

# Generalized Sheffer Sequences Satisfying Piecewise Functional Conditions

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## Abstract

The well known relationship between linear functionals and Sheffer sequences is extended to the case of piecewise functional conditions, where one functional determines the beginning, and a second functional shapes the remainder of a the sequence. The concept of Sheffer sequences is generalized from the Roman-Rota approach.

KEYWORDS: Umbral Calculus, operator equations, initial conditions.

## 1 Introduction

Suppose  $t_0(x), t_1(x), \dots$  is a sequence of polynomials,  $\deg t_n = n$ , solving a system of linear operator equations of the form

$$Bt_n(x) = t_{n-1}(x) \quad (1)$$

for all  $n = 1, 2, \dots$ . Examples are  $t_n(x) = x^n/n!$  solving  $D_x t_n(x) = t_{n-1}(x)$ , and  $t_n(x) = \binom{x+n}{n}$  solving the recursion  $t_n(x) - t_n(x-1) = t_{n-1}(x)$ . We show in this paper how to construct a second sequence  $p_0(x), p_1(x), \dots$  which solves the same system of operator equations as  $(t_n)$ , coincides with  $(t_n)$  for the first  $\ell$  terms, and satisfies a side condition of the form  $\langle L | p_n \rangle = 0$  after those first  $\ell$  terms, for all  $n \geq \ell$ , where  $L$  is a linear functional on polynomials. For example, the Bernoulli polynomials  $(\varphi_n)$  are the solution to the system of differential equations  $D_x p_n(x) = p_{n-1}(x)$  that satisfies the condition  $p_0(x) = 1$ , and  $\int_0^1 p_n(x) dx = 0$  for **all** positive integers  $n$ . Applying the method presented in this paper (Theorem 11) allows us to find the “delayed” Bernoulli polynomials where  $p_n(x) = x^n/n!$  for the first  $\ell$  polynomials, and  $\int_0^1 p_n(x) dx = 0$  only holds for  $n \geq \ell$  (Example 20). Instead of letting the beginning of the solution sequence coincide with a given sequence, we can as well demand a second functional condition,  $\langle J | p_n \rangle = \delta_{0,n}$ , say, for the first  $\ell$  terms. For example, in (23) we solve the problem  $\int_{-1}^0 p_n(x) = \delta_{0,n}$  for  $n = 0, \dots, \ell - 1$ , and  $\int_0^1 p_n(x) dx = 0$  for  $n \geq \ell$ . Iterating the process, we can solve systems under piecewise functional side conditions.

All linear functionals, operators, polynomial sequences and generating functions in this paper are related to each other in the framework of the “modern Umbral Calculus” as developed by Freeman [1] and Verde-Star [2] during the 1980’s; we show in Section 4.1 how to solve some basic problems in this general setting. However, except for Example 16, all of our detailed applications are written in the Roman-Rota Umbral Calculus [3] of the 1970’s. Finally, the more advanced functional conditions “along a line” (as in the last four examples) can only be treated in the “modern classical Umbral Calculus” of Rota, Kahaner, and Odlyzko [4, 1973].

## 2 The Problem and its Framework

Let  $K[x]$  be the ring of polynomials over some integral domain  $K$ . We denote the  $K$ -functionals on  $K[x]$  by  $K[x]^*$ . A *polynomial sequence*  $(p_n)_{n \in \mathbb{N}}$  in  $K[x]$  is a sequence of polynomials where  $\deg p_n = n$  for all non-negative integers  $n$ , and  $p_0 \neq 0$ . Let  $p_n$  be identically zero for negative  $n$ . The ring of formal power series over  $K$  is denoted by  $K[[t]]$ . The order of a formal power series  $\beta(t)$  is the smallest power of  $t$  that occurs in  $\beta(t)$ .

We will first formulate the “piecewise functional condition” in the language of “modern classical Umbral Calculus” [4, 1973]; in Section 3 we advance to a more general concept.

**Definition 1** *A delta series is a power series of order 1. A Sheffer sequence  $(s_n)$  has the generating function*

$$\sum_{n \geq 0} s_n(x)t^n = \rho(t)e^{x\beta(t)},$$

where  $\beta$  is a delta series, and  $\rho$  is a power series of order zero. If  $\rho(t) = 1$  then we call the resulting Sheffer sequence a basic sequence and denote it by  $(b_n)$ .

Obviously,  $\rho(t) = \sum_{n \geq 0} s_n(0)t^n$ , and therefore

$$s_n(x) = \sum_{i=0}^n s_i(0)b_{n-i}(x). \tag{2}$$

**Definition 2** *If  $(b_n)$  is a basic sequence then the linear operator  $B$  on  $K[x]$  defined by linear extension of*

$$Bb_n = b_{n-1}$$

*is called the delta operator (for  $(b_n)$ , or  $\beta(t)$ ).*

The expansion (2) shows that all Sheffer sequences  $(s_n)$  (with respect to the same delta series) are solutions of the system of equations  $Bs_n = s_{n-1}$ . We will call such solutions  $B$ -Sheffer sequences. We want to find the unique Sheffer sequence  $(p_n)$  which solves the system of delta operator equations

$$Bp_n(x) = p_{n-1}(x) \text{ for all } n = 1, 2, \dots$$

under the conditions

$$\begin{aligned} p_n(x) &= t_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1 \\ \langle L | p_n(x) \rangle &= 0 \text{ for all } n = \ell, \ell + 1, \dots \end{aligned} \quad (3)$$

where  $(t_n)$  is some given  $B$ -Sheffer sequence,  $\ell$  is a positive integer, and  $L$  is a linear functional on polynomials. In order to expand the solution  $p_\ell(x), p_{\ell+1}(x), \dots$  in terms of the basic sequence  $(b_n)$  associated with  $B$  we will need in Theorem 9 one more critical assumption,  $\langle L | 1 \rangle \neq 0$ , which implies the invertibility of  $L$  with respect to a product of functionals yet to be defined (in Section 3). We will see later, in Remark 8, that there exists an invertible functional  $J$  associated to  $(t_n)$  such that  $\langle J | t_n \rangle = \delta_{n,0}$  for all nonnegative integers. Given  $J$ , there is only one Sheffer sequence associated with  $B$  that satisfies this initial condition (by Corollary 10). Therefore the side conditions (3) can be rephrased as  $\langle J | p_n \rangle = \delta_{n,0}$  for all  $n = 0, 1, \dots, \ell - 1$ , and  $\langle L | p_n(x) \rangle = 0$  for all  $n = \ell, \ell + 1, \dots$ .

The following list shows some basic types of problems. They are solved in Section 4.1 in the general framework which we will introduce in the next section. In Section 6 we will return to the “modern classical Umbral Calculus” to solve more advanced versions of the basic problems.

**Problem I (Initial value)** *Let  $(t_n)$  be a given  $B$ -Sheffer sequence, and  $a \in K$ . Find the  $B$ -Sheffer sequence  $(p_n)$  satisfying the conditions*

$$\begin{aligned} p_n(x) &= t_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1 \\ p_n(a) &= 0 \text{ for all } n = \ell, \ell + 1, \dots \end{aligned} \quad (4)$$

**Problem II** *Find the  $B$ -Sheffer sequence  $(s_n)$  such that*

$$\begin{aligned} s_n(x) &= t_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1 \\ s_n(0) &= \sum_{i=1}^n (-1)^{i-1} c^i s_{n-i}(0) \text{ for all } n = \ell, \ell + 1, \dots \end{aligned} \quad (5)$$

for some given constant  $c \neq 0$ .

**Problem III (Recursive initial values)** *Find the  $B$ -Sheffer sequence  $(p_n)$  such that*

$$\begin{aligned} p_n(x) &= t_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1 \\ p_n(a) &= p_{n-1}(b) \text{ for all } n = \ell, \ell + 1, \dots \end{aligned} \quad (6)$$

where  $a$  and  $b$  are given constants.

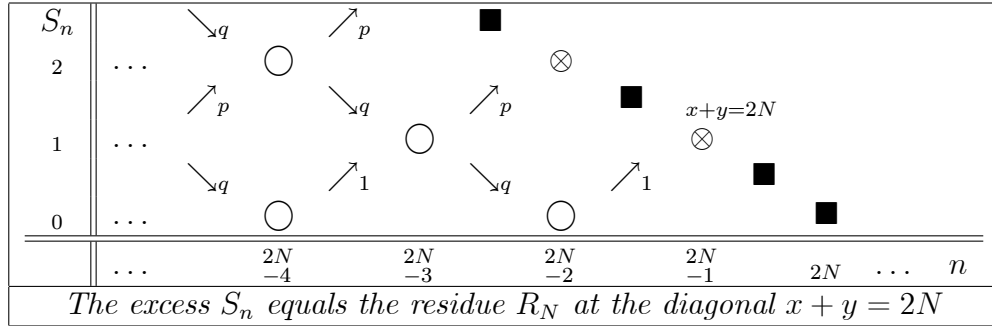
**Example 3 (Paper Towels)** *The paper towel problem, a variation of Banach’s match box problem, is a simple example for recursive initial values. Two rolls of paper towels are hanging side-by-side in a workshop; sheets are removed from the smaller roll with probability  $p$ , and from the larger roll with probability  $q = 1 - p$ . When both rolls are of the same size, they are equally likely selected. If originally both rolls had  $N$  towels, what is the distribution of the residue  $R_N$ , i.e., the number of sheets on the larger roll when the other roll becomes empty?*

In [5], Stirzacker derived the generating function of the probability generating functions for  $R_N$ ,  $N = 0, 1, \dots$ . However, the exact distribution of  $R_N$  is easily obtained. As in [5], we call  $S_n$  the excess of towels on the larger roll over the smaller after the  $n$ -th towel has been removed. Obviously  $S_n = R_N$  when one roll becomes empty. This happens when for the first time  $S_n + n = 2N$ , i.e., when the random walk (with steps  $\nearrow$  and  $\searrow$ ) through the points  $(n, S_n)$  reaches the diagonal  $x + y = 2N$  in a diagonally upwards step. Therefore

$$\Pr(R_N = r) = p \Pr(S_{2N-r-1} = r - 1)$$

when  $r > 1$  (the last towel came from the smaller roll), and

$$\Pr(R_N = 1) = \Pr(S_{2N-2} = 0).$$



The random walk  $S_n$  follows a distribution with backwards recursion

$$\pi_{n,m} := \Pr(S_n = m) = p\pi_{n-1,m-1} + q\pi_{n-1,m+1} \quad (7)$$

for  $n \geq m \geq 2$  and even  $n + m$ . If  $m = 1$  the recursion is slightly different because a zero excess certainly must increase to one when the next towel is removed,

$$\pi_{n,1} = q\pi_{n-2,1} + q\pi_{n-1,2} \text{ for } n \geq 1. \quad (8)$$

In addition,  $\pi_{0,0} = 1$ ,  $\pi_{m,m} = p^{m-1}$  for positive  $m$ , and  $\pi_{n,0} = q\pi_{n-1,1}$  for positive  $n \geq 2$  (because the excess cannot be negative). Now define  $s_n(m) := p^{-m-n+1}q^{-n}\pi_{2n+m,m}$  for  $n \geq 0$ ,  $m \geq 1$ , and  $s_n(m) = 0$  for negative  $n$ . The conditions on  $\pi_{n,m}$  easily translate into conditions on  $s_n(m)$ ,

$$\begin{aligned} s_n(m) &= s_n(m-1) + s_{n-1}(m+1) \text{ for } n \geq 0 \text{ and } m \geq 2 \text{ by (7),} \\ s_n(1) &= p^{-1}s_{n-1}(1) + s_{n-1}(2) \text{ for } n \geq 1 \text{ by (8), and} \\ s_0(m) &= 1 \text{ for } m > 0. \end{aligned}$$

The backwards difference operator  $\nabla : f(x) \mapsto f(x) - f(x-1)$  is a delta operator with basic sequence  $(\binom{x+n-1}{n})_{n \geq 0}$ . From  $\nabla s_n(m) = E^1 s_{n-1}(m)$  for all  $m \geq 2$  follows that  $s_n(m)$  can be extended to a Sheffer sequence  $(s_n)$  for the delta operator  $B = E^{-1}\nabla$ , with basic sequences  $b_n(x) = \frac{x}{x+2n} \binom{x+2n}{n}$  (see (20)). The recursive initial condition

$$0 = s_n(1) - p^{-1}s_{n-1}(1) - s_{n-1}(2) = s_n(0) - p^{-1}s_{n-1}(1)$$

is of the form (6), with  $\ell = 1$ . We will expand  $s_n(x)$  in terms of the basic sequence in Example 24.

**Problem IV (Zero average)** Find the B-Sheffer sequence  $(p_n)$  satisfying the conditions

$$p_n(x) = t_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1$$

$$\int_0^1 p_n(y) dy = 0 \text{ for all } n = \ell, \ell + 1, \dots$$

### 3 Transforms and Functionals

We will now define Sheffer sequences in a more general way than in the previous section, using the concepts introduced by Freeman [1], and Verde-Star [2]. For every  $a \in K$  the *evaluation functional*  $\varepsilon_a$  is defined as  $\langle \varepsilon_a | f(x) \rangle := f(a)$  for all  $f \in K[x]$ . The *augmentation*  $\varepsilon$  is the evaluation at 0. We call a polynomial sequence  $(e_n)$  a *reference sequence* iff  $\langle \varepsilon | e_n \rangle = \delta_{n,0}$ . In the following,  $(e_n)$  will always denote a reference sequence. The case  $e_n(x) = c_n x^n$  has been considered by Roman [6], and  $e_n(x) = x^n/n!$  leads to the ‘‘Finite Operator Calculus’’ [4] of the previous chapter.

Let  $(\sigma_n)$  be a sequence of formal power series, and  $order(\sigma_n) = n$  for all  $n \geq 0$ . There exists a unique polynomial sequence  $(s_n)$  such that

$$\sum_{n \geq 0} s_n(x) t^n = \sum_{n \geq 0} e_n(x) \sigma_n(t)$$

[1, Lemma 1.2]. We call  $(s_n)$  the *e-image* of  $(\sigma_n)$ . However, this approach is too general for specific results; we will investigate a more restricted setting.

**Definition 4** A (generalized) Sheffer sequence  $(s_n)$  is the e-image of  $(\rho(t)\beta(t)^n)$  where  $\beta$  is a delta series, and  $\rho$  is a power series of order zero. If  $\rho(t) = 1$  then we call the resulting Sheffer sequence  $(b_n)$  a (generalized) basic sequence.

The delta operator  $B$  is now defined via the generalized basic sequence  $(b_n)$ , and  $Bs_n = s_{n-1}$  still holds because the expansion (2) remains unchanged.

The solution of the general problem (3) requires the definition of the ‘‘multiplication operator’’  $\mu(L)$  on linear functionals. In this section we provide the necessary background.

Sheffer sequences are obtained by substituting a delta series  $\beta(t)$  for  $t$  into

$$e(x, t) := \sum_{n \geq 0} e_n(x) t^n \in K[x][[t]]$$

and multiplying by some power series  $\rho(t)$  of order 0. For this purpose the  $K[x]$ -linear operators *composition*  $C(\beta)$  and *multiplication*  $M(\rho)$  are defined on  $K[x][[t]]$  in [1] such that

$$M(\rho)C(\beta)e(x, t) = M(\rho)e(x, \beta(t)) = \sum_{n \geq 0} e_n(x) \rho(t) \beta(t)^n = \sum_{n \geq 0} s_n(x) t^n.$$

Any two multiplication operators commute. The  $K[x]$ -linear operators on  $K[x][[t]]$  are called *t-operators*; their ring is denoted by  $\mathcal{L}_t$ . Composition and multiplication are examples of

elements in  $\mathcal{L}_t$ . The  $e$ -transform of a  $t$ -operator  $T$  is the  $x$ -operator  $\check{T} \in \mathcal{L}_x$ , the ring of  $K[[t]]$ -linear operators on  $K[x][[t]]$ , such that

$$Te(x, t) = \check{T}e(x, t).$$

For example, the delta operator which maps  $e_n(x)$  into  $e_{n-1}(x)$  for all integers  $n$ , is the  $e$ -transform of  $M(t)$ . We denote this  $x$ -operator by  $\Delta_e$ ,

$$\Delta_e = M(t)^\smile.$$

All  $x$ -operators commute with all  $t$ -operators. If  $B$  is a delta operator associated to the delta series  $\beta$ , then

$$BC(\beta)e(x, t) = M(t)C(\beta)e(x, t). \quad (9)$$

It follows (Freeman [1, Proposition 4.9]) that

$$B = \beta^{-1}(M(t)^\smile) = \beta^{-1}(\Delta_e) \quad (10)$$

where  $\beta^{-1}$  is the compositional inverse of  $\beta$ , i.e.,  $\beta^{-1}(\beta(t)) = t$ . By definition, any  $B$ -Sheffer sequence  $(s_n)$  has a generating function of the form

$$M(\rho(t))C(\beta)e(x, t) = \rho(B)C(\beta)e(x, t)$$

(see (9)), and therefore

$$s_n(x) = \rho(B)b_n(x) = \rho(\beta^{-1}(\Delta_e))b_n(x), \quad (11)$$

where  $\rho$  has order 0.

An umbral operator  $U$  is an  $x$ -operator which maps basic sequences into basic sequences,  $UC(\beta)e(x, t) = C(\gamma)e(x, t)$  for a pair of delta series  $\beta$  and  $\gamma$ . Hence  $U = (C(\beta^{-1})C(\gamma))^\smile = C(\gamma \circ \beta^{-1})^\smile$ , which shows that composition operators are transforms of umbral operators.

The linear operator  $\tilde{\phantom{L}}$  maps the functionals  $K[x]^*$  into  $K[[t]]$  via  $e(x, t)$ ,

$$\tilde{L}(t) := Le(x, t) := \sum_{n \geq 0} \langle L | e_n \rangle t^n.$$

This mapping is a ring isomorphism if we define the product  $L\#N$  of two linear functionals  $L$  and  $N$  by linear extension of

$$\langle L\#N | e_n \rangle := \sum_{k=0}^n \langle L | e_k \rangle \langle N | e_{n-k} \rangle,$$

i.e.,

$$(L\#N)e(x, t) = (Le(x, t))(Ne(x, t)) = \tilde{L}(t)\tilde{N}(t).$$

We write  $L\#^k$  for the  $k$ -th power under this product. The augmentation  $\varepsilon$ , the evaluation at 0, is the multiplicative unit. A linear functional  $L$  is invertible (w.r.t. multiplication), iff  $\langle L | 1 \rangle \neq 0$ .

**Definition 5** The mapping  $\mu : K[x]^* \rightarrow \mathcal{L}_x$  is the  $e$ -transform of the multiplication operator  $M(\tilde{L}(t))$ , i.e.  $\mu(L) = M(\tilde{L}(t))^\smile = \tilde{L}(\Delta_e)$  for all  $L \in K[x]^*$ .

Note that  $\mu(L)$  commutes with delta operators, because they are transforms of  $t$ -multiplication operators.

**Lemma 6** The mapping  $\mu : K[x]^* \rightarrow \mathcal{L}_x$  is an isomorphism. For fixed  $L \in K[x]^*$  the adjoint  $\mu(L)^*$  of  $\mu(L)$  is the multiplication operator on  $K[x]^*$  which maps  $N$  into  $L\#N$  for all  $N \in K[x]^*$ ,

$$\langle L\#N \mid p \rangle = \langle \mu(L)^*N \mid p \rangle = \langle N \mid \mu(L)p \rangle$$

for all  $p \in K[x]$ .

**Proof.** Let  $J, L \in K[x]^*$ . The isomorphism follows from

$$\mu(J\#L) = M(\widetilde{J\#L}(t))^\smile = M(\widetilde{J\tilde{L}}(t))^\smile = M(\tilde{L})^\smile M(\tilde{J})^\smile = M(\tilde{J})^\smile M(\tilde{L})^\smile = \mu(J)\mu(L).$$

Next we have to show that

$$L\#Ne(x, t) = N\mu(L)e(x, t)$$

for all  $N, L \in K[x]^*$ . The  $\smile$ -operation is an isomorphism; it is sufficient to show that  $\widetilde{L\#N} = N\mu(L)$ . By proposition 6.1 in [1],  $\widetilde{N\mu(L)} = M(\tilde{L})\tilde{N}$ . ■

**Remark 7** For the evaluation functional  $\varepsilon_a$  we find

$$\mu(\varepsilon_a) = M(\tilde{\varepsilon}_a)^\smile = \sum_{n \geq 0} e_n(a)M(t^n)^\smile = \sum_{n \geq 0} e_n(a)\Delta_e^n = e(a, \Delta_e).$$

Solving the very basic Problem I will require the calculation of

$$\mu(\varepsilon_a^{-1}) = \mu(\varepsilon_a)^{-1} = \frac{1}{e(a, \Delta_e)}.$$

In most of the important examples for  $e(x, t)$  this is not difficult (see Example 19 in Section 5.1); the Rota-Kahaner-Odlyzko [4] setting, where  $e(x, t) = e^{xt}$ , is characterized by the property  $\mu(\varepsilon_a^{-1}) = \mu(\varepsilon_{-a})$ .

**Remark 8** If  $(b_n)$  is the basic sequence for the delta operator  $B$ , then we can define the functional  $I_B$  as

$$\langle I_B \mid b_n \rangle := \langle \varepsilon \mid b_{n-1} \rangle = \delta_{n,1}. \quad (12)$$

From  $I_B C(\beta)e(x, t) = t$  follows  $\tilde{I}_B(t) = \beta^{-1}(t)$ , and therefore by (10)  $\mu(I_B) = B$ . Suppose,  $(t_n)$  is the Sheffer sequence with generating function  $M(\rho(t))C(\beta)e(x, t)$ . If  $J$  is the invertible linear functional defined by  $\tilde{J}(t) = 1/\rho(\beta^{-1}(t))$  then

$$C(\beta)e(x, t) = C(\beta)M(\rho(\beta^{-1}(t)))M(\tilde{J}(t))e(x, t) = \mu(J)M(\rho(t))C(\beta)e(x, t)$$

and therefore  $b_n(x) = \mu(J)t_n(x)$  and  $\langle J \mid t_n \rangle = \langle \varepsilon \mid b_n \rangle = \delta_{0,n}$ . This relationship between functionals, basic sequences and Sheffer sequences is central to the Roman-Rota Umbral Calculus [3].

We can now formulate an “Expansion Theorem” which occurs in some form in most approaches to the Umbral Calculus.

**Theorem 9** *Suppose  $L$  is an invertible linear functional, and  $(b_n)$  is the basic sequence for the delta operator  $B$ . Any polynomial  $p(x)$  can be expanded in the form*

$$p(x) = \sum_{k \geq 0} \langle L \# I_B^{\#k} | p \rangle \mu(L)^{-1} b_k(x) = \sum_{k \geq 0} \langle L | B^k p \rangle \mu(L)^{-1} b_k(x).$$

**Proof.** The equivalence of both expansion formulas follows from (12),  $\mu(I_B^{\#k}) = B^k$  for all nonnegative integers  $k$ . Note that

$$\langle L | B^k \mu(L)^{-1} b_n \rangle = \langle \varepsilon | \mu(L) \mu(L)^{-1} B^k b_n \rangle = \langle \varepsilon | b_{n-k} \rangle = \delta_{n,k}.$$

Expand  $p(x)$  in terms of the basis  $(\mu(L)^{-1} b_n)$ . Apply the above formula to every term. ■

The following corollary can be seen as a generalization of the “Binomial Theorem for Sheffer sequences” (21).

**Corollary 10** *If  $(s_n)$  is a Sheffer sequence with associated basic sequence  $(b_n)$ , and  $L$  is an invertible functional, then*

$$s_n(x) = \sum_{i=0}^n \langle L | s_i \rangle \mu(L)^{-1} b_{n-i}(x)$$

for all  $n = 0, 1, \dots$

## 4 The Expansion

With the help of the transformed multiplication operator  $\mu(L) = M(\tilde{L}(t))^\sim$  we can expand the solution to our problem in terms of basic polynomials.

**Theorem 11** *Suppose  $L$  is an invertible linear functional, and  $\ell$  a positive integer. If  $B$  is a delta operator with basic sequence  $(b_n)$ , and  $(t_n)$  a  $B$ -Sheffer sequence, then*

$$\begin{aligned} p_n(x) &= t_n(x) - \sum_{i=\ell}^n \langle L | t_i(x) \rangle \mu(L^{-1}) b_{n-i}(x) \\ &= \sum_{i=0}^{\ell-1} \langle L | t_i(x) \rangle \mu(L^{-1}) b_{n-i}(x) \end{aligned} \tag{13}$$

is the solution to the initial value problem

$$\begin{aligned} Bp_n &= p_{n-1} \text{ for all } n = 1, 2, \dots \\ p_n &= t_n \text{ for all } n = 0, \dots, \ell - 1 \\ \langle L | p_n \rangle &= 0 \text{ for all } n = \ell, \ell + 1, \dots \end{aligned}$$

**Proof.** It is instructive to derive the Theorem from an “Ansatz” of the form

$$p_n(x) = t_n(x) - q_{n-\ell}(x)$$

where  $\{q_n(x)\}$  is also a  $B$ -Sheffer sequence, representing the correction terms which ensure the required side conditions on the solution. However, it is much faster to verify that  $(p_n)$  has the correct properties; the first two,  $Bp_n = p_{n-1}$  for all  $n = 1, 2, \dots$  and  $p_n = t_n$  for all  $n = 0, \dots, \ell - 1$ , are obvious. It remains to calculate for  $n \geq \ell$

$$\begin{aligned} \langle L | p_n \rangle &= \langle L | t_n \rangle - \sum_{i=\ell}^n \langle L | t_i(x) \rangle \langle L | \mu(L^{-1})b_{n-i}(x) \rangle \\ &= \langle L | t_n \rangle - \sum_{i=\ell}^n \langle L | t_i(x) \rangle \langle L \# L^{-1} | b_{n-i}(x) \rangle \\ &= \langle L | t_n \rangle - \langle L | t_n \rangle = 0. \end{aligned}$$

The second expansion of  $p_n(x)$  follows from Corollary 10. ■

If the initial value problem is phrased as

$$\begin{aligned} Bp_n &= p_{n-1} \text{ for all } n = 1, 2, \dots \\ \langle J | p_n \rangle &= \delta_{0,n} \text{ for all } n = 0, \dots, \ell - 1 \\ \langle L | p_n \rangle &= 0 \text{ for all } n = \ell, \ell + 1, \dots \end{aligned}$$

where  $J$  is some given invertible functional, then we can rephrase (13) as

$$p_n(x) = \sum_{i=0}^{\ell-1} \langle L | \mu(J^{-1})b_i(x) \rangle \mu(L^{-1})b_{n-i}(x)$$

(see Remark 8).

## 4.1 Some examples

The first three examples in this section are based on evaluation functionals. Example 15 constructs solutions with zero average on the unit interval. All examples are valid for general reference sequences. For special reference sequences see Section 5.2.

**Example 12 (Initial values)** *The initial value Problem I is solved by the  $B$ -Sheffer sequence*

$$p_n(x) = t_n(x) - \sum_{i=\ell}^n \langle \varepsilon_a | t_i(x) \rangle e(a, \Delta_e)^{-1} b_{n-i}(x)$$

(see Remark 7 for  $e(a, \Delta_e)^{-1}$ ).

**Example 13** *The functional condition in Problem II can be written as*

$$0 = \sum_{i=0}^n (-1)^i c^i s_{n-i}(0) = \left( \frac{1}{1 + cB} s_n \right) (0) = p_n(0)$$

if we define the polynomial sequence  $(p_n)$  by  $p_n := \frac{1}{1+cB}s_n$ . From (11) follows that  $(p_n)$  is again a  $B$ -Sheffer sequence. Hence condition (5) is a special case of Problem I,

$$p_n = \frac{1}{1+cB}t_n \text{ for } n = 0, \dots, \ell - 1$$

$$\langle \varepsilon \mid p_n(x) \rangle = 0 \text{ for all } n \geq \ell.$$

By Theorem 11,

$$s_n(x) = t_n(x) - \sum_{i=\ell}^n \left\langle \varepsilon_0 \mid \frac{1}{(1+cB)}t_i(x) \right\rangle (1+cB)e(0, \Delta_e)^{-1}b_{n-i}(x)$$

$$= t_n(x) - \sum_{i=\ell}^n (b_{n-i}(x) + cb_{n-i-1}(x)) \sum_{j=0}^i (-1)^j c^j t_{i-j}(0).$$

**Example 14 (Recursive initial values)** Let  $I_B$  be the delta functional such that  $B = \mu(I_B)$  (see Remark 8). The condition

$$\langle \varepsilon_a \mid p_n(x) \rangle - \langle \varepsilon_b \mid p_{n-1}(x) \rangle = 0$$

in Problem III can be expressed as

$$0 = \langle \varepsilon_a \mid p_n(x) \rangle - \langle \varepsilon_b \mid \mu(I_B)p_n(x) \rangle$$

$$= \langle \varepsilon_a - \varepsilon_b \# I_B \mid p_n(x) \rangle.$$

We choose  $L = \varepsilon_a - \varepsilon_b \# I_B$  in Theorem 11 and get

$$p_n(x) = t_n(x) - \sum_{i=\ell}^n \langle L \mid t_i(x) \rangle \mu(L)^{-1} b_{n-i}(x)$$

where

$$\mu(L) = \mu(\varepsilon_a) - \mu(\varepsilon_b) \mu(I_B) = e(a, \Delta_e) - e(b, \Delta_e)B$$

(see Remark 7). Hence

$$p_n(x) = t_n(x) - \sum_{i=\ell}^n (t_i(a) - t_{i-1}(b)) \frac{1}{e(a, \Delta_e) - e(b, \Delta_e)B} b_{n-i}(x). \quad (14)$$

The linear functional  $\langle \int_0^1 \mid p \rangle := \int_0^1 p(x)dx$  averages the polynomial  $p$  over the interval  $[0, 1]$ . We find  $\tilde{\int}_0^1 = \int_0^1 e(x, t)dx$ , and  $\mu(\int_0^1)^{-1} = 1/\int_0^1 e(x, \Delta_e)dx$ .

**Example 15 (Zero average)** In Problem IV we are looking for the solution with  $\langle \int_0^1 \mid p_n \rangle = 0$  for all  $n \geq \ell$ . By Theorem 11

$$p_n(x) = t_n(x) - \sum_{i=\ell}^n \left( \int_0^1 t_i(x)dx \right) \frac{1}{\int_0^1 e(x, \Delta_e)dx} b_{n-i}(x).$$

## 5 Additional Theory and Examples

In this section we continue the theory of transforms and functionals, and show how the basic problems can be solved under certain choices for the reference sequence. Except for the following example, all further applications are based on reference sequences which have generating functions of the form

$$e(x, t) = e(xt) \quad (15)$$

where  $e(t)$  is any power series of order 0 (with  $e^t$  as a guiding example for  $e(t)$ ). This case has been extensively studied by Roman ([3], [6], and subsequent papers). Polynomial sequences with a generating function of the form  $g(t)e(xt)$  for some  $g(t)$  of order 0 are called *Brenke type* sequences[7]. While Brenke type Sheffer sequences (with generating function  $g(t)e(x\beta(t))$ ) are abundant in the literature, there are also simple examples that fall outside this category and are therefore not covered by the Roman-Rota Umbral Calculus. The following problem is solved by a Sheffer sequence that is not Brenke type.

**Example 16** Find the polynomial sequence  $(p_n)$  which satisfies the recurrence

$$\chi p_n(x) = p_{n-1}(x) + p'_{n-1}(0) \text{ for } n = 1, 2, \dots$$

where

$$\chi f(x) := \frac{f(x) - f(0)}{x}$$

is the “division by  $x$ ” operator. We add the side conditions

$$\begin{aligned} p_n(x) &= x(1 - x^n)/(1 - x) \text{ for } n = 1, \dots, \ell - 1 \\ p_n(0) &= -p'_n(0) \text{ for } n \geq \ell. \end{aligned}$$

We attack the problem by first noting that  $e_0(x) := 1$ ,  $e_n(x) := x(1 - x^n)/(1 - x)$  for  $n > 0$ , is a reference (and therefore basic) sequence with generating function  $e(x, t) = 1 + \frac{x}{1-x} \left( \frac{t}{1-t} - \frac{xt}{1-xt} \right) = 1 + \frac{xt}{(1-t)(1-xt)}$ . It is easy to check that  $\chi e_n(x) = e_{n-1}(x) + e'_{n-1}(0)$  for  $n = 1, 2, \dots$ . There exists no delta series  $\beta(t)$  and no Brenke type sequence with generating function  $g(t)e(xt)$  such that  $e(x, t) = g(t)e(x\beta(t))$ .

Define the functional  $L$  such that  $\langle L | 1 \rangle = \langle L | x \rangle = 1$  and  $\langle L | x^n \rangle = 0$  for all  $n > 1$ . The functional condition  $\langle L | p \rangle = 0$  is equivalent to the condition that the constant term in  $p(x)$  is the negative of the coefficient of the linear term, or  $p(0) + p'(0) = 0$ . The side condition  $p_n(0) = -p'_n(0)$  can therefore be phrased as  $\langle L | p_n \rangle = 0$ . From

$$\tilde{L}(t) = Le(x, t) = \langle L | 1 \rangle + \sum_{n \geq 1} t^n \left\langle L \left| \sum_{j=1}^n x^j \right. \right\rangle = 1 + \sum_{n \geq 1} t^n = \frac{1}{1-t}$$

follows

$$\mu(L)^{-1} = M(1-t)^\smile = 1 - \Delta_e$$

and  $\mu(L)^{-1}e_n(x) = x^n$  for all  $n \neq 1$ ,  $\mu(L)^{-1}e_1(x) = x - 1$ . Hence, for  $n > \ell$ ,

$$p_n(x) = \sum_{i=0}^{\ell-1} \langle L | e_i \rangle \mu(L)^{-1}e_{n-i}(x) = \sum_{i=0}^{\ell-1} x^{n-i} = e_n(x) - e_{n-\ell}(x)$$

by Theorem 11 ( $p_n(x) = e_n(x) - e_{n-\ell}(x)$  holds also for  $n = \ell$ ).

## 5.1 Diagonal operators

A *diagonal* operator maps  $x^n$  into  $d(n)x^n$ , where  $d(n)$  is any  $K$ -valued function on  $\mathbb{N}$ . For diagonal operators we will use Freeman's notation [1]

$$d(\eta)(x^n) := d(n)x^n$$

for all  $n = 0, 1, \dots$ . For example,  $\eta(x^n) = nx^n$ , and  $\eta!(x^n) = n!x^n$ . The parentheses will be omitted; we write  $\eta!x^n$  instead of  $\eta!(x^n)$ . The ring  $\mathcal{D}_x$  of diagonal operators  $d(\eta)$  is an important commutative subring of  $\mathcal{L}_x$ .

**Proposition 17** (Freeman [1, Proposition 3.4]) *If  $e(x, t) = e(xt)$  and  $e_d(xt) = d(\eta)e(x, t)$  where  $d(\eta)$  is a unit in  $\mathcal{D}_x$ , then  $\Delta_{e_d} := d(\eta)d(\eta + 1)^{-1}\Delta_e$  is the  $e_d$ -transform of  $M(t)$ .*

For the following three examples in Table 1 (see also [1]) we choose  $e(xt) = (1 - xt)^{-1}$ , and select three different units in  $\mathcal{D}_x$ . Note that  $\Delta_{1/(1-xt)} = \chi$ , where the operator  $\chi$  was defined in Example 16,  $\chi : x^n \mapsto x^{n-1}$  for positive  $n$ , and  $\chi 1 = 0$ . When commuting  $\eta$  with  $\chi$ , the changes in degree have to be accounted for,

$$\chi\eta = (\eta + 1)\chi \quad (= D_x). \quad (16)$$

We use the standard notation  $(x; q)_n = \prod_{k=0}^{n-1} (1 - xq^k)$ .

Table 1			
	Binomial	Exponential	Eulerian
$d[\eta] =$	$\binom{\eta+\alpha-1}{\eta} \quad (\alpha \neq 0, -1, -2, \dots)$	$1/\eta!$	$(1 - q)^\eta / (q; q)_\eta$
$e_d(xt) =$	$(1 - xt)^{-\alpha}$	$e^{xt}$	$1 / ((1 - q)xt; q)_\infty$
$e_n(x) =$	$\binom{n+\alpha-1}{n} x^n$	$x^n/n!$	$((1 - q)x)^n / (q; q)_n$
$\Delta_{e_d} =$	$(\eta + \alpha)^{-1} D_x = \frac{\eta+1}{\eta+\alpha} \chi$	$D_x$	$D_q := \chi \frac{1-q^\eta}{1-q}$

The transform is a ring anti-isomorphism,  $(T_1 T_2)^\smile = \check{T}_2 \check{T}_1$ . For  $x$ -operators  $X$  we define the (inverse)  $e$ -transform as the  $t$ -operator  $\hat{X}$  such that

$$Xe(x, t) = \hat{X}e(x, t),$$

i.e.,  $(\hat{X})^\smile = X$ .

**Example 18** *Let  $e_d(xt) := (1 - xt)^{-\alpha}$ , and define the delta operator  $G = \beta^{-1}(\Delta_{e_d})$  by the composition  $C(\beta) = C(2t/(1 + t^2))$ . The compositional inverse  $\beta^{-1}(t)$  of the delta series  $\beta(t)$  must solve the equation  $t = 2\beta^{-1}(t) - t\beta^{-1}(t)^2$ , hence  $\Delta_{e_d} = 2G - \Delta_{e_d}G^2$ . From the table we find  $\Delta_{e_d} = \frac{\eta+1}{\eta+\alpha}\chi = (\eta + \alpha)^{-1} D_x$ , and therefore*

$$D_x = 2(\eta + \alpha)G - D_x G^2.$$

*Applying both sides of the equation to any  $G$ -Sheffer sequence  $(s_n)$  results in the recurrence formula  $D_x s_n = 2(\eta + \alpha)s_{n-1} - D_x s_{n-2}$ , or*

$$s'_n(x) - 2\alpha s_{n-1}(x) = 2x s'_{n-1}(x) - s'_{n-2}(x). \quad (17)$$

Suppose the side conditions are given by the linear functional

$$\langle L | x^n \rangle := \frac{1}{B(1/2, \alpha + 1/2)} \int_{-1}^1 x^n (1 - x^2)^{\alpha-1/2} dx$$

where  $B(v, w) = \int_0^1 u^{v-1} (1 - u)^{w-1} du$  is the beta function. This functional vanishes on odd powers of  $x$ , and takes the values

$$\langle L | x^{2m} \rangle = \frac{B(m + 1/2, \alpha + 1/2)}{B(1/2, \alpha + 1/2)} = 2^{-2m} \frac{(2m)!}{m! (\alpha + 1)_m}$$

on even powers. In order to apply Theorem 11 we have to find  $\mu(L)^{-1}$ , via

$$\begin{aligned} \tilde{L}(t) &= L(1 - xt)^{-\alpha} = \sum_{k \geq 0} \binom{2k + \alpha - 1}{2k} \langle L | x^{2k} \rangle t^{2k} \\ &= \sum_{k \geq 0} \binom{\alpha + 2k}{k} \frac{\alpha}{2k + \alpha} (t/2)^{2k} = 2^\alpha \left(1 + \sqrt{1 - t^2}\right)^{-\alpha}, \end{aligned}$$

and therefore  $\mu(L)^{-1} = M \left(2^{-\alpha} \left(1 + \sqrt{1 - t^2}\right)^\alpha\right)^\sim$ . We need to know how  $\mu(L)^{-1}$  acts on the basic sequence  $(g_n(x))$  for  $G$ , so we calculate its action on the generating function,

$$\begin{aligned} \mu(L)^{-1} C(2t/(1 + t^2)) e(xt) &= C(2t/(1 + t^2)) M \left(2^{-\alpha} \left(1 + \sqrt{1 - t^2}\right)^\alpha\right) e(xt) \\ &= 2^{-\alpha} \left(1 + \sqrt{1 - \left(\frac{2t}{1 + t^2}\right)^2}\right)^\alpha \left(1 - \frac{2xt}{1 + t^2}\right)^{-\alpha} \\ &= (1 + t^2 - 2xt)^{-\alpha}, \end{aligned}$$

and get the generating function for the Gegenbauer polynomials  $(\gamma_n(x))$ . Hence  $\mu(L)^{-1}g_n(x) = \gamma_n(x)$ , and  $\langle L | \gamma_n \rangle = \delta_{0,n}$ . Actually, a much stronger result holds,  $\langle L | \gamma_n \gamma_m \rangle = 0$  for all  $n \neq m$ , because  $(1 - x^2)^{\alpha-1/2}$  is the weight function on  $[-1, 1]$  for which the Gegenbauer polynomials are orthogonal [8]. Because of orthogonality, Gegenbauer polynomials also follow a three term recursion

$$n\gamma_n(x) = 2x(n + \alpha - 1)\gamma_{n-1}(x) - (n + 2\alpha - 2)\gamma_{n-2}(x)$$

which can be shown by comparing coefficients in the  $t$ -derivative of the Gegenbauer generating function  $(1 + t^2 - 2xt)^{-\alpha}$ .

By Theorem 11, the Sheffer polynomials  $(p_n)$  which follow the recurrence (17), agree with  $(2x)^n/n!$  for the first  $\ell$  degrees, and satisfy the condition  $\langle L | p_n \rangle = 0$  for all  $n \geq \ell$ , can be expanded as

$$p_n(x) = \sum_{i=0}^{\lfloor (\ell-1)/2 \rfloor} \frac{1}{i! (\alpha + 1)_i} \gamma_{n-2i}(x).$$

## 5.2 Functionals

Theorem 11 requires an explicit formula for  $\mu(L)^{-1}$ . We want to look at some diagonal examples in more detail. They can be used to find specific solutions in Section 4.1.

**Example 19 (Evaluation)** We saw already in Remark 7 that  $\mu(\varepsilon_a)^{-1} = e(a\Delta_e)^{-1}$ . With the help of Table 1 we calculate the following special diagonal cases:

**Exponential:** If  $e(xt) = e^{xt}$  then  $e(a\Delta_e)^{-1} = e^{-aD_x} = E^{-a}$ , where  $E^{-a} : f(x) \mapsto f(x - a)$ , the shift by  $-a$  (Taylor's formula).

**Binomial:** If  $e(xt) = (1 - xt)^{-\alpha}$  then the commutation rule (16) tells us that

$$\Delta_e^j = \left( \frac{\eta + 1}{\eta + \alpha} \chi \right)^j = \frac{(\eta + 1)_j}{(\eta + \alpha)_j} \chi^j = \frac{1}{(\eta + \alpha)_j} D_x^j.$$

Slightly abusing the hypergeometric series notation we can write  $e(a\Delta_e)^{-1} = F[-\alpha, \eta + 1; \eta + \alpha; a\chi]$ , or more precisely

$$e(a\Delta_e)^{-1} = (1 - a\Delta_e)^\alpha = \sum_{j \geq 0} \frac{(-\alpha)_j}{j!} \frac{(\eta + 1)_j}{(\eta + \alpha)_j} (a\chi)^j.$$

**Eulerian:** If  $e(xt) = 1/((1 - q)xt; q)_\infty$  then

$$\Delta_e^j = D_q^j = \left( \frac{1 - q^{\eta+1}}{1 - q} \chi \right)^j = \frac{(q^{\eta+1}; q)_j}{(1 - q)^j} \chi^j$$

and  $e(a\Delta_e)^{-1} = (a\chi; q)_\eta : x^n \mapsto (a/x; q)_n x^n$ , because of

$$\begin{aligned} e(a\Delta_e)^{-1} x^n &= ((1 - q) a D_q; q)_\infty x^n = \sum_{j=0}^n \frac{q^{\binom{j}{2}} (q - 1)^j a^j (q^{\eta+1-j}; q)_j}{(q; q)_j (1 - q)^j} x^{n-j} \\ &= x^n \sum_{j=0}^n (-1)^j q^{\binom{j}{2}} \frac{(q^{\eta+1-j}; q)_j}{(q; q)_j} (a/x)^j = (a/x; q)_n x^n \end{aligned}$$

by the  $q$ -binomial theorem.

**Example 20 (Average)** For the functional  $\int_0^1$  we found already  $\mu(\int_0^1)^{-1} = 1/\int_0^1 e(x\Delta_e)dx$  in Section 4.1. Some special cases:

**Exponential:** If  $e(xt) = e^{xt}$  then

$$\begin{aligned} \mu\left(\int_0^1\right)^{-1} &= \frac{D_x}{e^{D_x} - 1} = \sum_{k=0}^n \frac{B_k}{k!} D_x^k \\ \mu\left(\int_0^1\right)^{-1} x^n &= \sum_{k=0}^n \binom{n}{k} B_k x^{n-k} \end{aligned}$$

where  $B_k$  is the  $k$ -th Bernoulli number, and  $\mu(\int_0^1)^{-1}x^n/n!$  is the Bernoulli polynomial  $\varphi_n(x)$  [9, (1) § 85] (some authors define the Bernoulli polynomials as  $n!\varphi_n(x)$ ). We can now calculate the “delayed” Bernoulli polynomials, which we saw in the Introduction. If

$$\begin{aligned} p_n(x) &= \frac{x^n}{n!} - \sum_{i=\ell}^n \left( \int_0^1 \frac{x^i}{i!} dx \right) \frac{D_x}{e^{D_x} - 1} \frac{x^{n-i}}{(n-i)!} \\ &= \frac{x^n}{n!} - \sum_{i=\ell}^n \frac{1}{(i+1)!} \varphi_{n-i}(x) \end{aligned}$$

then  $p_n(x) = x^n/n!$  for the first  $\ell$  values of  $n$ , and  $\int_0^1 p_n(x) dx = 0$  for  $n \geq \ell$ .

**Binomial:** If  $e(xt) = (1 - xt)^{-\alpha}$  we must distinguish two cases. If  $\alpha = 1$  then

$$\begin{aligned} \widetilde{\int_0^1} (t) &= \int_0^1 (1 - xt)^{-1} dx = \frac{-1}{t} \ln(1 - t) \\ \mu(\widetilde{\int_0^1})^{-1} &= \frac{-\Delta_e}{\ln(1 - \Delta_e)} = - \sum_{j \geq 0} b_j \Delta_e^j = - \sum_{j \geq 0} b_j \chi^j \\ \mu(\widetilde{\int_0^1})^{-1} x^n &= - \sum_{j \geq 0} b_j x^{n-j} \end{aligned}$$

where  $b_0, b_1, \dots$  are the Bernoulli numbers of the second kind [9, (9) in par. 97].

If  $\alpha \neq 1, 0, -1, -2, \dots$  then

$$1/\widetilde{\int_0^1} (t) = \left( \int_0^1 (1 - xt)^{-\alpha} dx \right)^{-1} = \frac{t(1 - \alpha)}{1 - (1 - t)^{1-\alpha}} = \sum_{k \geq 0} \left( \sum_{j \geq 1} \binom{j + \alpha - 1}{j + 1} \frac{t^j}{1 - \alpha} \right)^k.$$

For example, if  $\alpha = 1/2$ , then  $1/\widetilde{\int_0^1} (t) = \frac{1}{2} (1 + \sqrt{1 - t})$ , and

$$\mu(\widetilde{\int_0^1})^{-1} = \frac{1}{2} \left( 1 + F \left[ -\frac{1}{2}, \eta + 1; \eta + \frac{1}{2}; \chi \right] \right).$$

## 6 Classical Sheffer Sequences

In this section we discuss some special results for classical Sheffer sequences, which means that the reference polynomials will always be  $e_n(x) = x^n/n!$ . Every classical basic sequence  $(b_n(x))$  has a generating function of the form

$$\sum_{n \geq 0} b_n(x) t^n = \exp(x\beta(t))$$

and is associated with the delta operator  $B = \beta^{-1}(D_x)$ . We need the following facts from the Finite Operator Calculus [4].

- All (classical) delta operators are translation invariant,

$$BE^{-c} = E^{-c}B,$$

and  $BE^{-c}$  is again a delta operator.

- If  $(s_n(x))$  is a  $B$ -Sheffer sequence and  $c$  and  $a$  are constants, then

$$\{s_n(cn + a + x)\} \text{ is a Sheffer sequence for } BE^{-c}. \quad (18)$$

- If  $(b_n(x))$  is the basic sequence for  $B$  then

$$\left(\frac{x - cn}{x}b_n(x)\right) \text{ is a } B\text{-Sheffer sequence.} \quad (19)$$

- It follows from the previous two results that

$$\left(\frac{x}{x + cn}b_n(x + cn)\right) \text{ is the basic sequence for } BE^{-c}. \quad (20)$$

- *Binomial Theorem for Sheffer Sequences* (apply Corollary 10 to  $L = \varepsilon_y$ ):

$$s_n(x + y) = \sum_{i=0}^n s_i(y)b_{n-i}(x). \quad (21)$$

**Example 21 (Zeroes along a boundary line)** *Suppose we want to find the  $B$ -Sheffer sequence  $(p_n)$  satisfying the conditions*

$$\begin{aligned} p_n(x) &= t_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1 \\ p_n(cn + \alpha) &= 0 \text{ for all } n = \ell, \ell + 1, \dots \end{aligned}$$

where  $(t_n)$  is a given  $B$ -Sheffer sequence, and  $c, \alpha$  are constants. Let  $\hat{p}_n(x) := p_n(cn + x)$  and  $\hat{t}_n(x) := t_n(cn + x)$ . We can rephrase the given problem as

$$\begin{aligned} \hat{p}_n(x) &= \hat{t}_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1 \\ \langle \varepsilon_\alpha \mid \hat{p}_n(x) \rangle &= 0 \text{ for all } n = \ell, \ell + 1, \dots \end{aligned}$$

We noted in (18) that the new polynomials are Sheffer polynomials for  $BE^{-c}$  in the classical setting. The delta operator  $BE^{-c}$  has the basic polynomials  $\hat{b}_n(x) = xb_n(x + cn)/(x + cn)$ . From  $\mu(\varepsilon_\alpha) = E^\alpha$  (see Section 5.2, or [3, p. 128]) and Theorem 11 follows

$$\hat{p}_n(x) = \hat{t}_n(x) - \sum_{i=\ell}^n \langle \varepsilon_\alpha \mid \hat{t}_i(x) \rangle E^{-\alpha} \hat{b}_{n-i}(x),$$

and

$$\begin{aligned}
p_n(x) &= t_n(x) - \sum_{i=\ell}^n t_i(ci + \alpha) \hat{b}_{n-i}(x - \alpha - cn) \\
&= t_n(x) - \sum_{i=\ell}^n t_i(ci + \alpha) \frac{x - \alpha - cn}{x - \alpha - ci} b_{n-i}(x - \alpha - ci) \\
&= \sum_{i=0}^{\ell-1} t_i(ci + \alpha) \frac{x - \alpha - cn}{x - \alpha - ci} b_{n-i}(x - \alpha - ci).
\end{aligned}$$

**Example 22** Suppose  $(s_n)$  is a  $B$ -Sheffer sequence,  $c \neq 0$  a given constant,

$$\begin{aligned}
s_n(x) &= t_n(x) \text{ for } n = 0, \dots, \ell - 1 \\
s_n(cn) &= \sum_{i=0}^{n-1} (-1)^i c^{i+1} s_{n-i-1}(cn) \text{ for all } n \geq \ell
\end{aligned}$$

Again, the side condition can be written as

$$0 = \sum_{i=0}^n (-1)^i c^i s_{n-i}(cn) = \left( \frac{1}{1 + cB} s_n \right) (cn) = p_n(cn)$$

if we define the  $B$ -Sheffer sequence  $(p_n)$  by  $p_n := \frac{1}{1+cB} s_n$ . From the previous example,  $(\alpha = 0)$  we see that

$$p_n(x) = \sum_{i=0}^{\ell-1} \left\langle \varepsilon_0 \left| \frac{1}{(1 + cB)} t_i(ci + x) \right. \right\rangle \frac{x - cn}{x - ci} b_{n-i}(x - ci)$$

and

$$s_n(x) = \sum_{i=0}^{\ell-1} \left( \frac{x - cn}{x - ci} b_{n-i}(x - ci) + c \frac{x - c(n-1)}{x - ci} b_{n-i-1}(x - ci) \right) \sum_{j=0}^i (-1)^j c^j t_{i-j}(ci).$$

**Example 23 (Recursive initial values)** Using the same notation as in example 21 we rephrase the recursive initial value problem

$$\begin{aligned}
p_n(x) &= t_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1 \\
p_n(cn + a) &= p_{n-1}(c(n-1) + b) \text{ for all } n = \ell, \ell + 1, \dots
\end{aligned}$$

as

$$\begin{aligned}
\hat{p}_n(x) &= \hat{t}_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1 \\
\langle \varepsilon_a | \hat{p}_n(x) \rangle - \langle \varepsilon_b | \hat{p}_{n-1}(x) \rangle &= 0 \text{ for all } n = \ell, \ell + 1, \dots
\end{aligned} \tag{22}$$

With  $e(a, \Delta_e) = E^a$  and (14) we get

$$\hat{p}_n(x) = \hat{t}_n(x) - \sum_{i=\ell}^n (\hat{t}_i(a) - \hat{t}_{i-1}(b)) \frac{E^{-a}}{1 - E^{b-a} B E^{-c}} \hat{b}_{n-i}(x).$$

Expanding

$$\begin{aligned} \frac{1}{1 - E^{b-a}BE^{-c}}\hat{b}_{n-i}(x) &= \sum_{j=0}^{n-i} E^{j(b-a)}\hat{b}_{n-i-j}(x) = \sum_{j=0}^{n-i} \hat{b}_{n-i-j}(x + j(b-a)) \\ &= \sum_{j=0}^{n-i} \frac{x + j(b-a)}{x + j(b-a) + c(n-i-j)} b_{n-i-j}(x + j(b-a) + c(n-i-j)) \end{aligned}$$

we get

$$\begin{aligned} p_n(x) &= \hat{p}_n(x - cn) = t_n(x) - \sum_{i=\ell}^n (t_i(a + ci) - t_{i-1}(b + c(i-1))) \\ &\quad \times \sum_{j=0}^{n-i} \frac{x - a + j(b-a) - cn}{x - a + j(b-a-c) - ci} b_{n-i-j}(x - a + j(b-a-c) - ci). \end{aligned}$$

Problems like this occur in the enumeration of lattice paths taking unit steps  $\rightarrow, \uparrow$  with weighted left turns. A path takes a left turn  $\rightarrow \overset{\uparrow}{\circ}$  at the lattice point  $(i, j)$  if it enters that point in a horizontal step  $\rightarrow$  and leaves it in a vertical step  $\uparrow$ ; the weight  $\mu$  is assigned to this left turn. The product of the weights is the total weight of the path, from  $(0, 0)$  to  $(n, m)$ , say. The number of weighted paths is denoted by  $d_n(m)$ ; we can think of  $d_n(m)$  as the generating function (in  $\mu$ ) of the number of left turns in lattice paths from  $(0, 0)$  to  $(n, m)$ . It is well known that  $d_n(m) = \sum_{j \geq 0} \binom{n}{j} \binom{m}{j} \mu^j$ ; we extend  $d_n(m)$  to  $d_n(x)$ , a polynomial with coefficients in  $\mathbb{R}[\mu]$ . It is shown in [10] that  $(d_n)$  is a Sheffer sequence for the delta operator  $\Delta/(\mu + \Delta)$ , where  $\Delta : f(x) \mapsto f(x+1) - f(x)$  is the forward difference operator. Suppose the path has to stay strictly above the line  $y = c(x - \ell)$  where  $c$  and  $\ell$  are given positive integers.

Lattice paths above $y = 2(x - 4)$							
$m$							
4	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$
3	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$
2	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$
1	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$	$\Rightarrow$	$\overset{\uparrow}{\circ}$	$\Rightarrow$	$\overset{\uparrow}{\circ}$
0	$\overset{\uparrow}{\circ}$	$\Rightarrow$	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$	$\rightarrow$	$\overset{\uparrow}{\circ}$
	0	1	2	3	4	5	6 ... $\rightarrow n$

The sample path  $\Rightarrow, \uparrow$  from  $(0, 0)$  to  $(5, 4)$  has weight  $\mu^2$ .

If  $r_n(m)$  denotes the generating function of left turns in the restricted paths, we obtain the recursive initial value problem

$$\begin{aligned} r_n(x) &= d_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1 \\ r_n(c(n - \ell) + 1) &= r_{n-1}(c(n - \ell) + 1) \text{ for all } n = \ell, \ell + 1, \dots \end{aligned}$$

(see [10] for details).

**Example 24 (Paper Towels Continued)** In Example 3 we had to find the Sheffer sequence  $(s_n)$  associated with the basic polynomials  $b_n(x) = \frac{x}{x+2n} \binom{x+2n}{n}$  that satisfies a recursive initial condition of the form (22),  $\langle \varepsilon \mid s_n(x) \rangle - \langle p^{-1}\varepsilon_1 \mid s_{n-1}(x) \rangle = 0$ , with  $\ell = 1$ . Hence  $s_n(x) = \sum_{k \geq 0} p^{-k} b_{n-k}(x+k)$  for  $n \geq m \geq 1$  and even  $n+m$ . Finally,

$$\begin{aligned} \Pr(R_N = r) &= p\pi_{2N-r-1, r-1} = p^{N-1}q^{N-r} s_{N-r}(r-1) \\ &= q^{N-r} \sum_{k=0}^{N-r} p^{k+r-1} \frac{N-k-1}{N+k-1} \binom{N+k-1}{k} \end{aligned}$$

for  $N \geq r > 1$ , but it is easily verified from  $\Pr(R_N = 1) = \pi_{2N-2, 0} = q\pi_{2N-3, 1}$  that this formula also holds for  $r = 1$ .

**Example 25** Suppose  $(p_n)$  is the classical B-Sheffer sequence such that

$$p_n(x) = t_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1$$

$$\int_0^1 p_n(y + cn) dy = 0 \text{ for all } n = \ell, \ell + 1, \dots$$

As before, we formulate this condition as

$$\hat{p}_n(x) = \hat{t}_n(x) \text{ for all } n = 0, 1, \dots, \ell - 1$$

$$\int_0^1 \hat{p}_n(y) dy = 0 \text{ for all } n = \ell, \ell + 1, \dots$$

using the same notation as in Example 21. Combining Examples 15 and 20 we find

$$\begin{aligned} \hat{p}_n(x) &= \hat{t}_n(x) - \sum_{i=\ell}^n \left( \int_0^1 \hat{t}_i(y) dy \right) \sum_{k=0}^{n-i} \frac{B_k}{k!} D_x^k \hat{b}_{n-i}(x) \\ p_n(x) &= t_n(x) - \sum_{i=\ell}^n \left( \int_0^1 t_i(ci + y) dy \right) \sum_{k=0}^{n-i} \frac{B_k}{k!} D_x^k \frac{x - cn}{x - ci} b_{n-i}(x - ci) \end{aligned}$$

We saw already in Example 20 how the Bernoulli polynomials  $\varphi_n(x)$  enter the solution when  $b_n(x) = x^n/n!$ . In this special case

$$\begin{aligned} p_n(x) &= t_n(x) - \sum_{i=\ell}^n \left( \int_0^1 t_i(ci + y) dy \right) \mu \left( \int_0^1 \right)^{-1} (x - cn) \frac{(x - ci)^{n-1-i}}{(n-i)!} \\ &= t_n(x) - \sum_{i=\ell}^n \left( \int_0^1 t_i(ci + y) dy \right) (\varphi_{n-i}(x - ci) - c\varphi_{n-1-i}(x - ci)). \end{aligned}$$

In the Introduction we mentioned (for  $c = 0$ ) the two-stage condition  $\int_{-1}^0 p_n(x) = \delta_{0,n}$  for  $n = 0, \dots, \ell - 1$ , and  $\int_0^1 p_n(cn + x) = 0$  for  $n \geq \ell$ . From  $\int_{-1}^0 \varphi_n(x + 1) = 0$  follows  $p_n(x) = \varphi_n(x + 1) =: t_n(x)$  for  $n = 0, \dots, \ell - 1$ , and therefore

$$\begin{aligned} p_n(x) &= \sum_{i=0}^{\ell} \left( \int_0^1 \varphi_i(ci + y + 1) dy \right) (\varphi_{n-i}(x - ci) - c\varphi_{n-1-i}(x - ci)) \\ &= \sum_{i=0}^{\ell} \frac{(ci + 1)^i}{i!} (\varphi_{n-i}(x - ci) - c\varphi_{n-1-i}(x - ci)). \end{aligned} \tag{23}$$

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