\\l\end{smallmatrix}\right]_q = \sum_{1\leq \eta_1 < \ldots < \eta_l \leq n} q^{\eta_1+\ldots+\eta_l}$ is the q-weighted count of the number of l-subsets (= η-coordinates of left turns) of $\{1,...,n\}$ (see [1]).

3.2.3 $k = 3$: Intersecting weighted paths

In subsection 4.3 the power series $\rho(t) = 1/(t;p)_3$ will be of some importance. In this case we get the polynomials

$$h_n(x;\mu,p,q) = \sum_{l=0}^{n} \mu^l (qp)^{\binom{l+1}{2}} \left[\begin{smallmatrix}x\\l\end{smallmatrix}\right]_q \left[\begin{smallmatrix}n+2\\l+2\end{smallmatrix}\right]_p. \tag{12}$$

4 Intersecting weighted lattice paths

We approach the enumeration of weighted intersecting pairs of lattice paths as an initial value problem in the framework of bivariate pq-Sheffer sequences. Special attention must be given to the question where such polynomial solutions exist. The end points of the two paths must stay apart far enough for guaranteeing an unhindered operation of the partial recurrence relations (14) and (15) which determine the class of polynomials supporting the weighted counts. A second recurrence in Lemma 5 determines the initial values of the system. We start out using different weights for the upper and lower paths. We cannot find a "closed form" solution for this very general case, but a numerical solution can be obtained from the recurrence relations.

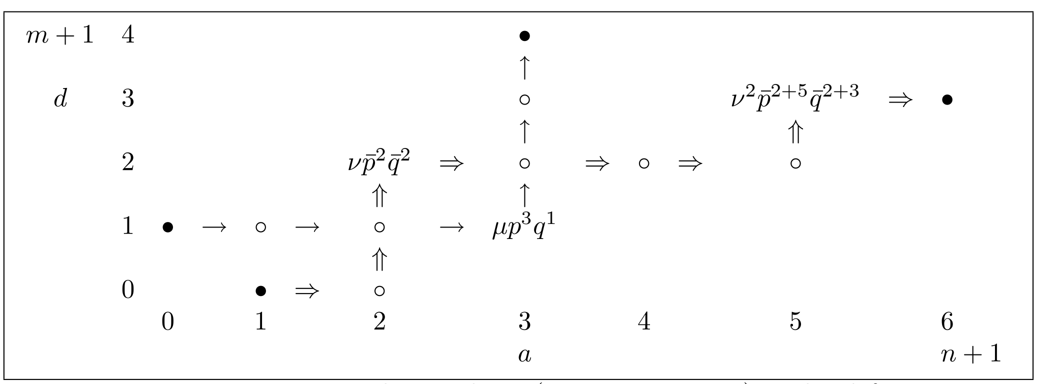


Figure 3: Intersecting paths, reaching $(a = 3,\ m+1 = 4)$ with 1 left turn (weight $\mu p^3 q^1$), and $(n+1 = 6,\ d = 3)$ with 2 right turns (weight $\nu^2 \bar{p}^7 \bar{q}^5$).

Let $I(a, m+1; n+1, d)$ denote the number of weighted *intersecting* pairs of paths reaching $(a, m+1)$ from $(0,1)$ with μpq-weighted left turns, and $(n+1, d)$ from $(1,0)$ with $\nu\bar{p}\bar{q}$-weighted right turns, where $0 \leq a \leq n$ and $0 \leq d \leq m$. Because of the order among starting and end points we can speak of an upper path and a lower path. The number of *nonintersecting* paths $N(a, m+1; n+1, d)$ plus $I(a, m+1; n+1, d)$ sum up to the number of all 'unrestricted' paths,

$$A(a, m+1; n+1, d) = \sum_{l,r\geq 0} \mu^l (pq)^{l(l+1)/2} \begin{bmatrix}a\\l\end{bmatrix}_p \begin{bmatrix}m\\l\end{bmatrix}_q \begin{bmatrix}n\\r\end{bmatrix}_{\bar{p}} \begin{bmatrix}d\\r\end{bmatrix}_{\bar{q}} \nu^r (\bar{p}\bar{q})^{r(r+1)/2},$$

(see (11)). If we interchange the weight μ with ν, and the weight p with $\bar{q}$, $\bar{p}$ with q, we call the resulting weighted counts $\bar{I}$. It is easy to see by reflection along the diagonal that

$$I(a, m+1; n+1, d) = \bar{I}(d, n+1; m+1, a) \tag{13}$$

4.1 Combinatorial information

Our approach requires that we extract two pieces of information from the combinatorial problem: Recurrences and initial values. If we keep the two end points far enough apart and one of them fix, the weighted counts $I(a, m+1; n+1, d)$ follow the same recurrence relation (2) as $D_{\mu pq}(a, n+1)$ and $D_{\nu\bar{p}\bar{q}}(d, m+1)$. Define $I(a, m+1; n+1, d) = 0$ for a or d equal to 0 or -1. Then

$$\begin{aligned} I(a, m+1; n+1, d) &= I(a-1, m+1; n+1, d) \\ &+ (\mu p^a q^m - 1)\, I(a-1, m; n+1, d) + I(a, m; n+1, d) \end{aligned} \tag{14}$$

for all $m > d \geq 0,\ n > 0$. The symmetry (13) implies that

$$\begin{aligned} I(a, m+1; n+1, d) &= I(a, m+1; n+1, d-1) \\ &+ \left(\nu \bar{q}^d \bar{p}^n - 1\right) I(a, m+1; n, d-1) + I(a, m+1; n, d) \end{aligned} \tag{15}$$

for all $n > a \geq 0,\ m > 0$. Obvious initial values are

$$I(1, m+1; n+1, 1) = \mu\nu pq\bar{p}\bar{q} \tag{16}$$

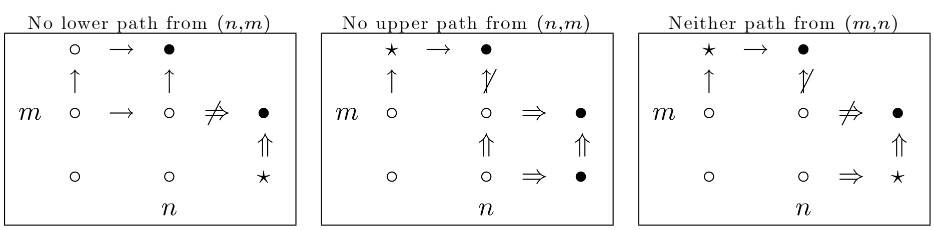
for all positive integers m and n.

Neither of the two above recurrences can be applied if $m = d$ and $n = a$. In this case the following lemma will be needed to calculate a new initial value $I(n, m+1; n+1, m)$ from previously found numbers.

Lemma 4 *Let m and n be nonnegative integers. The weighted number of nonintersecting lattice paths ending at the neighboring points $(n, m+1)$ and $(n+1, m)$ satisfies the recursion*

$$\begin{aligned} N(n, m+1; n+1, m) &= N(n, m+1; n+1, m-1) \\ &+ N(n-1, m+1; n+1, m) - N(n-1, m+1; n+1, m-1) \end{aligned}$$

Proof. At most one of the two paths can go through the point (n, m). Hence
$N(n, m+1; n+1, m) =$



$$N(n, m+1; n+1, m-1) + N(n-1, m+1; n+1, m) - N(n-1, m+1; n+1, m-1)$$

Figure 4

■

Note that none of the 6 weights enters this recurrence. But it is essential for this convenience, that the upper path has its left turns weighted, and the lower path its right turns. We obtain a recurrence for intersecting pairs if we use that $I(a, m+1; n+1, d) + N(a, m+1; n+1, d) = A(a, m+1; n+1, d)$. Collecting equal powers of $\mu^l \nu^r$ gives the following result.

Corollary 5 *Let m and n be nonnegative integers. The weighted number of intersecting lattice paths ending at the neighboring points $(n, m+1)$ and $(n+1, m)$ satisfies the recursion*

$$\begin{aligned} I(n, m+1; n+1, m) &= I(n-1, m+1; n+1, m) \\ &+ I(n, m+1; n+1, m-1) - I(n-1, m+1; n+1, m-1) \\ &+ \textstyle\sum_{l,r\geq 1} \mu^l (pq)^{l(l+1)/2} \nu^r (\bar{p}\bar{q})^{r(r+1)/2} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_p \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_{\bar{p}} \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_{\bar{q}} p^{n-l} \bar{q}^{m-r}. \end{aligned}$$

Proof.

$$\begin{aligned}
& A(n,m+1;n+1,m) - I(n,m+1;n+1,m) \\
= \;& N(n,m+1;n+1,m) \\
= \;& N(n,m+1;n+1,m-1) + N(n-1,m+1;n+1,m) - N(n-1,m+1;n+1,m-1) \\
= \;& A(n,m+1;n+1,m-1) - I(n,m+1;n+1,m-1) \\
& +A(n-1,m+1;n+1,m) - I(n-1,m+1;n+1,m) \\
& -A(n-1,m+1;n+1,m-1) + I(n-1,m+1;n+1,m-1)
\end{aligned}$$

The expression $A(n,m+1;n+1,m) - A(n-1,m+1;n+1,m) - A(n,m+1;n+1,m-1) + A(n-1,m+1;n+1,m-1)$ simplifies a little:

$$\begin{aligned}
& A(n,m+1;n+1,m) - A(n-1,m+1;n+1,m) \\
& -A(n,m+1;n+1,m-1) + A(n-1,m+1;n+1,m-1) \\
& = \textstyle\sum_{l,r\geq 0} \mu^l (pq)^{l(l+1)/2} \nu^r (\bar p \bar q)^{r(r+1)/2} \times \\
& \left[\begin{bmatrix} n \\ l \end{bmatrix}_p \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_{\bar p} \begin{bmatrix} m \\ r \end{bmatrix}_{\bar q} - \begin{bmatrix} n-1 \\ l \end{bmatrix}_p \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_{\bar p} \begin{bmatrix} m \\ r \end{bmatrix}_{\bar q} - \begin{bmatrix} n \\ l \end{bmatrix}_p \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_{\bar p} \begin{bmatrix} m-1 \\ r \end{bmatrix}_{\bar q} + \begin{bmatrix} n-1 \\ l \end{bmatrix}_p \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_{\bar p} \begin{bmatrix} m-1 \\ r \end{bmatrix}_{\bar q} \right] \\
& = \textstyle\sum_{l,r\geq 0} \mu^l (pq)^{l(l+1)/2} \nu^r (\bar p \bar q)^{r(r+1)/2} \left[p^{n-l} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_p \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_{\bar p} \begin{bmatrix} m \\ r \end{bmatrix}_{\bar q} - p^{n-l} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_p \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_{\bar p} \begin{bmatrix} m-1 \\ r \end{bmatrix}_{\bar q} \right] \\
& = \textstyle\sum_{l,r\geq 1} \mu^l (pq)^{l(l+1)/2} \nu^r (\bar p \bar q)^{r(r+1)/2} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_p \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_{\bar p} \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_{\bar q} p^{n-l} \bar q^{m-r} \;\blacksquare
\end{aligned}$$

4.2 Uniqueness

We claim that the initial values (16), Corollary 5 and the recurrences (14),(15) uniquely define the numbers $I(a,m+1;n+1,d)$ for all $m \geq d \geq 0$ and $n \geq a \geq 0$. This follows easily by induction over $k := a+m+n+d$. If $k = 2$ we obtain $I(a,m+1;n+1,d)$ from the initial values. Suppose $a+m+n+d > k \geq 2$. If $n = 0$ or $m = 0$ then $a = 0$ or $d = 0$, and thus $I(a,m+1;n+1,d) = 0$. Hence we can assume that both, n and m, are positive. If $m > d \geq 0$, the recurrence (14) reduces the sum of the parameters below or equal to k. If $n > a \geq 0$, recurrence (15) does the same. They can be applied in any order, until $m = d$ and $n = a$. In that case, Corollary 5 reduces the sum of the parameters below or equal to k.

4.3 Closed forms in special cases

It is of course straight forward to calculate $I(1,m+1;2,d)$ and $I(a,2;n+1,1)$ from Fig. 5.

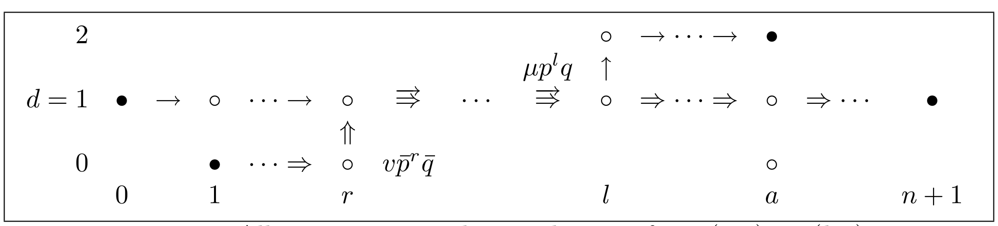


Figure 5: All intersecting paths, overlapping from $(r,1)$ to $(l,1)$.

Lemma 6 *Let $m \geq d \geq 0$ and $n \geq a \geq 0$. Then*

$$\begin{aligned}
I(1,m+1;2,d) &= \tfrac{\mu\nu pq\bar p\bar q}{1-q} \left(\tfrac{1-\bar q^d}{1-\bar q} - q \tfrac{1-(q\bar q)^a}{1-q\bar q} \right) \\
I(a,2;n+1,1) &= \tfrac{\mu\nu pq\bar p\bar q}{1-\bar p} \left(\tfrac{1-p^a}{1-p} - \bar p \tfrac{1-(p\bar p)^a}{1-p\bar p} \right).
\end{aligned}$$

If $p = \bar p$ and $q = \bar q$ then

$$I(1,m+1;2,d) = \mu\nu\,(pq)^2 \begin{bmatrix} d+1 \\ 2 \end{bmatrix}_q \quad \textit{and} \quad I(a,2;n+1,1) = \mu\nu\,(pq)^2 \begin{bmatrix} a+1 \\ 2 \end{bmatrix}_p .$$

Proof. We show the second identity (the first follows in the same way).

$$\begin{aligned} I(a,2;n+1,1) &= \textstyle\sum_{l=1}^{a} \mu p^l q \sum_{r=1}^{l} v\bar{p}^r \bar{q} = \sum_{l=1}^{a} \mu p^l q \bar{q}\bar{p} v \frac{1-\bar{p}^l}{1-\bar{p}} \\ &= \frac{\mu v p q \bar{p}\bar{q}}{1-\bar{p}} \left(\frac{1-p^a}{1-p} - \bar{p}\frac{1-(p\bar{p})^a}{1-p\bar{p}} \right) \end{aligned}$$

If $p=\bar{p}$ we get

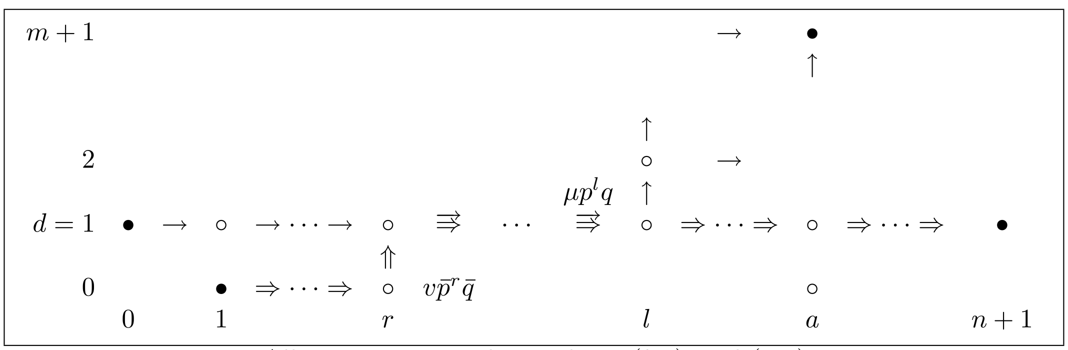
$$\begin{aligned} I(a,2;n+1,1) &= \frac{\mu v p^2 q\bar{q}}{1-p} \left(\frac{1-p^a}{1-p} - p\frac{1-p^{2a}}{1-p^2} \right) \\ &= \frac{\mu v p^2 q\bar{q}}{1-p}(1-p^a) \left(\frac{1+p}{1-p^2} - p\frac{1+p^a}{1-p^2} \right) = \mu v p^2 q\bar{q} \begin{bmatrix} a+1 \\ 2 \end{bmatrix}_p. \end{aligned}$$

■

Similar elementary arguments show that

$$\begin{aligned} I(1,m+1;n+1,d) &= pq\mu \textstyle\sum_{r=1}^{d} \nu^r \bar{p}^r \,(\bar{p}q)^{\binom{r}{2}} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{\bar{p}} \sum_{i\geq 1} \begin{bmatrix} d-i \\ r-1 \end{bmatrix}_{\bar{q}} \bar{q}^{ir} \begin{bmatrix} i \\ i-1 \end{bmatrix}_q \\ I(a,m+1;n+1,1) &= \bar{q}\bar{p}\nu \textstyle\sum_{l=1}^{a} \mu^l q^l \,(pq)^{\binom{l}{2}} \begin{bmatrix} m-1 \\ l-1 \end{bmatrix}_q \sum_{i\geq 1} \begin{bmatrix} a-i \\ l-1 \end{bmatrix}_p p^{il} \begin{bmatrix} i \\ i-1 \end{bmatrix}_{\bar{p}} \end{aligned} \tag{17}$$

Proof.
We make use of the fact that we can count unrestricted paths from $(l,2)$ to $(a,m+1)$ by translating them to a path from $(0,1)$ to $(a-l,m)$. The translated path needs a weight adjustment at the left turns (ξ,η). Instead of $p^\xi q^\eta$ we have to give the weight $p^{\xi+l}q^{\eta+1}$
$\sum_{j=0}^{a-l} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \begin{bmatrix} a-l \\ j \end{bmatrix}_p (pq)^{\binom{j+1}{2}} p^{jl} q^j \mu^j$ for all $a,m\geq 0$.



All intersecting paths reaching $(l,2)$ and $(r,1)$.

We show the second identity (the first follows in the same way).

$$\begin{aligned} I(a,m+1;n+1,1) &= \textstyle\sum_{l=1}^{a} \mu p^l q \sum_{j=0}^{a-l} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q \begin{bmatrix} a-l \\ j \end{bmatrix}_p (pq)^{\binom{j+1}{2}} \mu^j q^j p^{lj} \sum_{r=1}^{l} v\bar{p}^r \bar{q} \\ &= \textstyle\sum_{L=1}^{a} \mu^L q^L \sum_{l\geq 1} \begin{bmatrix} m-1 \\ L-1 \end{bmatrix}_q \begin{bmatrix} a-l \\ L-1 \end{bmatrix}_p (pq)^{\binom{L}{2}} p^{lL} \sum_{r=1}^{l} v\bar{p}^r \bar{q} \\ &= \frac{\bar{q}\bar{p}v}{1-\bar{p}} \textstyle\sum_{L=1}^{a} \mu^L q^L \sum_{i\geq 1} \begin{bmatrix} m-1 \\ L-1 \end{bmatrix}_q \begin{bmatrix} a-i \\ L-1 \end{bmatrix}_p (pq)^{\binom{L}{2}} p^{iL} \left(1-\bar{p}^i\right) \\ &= \bar{q}\bar{p}v \textstyle\sum_{L=1}^{a} \mu^L q^L \,(pq)^{\binom{L}{2}} \begin{bmatrix} m-1 \\ L-1 \end{bmatrix}_q \sum_{i\geq 1} \begin{bmatrix} a-i \\ L-1 \end{bmatrix}_p p^{iL} \begin{bmatrix} i \\ i-1 \end{bmatrix}_{\bar{p}} \\ &= \frac{\bar{q}\bar{p}v}{1-\bar{p}} \textstyle\sum_{j=0}^{a-1} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q q^{j(j+3)/2} p^{\binom{j+1}{2}} \mu^{j+1} \sum_{l=1}^{a-j} \begin{bmatrix} a-l \\ j \end{bmatrix}_p p^{(j+1)l} \left(1-\bar{p}^l\right) \\ &= \frac{\bar{q}\bar{p}v}{1-\bar{p}} \textstyle\sum_{j=0}^{a-1} \begin{bmatrix} m-1 \\ j \end{bmatrix}_q q^{j(j+3)/2} p^{\binom{j+1}{2}} \mu^{j+1} \sum_{l=j}^{a-1} \begin{bmatrix} l \\ j \end{bmatrix}_p p^{(j+1)(a-l)} \left(1-\bar{p}^{a-l}\right) \end{aligned}$$

Check: If $m=1$ then $I(a,2;n+1,1) = \frac{\bar{q}\bar{p}v}{1-\bar{p}} \mu q \sum_{i\geq 1} p^i \left(1-\bar{p}^i\right)$.
If $p=\bar{p}$ we get by Chu-Vandermonde convolution

$$\begin{aligned} \textstyle\sum_{i\geq 1} \begin{bmatrix} a-i \\ L-1 \end{bmatrix}_p p^{iL} \begin{bmatrix} i \\ i-1 \end{bmatrix}_p &= \textstyle\sum_{i=0}^{a-L} \begin{bmatrix} a-1-i \\ a-L-i \end{bmatrix}_p \begin{bmatrix} i+1 \\ i \end{bmatrix}_p p^{L(i+1)} \\ &= p^L \textstyle\sum_{i=0}^{a-L} \begin{bmatrix} L+a-L-i-1 \\ a-L-i \end{bmatrix}_p \begin{bmatrix} 2+i-1 \\ i \end{bmatrix}_p p^{Li} = p^L \begin{bmatrix} 2+L+a-L-1 \\ a-L \end{bmatrix}_p \end{aligned}$$

and therefore

$$\begin{aligned} I(a,m+1;n+1,1) &= \bar{q}pv \sum_{L=1}^{a} \mu^L q^L (pq)^{\binom{L}{2}} \begin{bmatrix} m-1 \\ L-1 \end{bmatrix}_q p^L \begin{bmatrix} 2+L+a-L-1 \\ a-L \end{bmatrix}_p \\ &= \bar{q}pv \sum_{L=1}^{a} \mu^L (pq)^{\binom{L+1}{2}} \begin{bmatrix} m-1 \\ L-1 \end{bmatrix}_q \begin{bmatrix} a+1 \\ L+1 \end{bmatrix}_p \end{aligned}$$

This agrees with expectations. ■

From now on we will always assume that $p = \bar{p}$ and $q = \bar{q}$. Note that $I(a,2;n+1,1) = \mu\nu (pq)^2 \begin{bmatrix} a+1 \\ a-1 \end{bmatrix}_p$ does not explicitly depend on n. We interpret $\mu\nu (pq)^2 \begin{bmatrix} a+1 \\ a-1 \end{bmatrix}_p$, $a = 1, 2, \ldots$ as a sequence of initial values $d_{a-1}(0)$ for a polynomial sequence $\{d_n(x)\}$ that represents $I(a,m+1;n+1,1)$,

$$d_{a-1}(m-1) = I(a,m+1;n+1,1)$$

for all $n \geq a$, $m \geq 0$. Because of (14), this polynomial sequence must follow the recurrence

$$d_a(m) = d_{a-1}(m) + (\mu p^{a+1} q^{m+1} - 1) d_{a-1}(m-1) + d_a(m-1).$$

This recurrence equals recurrence (5) if we replace μ by μpq in the latter. The given initial values for $\{d_n\}$ allow us to find

$$\rho(t) = \sum_{a\geq 0} d_a(0) t^a = \mu\nu (pq)^2 \sum_{a\geq 0} \begin{bmatrix} a+2 \\ a \end{bmatrix}_p t^a = \mu\nu (pq)^2 / (t;p)_3$$

It follows from (12) that

$$\begin{aligned} I(a,m+1;n+1,1) &= d_{a-1}(m-1) = \mu\nu (pq)^2 h_{a-1}(m-1;\mu pq,p,q) \\ &= \nu pq \sum_{l=1}^{a} \mu^l (qp)^{\binom{l+1}{2}} \begin{bmatrix} m-1 \\ l-1 \end{bmatrix}_q \begin{bmatrix} a+1 \\ l+1 \end{bmatrix}_p, \end{aligned}$$

in agreement with (17).

By symmetry (13),

$$I(1,m+1;n+1,d) = \mu\nu (pq)^2 h_{d-1}(n-1; vqp, q, p).$$

For arbitrary values of $a \leq n$ and $d \leq m$ it is not far fetched to conjecture that $I(a,m+1;n+1,d)$ is just the product of $\mu pq h_{a-1}(m-1;\mu pq,p,q)$ and $\nu pq h_{d-1}(n-1;vpq,q,p)$. But an elementary check for $a = d = 2$ already shows that this can only be true if $p = q$.

Include subdocument examples

Theorem 7 *If $p = \bar{p} = q = \bar{q}$ then for all $n \geq a \geq 0$ and $m \geq d \geq 0$,*

$$I(a,m+1;n+1,d) = \sum_{l,r\geq 1} \mu^l \nu^r q^{l(l+1)+r(r+1)} \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} a+1 \\ l+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} d+1 \\ r+1 \end{bmatrix}_q.$$

Proof. In terms of qq-Sheffer sequences we want to show that

$$I(a,m+1;n+1,d) = \mu q^2 h_{a-1}(m-1;\mu q^2,q,q) \cdot \nu q^2 h_{d-1}(n-1;vq^2,q,q)$$

for all $n \geq a \geq 0$ and $m \geq d \geq 0$. The recurrences (14) and (15) hold for the right hand side. The main work consists in verifying the recurrence in Corollary 5. Sorting the sums in terms of powers $\mu^l v^r q^{l(l+1)+r(r+1)}$ leaves to prove the identity

$$\begin{aligned} &\begin{bmatrix} m-1 \\ l-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ l+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m+1 \\ r+1 \end{bmatrix}_q \\ &\quad = \begin{bmatrix} m-1 \\ l-1 \end{bmatrix}_q \begin{bmatrix} n \\ l+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m+1 \\ r+1 \end{bmatrix}_q + \begin{bmatrix} m-1 \\ l-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ l+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m \\ r+1 \end{bmatrix}_q \\ &\quad\quad - \begin{bmatrix} m-1 \\ l-1 \end{bmatrix}_q \begin{bmatrix} n \\ l+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m \\ r+1 \end{bmatrix}_q + q^{n+m-l-r} \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_q \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q, \end{aligned}$$

which is elementary. ■

Details: Show that

$$\begin{aligned}
&\mu q^2 h_{n-1}(m-1;\mu q^2,q,q)\cdot \nu q^2 h_{m-1}(n-1;vq^2,q,q)\\
&= \mu q^2 h_{n-2}(m-1;\mu q^2,q,q)\cdot \nu q^2 h_{m-1}(n-1;vq^2,q,q)\\
&+\mu q^2 h_{n-1}(m-1;\mu q^2,q,q)\cdot \nu q^2 h_{m-2}(n-1;vq^2,q,q)\\
&-\mu q^2 h_{n-2}(m-1;\mu q^2,q,q)\cdot \nu q^2 h_{m-2}(n-1;vq^2,q,q)\\
&+\textstyle\sum_{l,r\geq 1}\mu^l q^{l(l+1)+r(r+1)}\nu^r \begin{bmatrix} n-1 \\ l-1 \end{bmatrix}_q \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q q^{n+m-l-r}
\end{aligned}$$

The right-hand side of this equation can be written as

$$\begin{aligned}
&\begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n \\ l+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m+1 \\ r+1 \end{bmatrix}_q + q^{n-l} \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m \\ r+1 \end{bmatrix}_q + q^{n+m-l-r} \begin{bmatrix} n-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q\\
&= \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ l+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m+1 \\ r+1 \end{bmatrix}_q + q^{n+m-l-r} \begin{bmatrix} n-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q\\
&+q^{n-l}\left(-\begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m+1 \\ r+1 \end{bmatrix}_q + \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m \\ r+1 \end{bmatrix}_q\right)\\
&= \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ l+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m+1 \\ r+1 \end{bmatrix}_q + q^{n+m-l-r} \begin{bmatrix} n-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q - q^{n-l}q^{m-r} \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m \\ r \end{bmatrix}_q\\
&= \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n+1 \\ l+1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m+1 \\ r+1 \end{bmatrix}_q + q^{n+m-l-r}\left(\begin{bmatrix} n-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q - \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m \\ r \end{bmatrix}_q\right)
\end{aligned}$$

where the expression in parenthesis equals

$$\begin{aligned}
&\begin{bmatrix} n-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} m \\ l \end{bmatrix}_q \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q - \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n \\ l \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m \\ r \end{bmatrix}_q\\
&= \begin{bmatrix} n-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} m-1 \\ l-1 \end{bmatrix} \frac{1-q^m}{1-q^l}{}_q \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \frac{1-q^n}{1-q^r} - \begin{bmatrix} m-1 \\ 1-1 \end{bmatrix}_q \begin{bmatrix} n-1 \\ l-1 \end{bmatrix} \frac{1-q^n}{1-q^l} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_q \begin{bmatrix} m-1 \\ r-1 \end{bmatrix}_q \frac{1-q^m}{1-q^r} = 0
\end{aligned}$$

If $\mu = v = 1/q$ we arrive at the equivalent result of R. Sulanke and C. Krattenthaler, who used the "rotation method" in [4]. In this case

$$I(a,m+1;n+1,d) = \sum_{l=0}^{m}\sum_{r=0}^{n} \begin{bmatrix} m-1 \\ l-1 \end{bmatrix} \begin{bmatrix} a+1 \\ l+1 \end{bmatrix} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix} \begin{bmatrix} d+1 \\ r+1 \end{bmatrix} q^{r^2+l^2} = q^2 \begin{bmatrix} m+a \\ m+1 \end{bmatrix} \begin{bmatrix} n+d \\ n+1 \end{bmatrix}$$

by q-Chu-Vandermonde summation (see [1, (3.3.10)]). If $q=1$ then $\begin{bmatrix} x \\ n \end{bmatrix} = \binom{x}{n}$, and

$$I(a,m+1;n+1,d) = \sum_{l=0}^{m}\sum_{r=0}^{n} \mu^l v^r \binom{m-1}{l-1}\binom{a+1}{l+1}\binom{n-1}{r-1}\binom{d+1}{r+1}.$$

This is the "Refinement of Narayana numbers" that R. Sulanke constructed in [8]. If $\mu = \nu = 1$ then

$$I(a,m+1;n+1,d) = \binom{m+a}{m+1}\binom{n+d}{n+1},$$

the Narayana numbers.

5 The log cabin bijection

Beneath each horizontal step of a lattice path there is a vertical column of nonnegative height. The sum of these heights is called the area covered by the path. There is an interesting bijection between lattice paths with weighted left turns and lattice paths over a given area.

Theorem 8 *Denote the total weight of all lattice paths that start at* $(0,0)$ *and reach the point* (a,m) *with exactly* l *left turns by* $L(a,m,l)$*, where a left turn at* (ξ,η) *gets the weight* $q^{\xi+\eta}$*. Denote the number of all lattice paths that start at* $(0,0)$ *and reach the point* (a,m) *covering the area* f *by* $A(a,m,f)$*.*

$$\sum_{l\geq 0} L(a,m,l) = \sum_{f\geq 0} A(a,m,f)q^f$$

Proof. The figure shows a lattice path, its vertical columns (∘), and its vertical and horizontal weights ($\bullet_i$and $\star_i$) at left turn i.

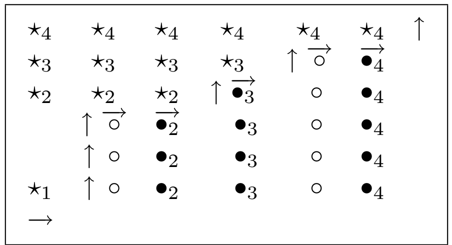


Figure 6: Path with left turn weight $q^{(1+0)+(3+3)+(4+4)+(6+5)}$

We rearrange the weights in log cabin fashion, constructing a new path ending at the same point, whose area is the sum of the former left turn coordinates.

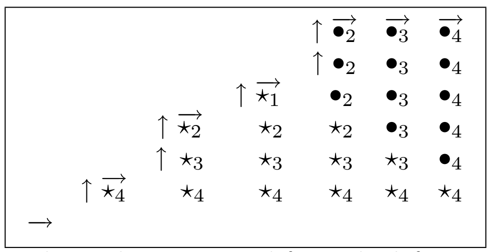


Figure 7: The path constructed from the left turn weights

The construction is obvious from Figure 7: The right angles representing the left turn weights are first shifted together and then all together turned upside down. This mapping between lattice paths is bijective, because the left turn coordinates of the path in Figure 6 can be recovered from the diagonal starting at the bottom right of the last vertical column.

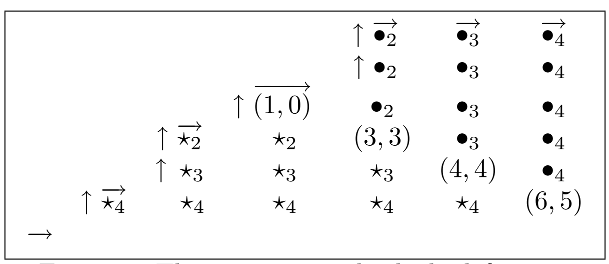


Figure 8: The corners give back the left turns

■

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E-mail address of the author: `niederha@acc.fau.edu`