

# A Formula for Explicit Solutions of Certain Linear Recursions on Polynomial Sequences

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This paper appeared in CONGRESSUS NUMERANTIUM, 49 (1985), pp.87-98, and was retyped in  $\text{\LaTeX}$  to make it available on the internet.

## 1 Introduction

A central question of the Umbral Calculus is the explicit calculation of connection coefficients between two polynomial sequences. Solutions to this problem have immediate applications to solving certain linear recurrences. For example, let  $(p_n)$  be a polynomial sequence, i.e.  $\deg(p_n) = n$  and  $p_0 \neq 0$ , such that

$$p_n(x) = p_n(x-1) + 2p_{n-1}(x) + p_{n-1}(x-1) - p_{n-3}(x) \quad (1)$$

for all  $n \geq 1$ . Such a recursion describes the number of arrangements of  $n$  dumbbells on a  $2*(n-1+x)$  array of compartments (McQuistan and Lichtman, [2, 1970]). Umbral Calculus can be helpful in solving such a recursion, if we find a suitable, known polynomial sequence to connect to  $(p_n)$ .

The problem becomes usually more transparent if we introduce the associated linear operators. In our example, define  $P$  by  $Pp_n = p_{n-1}$  for all  $n \geq 0$  ( $p_m = 0$  for  $m < 0$ ). Furthermore, we make use of the "shift" operator

$$E^a f(x) = f(x+a),$$

and the backwards difference operator  $\nabla = 1 - E^{-1}$ . Now we can write the recursion (1) as

$$\nabla = (2 + E^{-1})P - P^3. \quad (2)$$

The purpose of this paper is to establish a formula which explicitly solves the recurrence relation by transforming the connection between the delta operators  $\nabla$  and  $P$  into a connection between their basic sequences  $\left(\binom{n-1+x}{n}\right)_{n \geq 0}$  and  $(p_n)$ . The tools for this process are the Transfer Formula of the Umbral Calculus (cf. Roman, 1982, [4, Section 7]) and a generalized Lagrange-Bürmann inversion formula of Barnabei, Brini and Nicoletti (1982) [1]. The solution  $(p_n)$  which we obtain is a so called "basic sequence", satisfying the side conditions  $p_n(0) = \delta_{0,n}$  for all  $n$ . In the above combinatorial problem we actually have to calculate a solution  $(s_n)$  under the side conditions  $s_0(0) = 1$  and  $s_n(0) = 2(-1)^n$  for  $n \geq 1$ . How to

get from  $(p_n)$  other solutions satisfying different side conditions has been discussed in an earlier paper (Niederhausen, 1980 [3]) (see also Section 5). In general, we are investigating recursions which are described by  $R = A_1Q + A_2Q^3 + A_3Q^3 + \dots$  where  $R$  and  $Q$  are delta operators and  $A_i \in K[[Q]]$  for  $i \geq 0$ ,  $A_1$  invertible (see Section 2 for definitions). We derive an applicable formula (10) for the special case  $(b, c \geq 1)$   $R = AQ + BQ^b + CQ^{b+c}$ . To further illustrate the applications, we show in Section 5 how to solve the recursions

$$g'_n(x) = 2\lambda g_{n-1}(x) + 2xg'_{n-1}(x) - g'_{n-2}(x), \quad (3)$$

and

$$t_n(x) = t_n(x-1) + t_{n-1}(x) + t_{n-b}(x) + t_{n-b-c}(x). \quad (4)$$

## 2 Definitions and basic theorems

Let  $K$  be a field of characteristic 0, and  $K[x]$  the algebra of polynomials in  $x$  over  $K$ . We single out one linear operator  $\tau$  on  $K[x]$ , which we define by  $\tau x^n / c_n = x^{n-1} / c_{n-1}$  for all  $n \geq 1$ ,  $\tau 1 = 0$ , where  $c_0, c_1, \dots$  is a given sequence of nonzero constants. The choice of these constants depends on the applications.

Examples:

- (a)  $c_i = i!$  implies  $\tau = D$ , the derivative operator
- (b)  $c_i = \binom{-\lambda}{i}$  implies  $\tau = (\lambda + xD)^{-1}D$  (Roman, (1982), [4, (9.7)]).

A formal power series  $Q(t) \in K[[t]]$  is of the form  $\sum_{i \geq 0} k_i t^i$  for some formal parameter  $t$ . By choosing  $t = \tau$  we can define a linear operator  $Q = \tilde{Q}(\tau)$  on  $K[x]$  as follows,

$$Q \frac{x^n}{c_n} = \sum_{i \geq 0} k_i \tau^i \frac{x^n}{c_n} = \sum_{i=0}^n k_i \frac{x^{n-i}}{c_{n-i}}.$$

It is not difficult to see that products of formal power series correspond to composition of operators (Roman (1982), [4, Chapter 4]), so that  $K[[\tau]]$  is a commutative integral domain of operators. Sometimes it can be preferable to distinguish between the interpretation of  $Q$  as an operator or as a formal power series. To stress the operator point of view we also write  $\Sigma_\tau$  for  $K[[\tau]]$ . Observe that  $\tau = D$  gives  $\Sigma = \Sigma_D$ , the shift-invariant operators of the classical "Finite Operator Calculus" (Rota et al., [5]).

By  $U_\tau$  we denote the group of units in  $\Sigma_\tau$ . In this view,  $U_\tau$  is the group of operators  $Q$  in  $\Sigma_\tau$  which have a (compositional) inverse  $Q^{-1}$ . In  $K[[t]]$ ,  $U_t$  is the group of power series  $Q(t)$  which admit an (algebraic) reciprocal  $Q(t)^{-1}$ . Hence,  $Q(t) = \sum_{i \geq 0} k_i t^i$ , where  $k_0 \neq 0$ .

A power series  $Q(t) = \sum_{i \geq 0} k_i t^i$  has a (formal) compositional inverse  $\tilde{Q}(t)$  such that

$$Q(\tilde{Q}(t)) = t$$

iff  $k_0 = 0$  and  $k_1 \neq 0$ . Such an invertible series is called a delta series, and the corresponding operator  $Q(\tau)$  is a delta operator. The delta operators generate  $\Sigma_\tau$  in the sense that for any

delta operator  $Q \in \Sigma_\tau$  the integral domains  $\Sigma_\tau$  and  $\Sigma_Q$  are isomorphic. Also very important for the Umbral Calculus is the characterization of delta operators as the center of  $\Sigma_\tau$  (cf. Roman, 1982, [4, Chapter 4]).

Let  $Q$  be a delta operator in  $\Sigma_\tau$  and  $T = \sum_{i \geq 0} k_i Q^i$  be an element in  $\Sigma_Q$ . Let  $(a_{i,j})_{i,j \geq 0}$  be any double sequence in  $K$  such that  $k_i = \sum_{n=0}^i a_{n,i-n}$ . Of course, such a double sequence is not unique. We obtain

$$T = \sum_{i \geq 0} \sum_{n=0}^i a_{n,i-n} Q^i = \sum_{n \geq 0} Q^n \sum_{i \geq 0} a_{n,i} Q^i.$$

The operators  $A_n = \sum_{i \geq 0} a_{n,i} Q^i$  are also elements of  $\Sigma_Q$  and therefore commute with  $Q$ . Thus

$$T = \sum_{n \geq 0} A_n Q^n$$

i.e., every  $T \in \Sigma_Q$  can be written as a formal power series in  $Q$  with coefficients in  $\Sigma_Q$  and vice versa. This representation is not unique.

If  $T$  is a delta operator itself, we need  $k_0 = 0$  and  $k_1 \neq 0$ , hence

$$a_{0,0} = k_0 = 0$$

and

$$a_{0,1} + a_{1,0} = k_1 \neq 0.$$

It will be convenient to choose  $A_0 = 0$  in this case, so that  $A_1$  becomes a unit in  $\Sigma_Q$ .

### 3 Laurent series

Let  $Q$  and  $R$  be two delta operators in  $\Sigma_\tau$ . In the previous chapter we have seen that  $R$  can be understood as a formal power series in  $\Sigma_Q[[Q]]$ . We want to invert this series, i.e., express  $Q$  in terms of  $R$ . For this purpose it is convenient to view  $R$  as a Laurent series over  $\Sigma_Q$ .

Let  $\mathbb{Z}$  be the ring of integers, and  $A$  be a commutative integral domain. In our application,  $A = \Sigma_Q$ . A Laurent sequence  $\alpha = (A_i)$  of degree  $n$  with  $A_i \in A$  for every  $i \in \mathbb{Z}$  has to satisfy the condition

$$A_i = 0 \text{ for all } i < n, \text{ and } A_n \neq 0.$$

The set of all Laurent series is denoted by  $\mathcal{L}^+$  (suppressing the dependence on  $A$  for notational convenience). Delta operators  $R = \sum_{i \geq 0} A_i Q^i$  give rise to Laurent series of degree 1. We also noted that  $A_1$  is a unit in such a representation. Hence  $R$  has a reciprocal in  $\mathcal{L}^+$ , and therefore belongs to  $\mathcal{I}^+$ , the set of generators of  $\mathcal{L}^+$ . Of course, this reciprocal is no longer an operator in  $\Sigma_\tau$ .

The generalized Lagrange Birrmann inversion formula of Barnabei, Brini and Nicoletti (1982) [1] can be stated as follows. For every  $R \in \mathcal{I}^+$  and  $B \in \mathcal{L}^+$ , we have

$$((\beta \circ \tilde{R}) D \tilde{R}) = \sum_{i \in \mathbb{Z}} \text{Res}(R^{-i-1} \beta) t^i, \quad (5)$$

where

$$D \sum_{i \in \mathbb{Z}} A_i t^i = \sum_{i \in \mathbb{Z}} i A_i t^{i-1}.$$

Using that

$$1 = D(R \circ \tilde{R}) = ((DR) \circ \tilde{R}) D\tilde{R}$$

we get

$$((\beta \circ \tilde{R}) D\tilde{R}) = (\beta \circ \tilde{R}) ((DR) \circ \tilde{R})^{-1} = (\beta \circ \tilde{R}) ((DR)^{-1} \circ \tilde{R}) = (\beta(DR)^{-1}) \circ \tilde{R}.$$

Hence, we obtain (5) in the better known form

$$\beta \circ \tilde{R} = \sum_{i \in \mathbb{Z}} Res(R^{-i-1} \beta DR) t^i \quad (6)$$

if we replace  $\beta$  by  $\beta DR$  in (5).

As a first application, we use Lagrange inversion to express  $Q$  in terms of  $A$ .

**Lemma 1** *Let  $R \in \Sigma_\tau$  be a delta operator such that*

$$R = \sum_{i \geq 1} A_i Q^i$$

*for  $Q, A_i \in \Sigma_\tau$ ,  $i \geq 1$ ,  $A_1$  invertible, then  $Q$  is also a delta operator, and*

$$Q = \sum_{i \geq 1} Res(R^{-i-1} Q DR) R^i. \quad (7)$$

**Proof.** Apply the inversion formula (6) to  $R(t)$ , using  $\beta = t$ ,

$$R(t) = \sum_{i \in \mathbb{Z}} Res(R^{-i-1} t DR) t^i.$$

From  $deg(R) = 1$  we see that the range of the summation can be restricted to the positive integers. Now replace  $t$  by  $Q$  and calculate  $R(\tilde{R}) = Q$  to obtain (7). In (7), we get for  $i = 1$  the coefficient  $Res(R^{-2} Q DR) = A_1^{-2} A_1 = A_1^{-1}$  of  $R^1$  in  $Q(R)$ , which shows that  $Q$  is also a delta operator in  $\Sigma_\tau$ . ■

## 4 The connection coefficients

Let  $(q_n)$  and  $(r_n)$  be the basic sequences of the delta operators  $Q$  and  $R$ , resp. From the Transfer Formula (cf. Roman, 1982 [4, chapter 7]) follows a representation of  $q_n$  in terms of  $r_n$

$$q_n = \theta_\tau(Q R^{-1}(Q))^{-n} \theta_\tau^- r_n, \quad (8)$$

where the umbral shift associated to  $\tau$  is denoted by  $\theta_\tau$

$$\theta_\tau \frac{x^n}{c_n} = (n+1) \frac{x^{n+1}}{c_{n+1}} \text{ for } n \geq 0.$$

$\theta_\tau^-$  stands for the left inverse of  $\theta_\tau$

$$\theta_\tau^- \frac{x^n}{c_n} = \frac{1}{n} \frac{x^{n-1}}{c_{n-1}} \text{ for } n \geq 1, \text{ and}$$

$$\theta_\tau^- 1 = 0.$$

**Remark 2**  $R^{-1}(Q)$  in (8) does not represent a linear operator, but a Laurent series of degree  $-1$ . The product  $QR^{-1}(Q)$  has degree 0 and corresponds to an invertible operator in  $\Sigma_\tau$ .

Examples for  $\theta_\tau$ :

- (a) From  $\theta_D \frac{x^n}{n!} = \frac{x^n}{n!}$  we get  $\theta_D = X$  (multiplication by  $x$ ), and  $\theta_D^- = X^-$ , where  $X^- x^n = x^{n-1}$  for  $n \geq 1$ .
- (b) Let  $\tau = -(\lambda + xD)^{-1}D$ . Instead of deriving  $\theta_\tau$  directly from the definition we use that always  $\theta_\tau \tau = XD$  (Roman, 1982, [4, p. 79]). Therefore, we find from  $(\lambda + xD)\tau = -D$  that  $\theta_\tau = -x(\lambda + xD)$  (see also Roman, 1982, [4, p. 96]).

To determine the operator

$$(QR^{-1}(Q))^{-n} = Q^{-n}R^n$$

in (8) we apply again the inversion formula (6), but now with  $\beta = t^{-n}$ ,

$$(\tilde{R}(t))^{-n} = \sum_{i \in \mathbb{Z}} \text{Res}(R^{-i-1}t^{-n}DR)t^i,$$

and using  $t = Q$

$$Q^{-n}R^n = (\tilde{R}(R))^{-n}R^n = \sum_{i \in \mathbb{Z}} \text{Res}(R^{-i-1}Q^{-n}DR)R^{i+n}.$$

**Corollary 3** If the delta operator  $R \in \Sigma_\tau$  can be written as  $R(Q) = \sum_{i \geq 1} A_i Q^i$  such that  $R(t) \in \mathcal{I}^+$ , then  $Q$  is also a delta operator, and the basic sequence  $(q_n)$  of  $Q$  can be expressed in terms of the basic sequence  $(r_n)$  of  $R$  as follows

$$q_n = \theta_\tau \sum_{i \geq 0} \text{Res}(R^{-i-1}Q^{-n}DR)R^i \theta_\tau^- r_n. \quad (9)$$

Finally, we explicitly calculate the connection coefficients for the case  $R = AQ + BQ^b + cQ^{b+c}$  where  $A$  is invertible, and  $b$  and  $c$  are positive integers. One gets from (9)

$$q_n = n\theta_\tau \sum_{j,k \geq 0} \frac{(n - jb - kc + 1)_{j-1}}{k!(j-k)!} A^{n-jb-kc} B^{j-k} C^k Q^{j(b-1)+kc} \theta_\tau^- r_n \quad (10)$$

where  $(x)_j = x(x+1)_{j-1}$ , and  $(x+1)_{-1} = 1/x$ . The simple  $R = AQ + BQ^b$  can be obtained from (10) by keeping  $k = 0$ .

## 5 Examples

- (a) The introductory example of the enumeration of dumbbells gave rise to the equation (2)  $\nabla = (2 + E^{-1})P - P^3$ . We want to apply formula (10) to calculate the basic sequence  $(P_n)$  of  $P$ .  $\nabla$  and  $E^{-1}$  are both in  $\Sigma_D$ , so we choose  $\tau = D$ , hence  $\theta_\tau = X$ . Of course,  $A = 2 + E^{-1}$ , but we can also write  $A$  as  $A = 3 - \nabla$ , or  $A = 2\nabla + 3E^{-1}$ . These three forms of  $A$  produce three equivalent solutions for  $p_n$ . Using that  $R = \nabla$  has the basic polynomials  $r_n(x) = \binom{n-1+x}{n}$ , we get for  $A = 3 - \nabla$

$$p_n(x) = (-1)^n \sum_{i,j \geq 0} \binom{n-2j-1}{n-3j-i} \binom{-x-j}{i} \binom{j+x-1}{j} 3^i.$$

The given combinatorial problem is not solved by this basic solution. Instead, we have to find a Sheffer sequence  $(s_n)$  for  $P$ , which also follows the recursion (1), and satisfies the initial condition  $s_n(1) = f_{n+1}$  the  $n+1$ -th Fibonacci number (McQuistan and Lichtman, 1970 [2]). It follows immediately from (1) that  $s_n(0) = 2(-1)^n$  for  $n \geq 1$ , and  $s_0(0) = 1$ . Using these initial values and the Binomial Theorem for Sheffer sequences (Rota et al., 1973, [5, p. 703]) we get

$$\begin{aligned} s_n(x) &= \sum_{k=0}^n s_{n-k}(0)p_k(x) \\ &= (-1)^n \sum_{\substack{0 \leq j \leq n/3 \\ 0 \leq i \leq n-3j}} \binom{n-2j}{i+j} \binom{-x-j}{i} \binom{j+x-1}{j} 3^i \left(2 - \frac{i+j}{n-2j}\right) \end{aligned}$$

for  $n \geq 0$ . The choice  $x = 1$  expresses the Fibonacci numbers as

$$f_n = s_{n-1}(1) = \sum_{j \geq 0} \binom{n-1-2j}{j} (-1)^j \frac{n-1-j}{n-1-2j} 2^{n-2-3j}.$$

- (b) In terms of linear operators, the recursion (3) can be written as

$$D = 2\lambda G + 2XDG - DG^2, \tag{11}$$

where  $Gg_n = g_{n-1}$  for all  $n \geq 0$ . The operator  $2(\lambda I + XD)$  is not in  $\Sigma_D$ . Thus, the choice  $\tau = D$  will not work. But (11) can be formulated as

$$\tau = -G - \tau G^2$$

with  $\tau = -(\lambda I + xD)^{-1}D$ .  $(\binom{-\lambda}{n}x^n)_{n \geq 0}$  is the basic sequence for  $\tau$ , and  $\theta_\tau = -X(\lambda I + XD)$ . (See example (b) in chapters 2 and 4.) Hence, we obtain from formula (10)

$$g_n(x) = \sum_{j \geq 0} \binom{-\lambda}{n-2j} \binom{2j-n}{j} (-2x)^{n-2j}.$$

The Sheffer sequence  $(s_n)$  for  $G$  with initial values  $s_n(0) = \binom{-\lambda}{n/2}$  if  $n$  is even, and 0 else, has the representation

$$s_n(x) = \sum_{i \geq 0} \binom{-\lambda}{i} g_{n-2i}(x),$$

which is equal to  $P_n^{(-\lambda)}(x)$ , the Gegenbauer polynomials.

- (c) Consider a random walk on the integers  $\mathbb{Z}$ . For some positive integers  $b$  and  $c$ , we define the transition probabilities

$$p_b = P_{i,i+b-1}, \quad p_c = P_{i,i+b+c-1},$$

and

$$q = 1 - p_b - p_c = P_{i,i-1},$$

for all  $i \in \mathbb{Z}$ . All non-positive states, and all states  $i \geq N$  for some given positive integer  $N$ , are assumed to be absorbing. We want to calculate the probability  $q(n)$  of entering the non-positive states (the gambler's ruin), given the random walk is in state  $n$ . Of course,  $q(0) = 1$ , and  $q(n) = 0$  for all  $n \geq N$ . Furthermore,  $q(n)$  follows the recursion

$$q(n) = p_b q(n+b-1) + p_c q(n+b+c-1) + q q(n-1).$$

Let  $t_n = q(N-1-n)$ . Hence,

$$t_n = \frac{1}{q} t_{n-1} - \frac{p_b}{q} t_{n-b} - \frac{p_c}{q} t_{n-b-c}, \quad (12)$$

and  $t_n = 0$  for  $n < 0$ ,  $t_{N-1} = 1$ . To solve this recursion, we view the numbers  $t_n$  as values  $t_n(1) = t_n$  of a polynomial sequence, such that

$$t_n(x) - t_n(x-1) = \frac{1}{q} t_{n-1}(x) - \frac{p_b}{q} t_{n-b}(x) - \frac{p_c}{q} t_{n-b-c}(x). \quad (13)$$

To comply with (12), we have to impose the side conditions  $t_n(0) = 0$  for all  $1 \leq n \leq N-1$ , and  $t_{N-1}(1) = 1$ . In terms of operators, (13) becomes

$$\nabla = \frac{1}{q} H - \frac{p_b}{q} H^b - \frac{p_c}{q} H^{b+c},$$

where  $H t_n = t_{n-1}$  for all  $n \geq 0$ . First, we find the basic sequence  $(h_n)$  for  $H$  from formula (10), using  $\tau = D$ ,  $R = \nabla$  and  $r_n(x) = \binom{n-1+x}{n}$

$$h_n(x) = \sum_{j,k \geq 0} \binom{x+k-1}{k} \binom{x+j-1}{j-k} \binom{-x-j}{n-jb-kc} (p_b + p_c - 1)^{j(b-1)+kc-n} p_b^{j-k} p_c^k.$$

This basic sequence satisfies already the side condition  $h_n(0) = 0$  for all  $n \geq 1$ . So we find  $(t_n)$  as the Sheffer sequence

$$t_n(x) = h_n(x)/h_{N-1}(1) \quad \text{for all } n \geq 0.$$

Finally,

$$q(n) = t_{N-1-n} = t_{N-1-n}(1) = h_{N-1-n}(1)/h_{N-1}(1).$$

**Acknowledgement** I want to thank Jack Freeman, Florida Atlantic University, for many helpful discussions. The paper "Unimodality properties of triangular arrays" by E. R. Canfield [Congr. Num. 43 (1984)], introduced me to the dumbbell example.

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