

m														
7							1		8		44		208	
6						1		7		35		154		
5					1		6		27		110		423	
4				1		5		20		75		270		
3			1		4		14		48		161		536	
2		1		3		9		28		87		273		
1	1		2		5		14		40		118		357	
0	1	1	2		5		13		36		104			
$n :$	0	1	2	3	4	5	6	7	8	9	10	11	12	13

The number of Dyck paths avoiding $dddd$

The two tables indicate the differences between the two problems, both starting out from equal counts on the x -axis ($m = 0$). Because only points (n, m) with $n + m = 0 \pmod{2}$ can be reached by a Dyck path, we consider the lattice points $(2n + b, 2m + b)$, for $b = 0, 1$. We first show that the number of Dyck paths to $(2n + b, 2m + b)$ avoiding d^r equals

$$Dyck(2n + b, 2m + b; d^r) = \frac{2m + b + 1}{n + m + b + 1} \binom{n + m + b + 1}{n - m}_r,$$

where the Euler coefficient [2] is denoted by

$$\binom{n + m + b + 1}{n - m}_r = \sum_{i=0}^{\lfloor (n-m)/r \rfloor} (-1)^i \binom{n + m + b + 1}{i} \binom{2n + b - ri}{n - m - ri}$$

(see Definition 4 and expansion (12)). More about Euler coefficients can be found in Section 4. For given m , the number of Dyck paths

$Dyck(n + m, m - n; d^r)$ to $(n + m, m - n)$ avoiding d^r has the generating function (over n) $(1 - t^r)^m (1 - t)^{-m-2} (rt^r(1 - t) - (1 - t^r)(2t - 1))$, as shown in (11). Note that the coefficient of t^m in this generating function equals the original $Dyck(2m, 0; d^r)$.

Next we show that the number of Dyck paths to $(2n + b, 2m + b)$ avoiding u^r equals

$$D(2m + b, 2m + b; u^4) = \sum_{i=0}^{2m+b-1} \frac{1}{n + m + b + 1 - i} \binom{i - 2m - b}{i}_r \binom{n + m + b + 1 - i}{n + m + b - i}_r,$$

except for the original Dyck path counts to $(2n, 0)$, which either must be gotten from those to $(2n - 1, 1)$, or from the Dyck paths to $(2n, 0)$ avoiding

d^r . The case $r = 4$ seems to be very special. We conjecture in Section 3.1 that in this case the generating function for the Dyck paths equals

$$\begin{aligned} & \sum_{n \geq 0} Dyck(4m - n - 1, 2m - n + 1; u^4) \\ &= \left(3 + t - \sqrt{(1+t)^2 + 4t^3} \right) \left(\frac{1-t^4}{1-t} \right)^m / 2, \end{aligned}$$

$$\text{hence } Dyck(2m, 0; u^4) = [t^{2m}] \left(3 + t - \sqrt{(1+t)^2 + 4t^3} \right) \left(\frac{1-t^4}{1-t} \right)^m / 2.$$

Throughout the following sections we will discuss ballot paths (weakly above $y = x$), with steps \uparrow and \rightarrow , instead of Dyck paths. The transformations $D(n, m) = Dyck(n + m, m - n)$ and $Dyck(2n + b, 2m + b) = D(n - m, n + m + b)$, with $D(n, m)$ counting ballot path to (n, m) , go back and forth between the two equivalent setups. Of course, the pattern u^r becomes the pattern \uparrow^r , or N^r , and d^r becomes \rightarrow^r , or E^r .

2 Ballot paths without the pattern \rightarrow^r

Definition 1 $s_n(m; r) = s_n(m)$ is the number of $\{\uparrow, \rightarrow\}$ paths staying weakly above the diagonal $y = x$ from $(0, 0)$ to $(n, m) \in \mathbb{Z}^2$ avoiding a sequence of $r > 0$ consecutive \rightarrow steps. We get $s_0(m) = 1$ for all $m \geq 0$. We set $s_n(m) = 0$ if $n < 0$ or if $m + 1 = n > 0$.

Lemma 2 The following recurrence holds for all $m \geq n > 0$:

$$s_n(m) = s_{n-1}(m) + s_n(m-1) - s_{n-r}(m-1). \quad (1)$$

Proof: The number of paths reaching (n, m) is obtained by adding the number of paths reaching $(n-1, m)$ and $(n, m-1)$, but subtracting paths that would have exactly $r \rightarrow$ steps. Those forbidden steps occur necessarily at the end of the path, so they are preceded by an up step, and must come from $(n-r, m-1)$. ■

We now extend $s_n(m)$ to all integers m by first setting $s_0(m) = 1$ and using (1) to define the remaining $s_n(m)$ for $m < n-1$.

Lemma 3 (s_n) is a polynomial sequence with $\deg s_n = n$.

Proof: We proceed by induction on n . Clearly, $\deg(s_0) = 0$. Suppose $s_k(m)$ is a polynomial of degree k for $0 \leq k \leq l$. Then $s_{l+1}(m) - s_{l+1}(m-1) = s_l(m) - s_{l-r+1}(m-1)$, which implies the first difference of $s_{l+1}(m)$ is a polynomial of degree l . Thus, $s_{l+1}(m)$ is a polynomial of degree $l+1$. ■

By interpolation we can define (s_n) on all real numbers.

Definition 4 The Eulerian Coefficient is defined as

$$\begin{aligned} \binom{x}{n}_r &= [t^n](1+t+\dots+t^{r-1})^x \\ &= \sum_{i=0}^{\lfloor n/r \rfloor} (-1)^i \binom{x}{i} \binom{x+n-ri-1}{n-ri} \end{aligned}$$

(see (12)). Note that for $r = 2$ the Euler coefficient equals the binomial coefficient $\binom{x}{n}$.

The following table shows the polynomial extension of $s_n(m)$. The number of $\{\uparrow, \rightarrow\}$ paths to (n, m) avoiding a sequence of 4 \rightarrow steps appear above the $y = x$ diagonal. The numbers on the diagonal (n, n) , $1, 1, 2, 5, 13, 36, \dots$, are the number of Dyck paths to $(2n, 0)$. Of course, $s_n(n) \leq C_n$, the n -th Catalan number.

m	1	7	27	75	161	273	357	309	0
6	1	6	20	48	87	118	104	0	-222
5	1	5	14	28	40	36	0	-76	-182
4	1	4	9	14	13	0	-27	-62	-93
3	1	3	5	5	0	-10	-22	-30	-31
2	1	2	2	0	-4	-8	-10	-8	-5
1	1	1	0	-2	-3	-3	-2	0	0
0	1	0	-1	-2	0	0	0	0	0
-1	1	-1	-1	-1	3	-1	-1	-1	3
$n :$	0	1	2	3	4	5	6	7	8

The path counts $s_n(m)$ and their polynomial extension ($r = 4$)

Theorem 5

$$s_n(x) = \frac{x-n+1}{x+1} \binom{x+1}{n}_r = \frac{x-n+1}{x+1} \sum_{i=0}^{\lfloor n/r \rfloor} (-1)^i \binom{x+1}{i} \binom{x+n-ri}{n-ri}$$

Proof: We saw that $(s_n(x))$ is a basis for the vector space of polynomials. Using operators on polynomials, we can write the recurrence relation as

$$1 - E^{-1} = B - B^r E^{-1} \tag{2}$$

where B and E^a are defined by linear extension of $Bs_n(x) = s_{n-1}(x)$ and $E^a s_n(x) = s_n(x+a)$, the shift by a . The operators $\nabla = 1 - E^{-1}$ and E^{-1} both have power series expansions in D , the derivative operator. Hence

B must have such an expansion too, and therefore commutes with ∇ and E^a . The power series for B must be of order 1, because B reduces degrees by 1. Such linear operators are called *delta operators*. The basic sequence $(b_n(x))_{n \geq 0}$ of a delta operator B is a sequence of polynomials such that $\deg b_n = n$, $Bb_n(x) = b_{n-1}(x)$ (like the *Sheffer sequence* $s_n(x)$ for B), and initial conditions $b_n(0) = \delta_{0,n}$ for all $n \in \mathbb{N}_0$. In our special case, the basic sequence is easily determined. Solving for E^1 in (2) shows that

$$E^1 = \sum_{i=0}^{r-1} B^i.$$

Finite Operator Calculus tells us that if $E^1 = 1 + \sigma(B)$, where $\sigma(t)$ is a power series of order 1 [3, (2.5)], then the basic sequence $b_n(x)$ of B has the generating function

$$\sum_{n \geq 0} b_n(x)t^n = (1 + \sigma(t))^x.$$

Thus, in our case $b_n(x) = [t^n](1 + t + t^2 + \dots + t^{r-1})^x = \binom{x}{n}_r$. Since the Sheffer sequence (s_n) has initial values $s_n(n-1) = \delta_{n,0}$, using Abelization [3] gives us

$$s_n(x) = \frac{x-n+1}{x+1} b_n(x+1) = \frac{x-n+1}{x+1} \binom{x+1}{n}_r. \quad (3)$$

■

Corollary 6 *The number of Dyck paths to $(2n, 0)$ avoiding r down steps is*

$$s_n(n) = \frac{1}{n+1} \binom{n+1}{n}_r. \quad (4)$$

3 Ballot paths without the pattern \uparrow^r

Definition 7 $t_n(m; r) = t_n(m)$ is the number of $\{\uparrow, \rightarrow\}$ paths staying weakly above the line $y = x$ from $(0, 0)$ to (n, m) avoiding a sequence of $r > 0$ consecutive \uparrow steps. We set $t_n(m) = 0$ if $n < 0$ or $m + 1 = n > 0$.

This time we do not immediately have a polynomial sequence, as the table below shows. The path $N^{r-1}(EN^{r-1})^k$ to $(k, (r-1)(k+1))$ is the only admissible path reaching the point $(k, (r-1)(k+1))$ (all others would have r or more N -steps). Hence $t_{n-1}((r-1)n) = 1$ for all $n \geq 1$, and $t_{n-1}(m) = 0$ for $m > (r-1)n$. The only other 1's in the table occur in column 0, $t_0(m) = 1$ for $m = 0, \dots, r-1$, and 0 for all other values of m .

The table contains a strip weakly above the diagonal $y = x$ where

$$t_n(m) = t_n(m-1) + t_{n-1}(m) \quad (5)$$

This happens for $0 < n \leq m < n+r$ because paths in this strip cannot have r consecutive vertical steps. All paths that reach a point (n, m) for $m \geq n+r$ and violate the condition of not containing N^r must have this pattern **exactly** at the end of the path, which means that they end in the pattern EN^r . Hence for $m \geq n+r$ we get the recurrence

$$t_n(m) = t_n(m-1) + t_{n-1}(m) - t_{n-1}(m-r) \quad (6)$$

We assume that $t_n(m) = 0$ for all $m < n$ (also for $n = 0$).

We can find a recursion that holds for all $m \geq n$ as follows: For $n \geq 1$ we always have $t_n(n) = t_{n-1}(n)$, because $t_n(n-1) = 0$. From (5) follows by induction (inside the exceptional strip) that $t_n(m) = \sum_{i=n}^m t_{n-1}(i)$ for all $n \leq m < n+r$. For the values of m on the boundary of the strip we have

$$t_n(n+r) = \sum_{i=n}^{n+r} t_{n-1}(i) - t_{n-1}(n) = \sum_{i=n+1}^{n+r} t_{n-1}(i)$$

from (5) and (6), and after that by induction using (6),

$t_n(m) = \sum_{i=m+1-r}^m t_{n-1}(i)$ for all $m \geq n+r$. We can write both recursions together as

$$t_n(m) = \sum_{i=\max\{n, m+1-r\}}^m t_{n-1}(i) \quad (7)$$

for all $m \geq n$. We can avoid the difficulty with the lower bound in the summation by setting $t_n(m) = 0$ for all $m \leq n$. Call the modified numbers $t'_n(m)$. The new table follows the recursion

$$t'_n(m) = \sum_{i=m+1-r}^m t'_{n-1}(i)$$

for all $m > n > 0$. The 'lost' value $t_n(n)$ can be easily recovered, because $t_n(n) = t_{n-1}(n) = t'_{n-1}(n)$.

m	0	0	1	19	112	397	1027	1966	2905
8	0	0	3	28	116	321	630	939	(939)
7	0	0	6	33	101	205	309	(309)	0
6	0	1	9	32	68	104	(104)	0	0
5	0	2	10	23	36	(36)	0	0	0
4	0	3	8	13	(13)	0	0	0	0
3	1	3	5	(5)	0	0	0	0	0
2	1	2	(2)	0	0	0	0	0	0
1	1	(1)	0	0	0	0	0	0	0
0	(1)	0	0	The values in parentheses are 0 in $t'_n(m)$					
$n :$	0	1	2	3	4	5	6	7	8

The restricted ballot path counts $t_n(m)$ ($r = 4$).

In order to show the polynomial structure in the above table, we transform it into the table below by a 90° counterclockwise turn, and shifting the top 1's flush against the y -axis. In formulas, we define $p_n(m) = t'_{m-1}((r-1)m - n)$ for $m(r-1) \geq n \geq 0$ (or $t'_n(m) = p_{(r-1)(n+1)-m}(n+1)$). The recursion $t'_n(m) = \sum_{i=m+1-r}^m t'_{n-1}(i)$ 'along the previous column' becomes now a recursion $p_n(m) = \sum_{i=0}^{r-1} p_{n-j}(m-1)$ 'along the previous row'. More precisely, for $(r-1)m - n > m - 1 \geq 1$, i.e., $m \geq 2$ and $n \leq (r-2)m$ holds

$$\begin{aligned}
p_n(m) &= t'_{m-1}((r-1)m - n) = \sum_{i=(r-1)(m-1)-n}^{(r-1)m-n} t'_{m-2}(i) \\
&= \sum_{i=(r-1)(m-1)-n}^{(r-1)m-n} p_{(r-1)(m-1)-i}(m-1) = \sum_{i=0}^{r-1} p_{n-j}(m-1) \quad (8)
\end{aligned}$$

The numbers $p_n(m)$ for $0 \leq n \leq (r-2)m$ are exactly the cases where $t'_n(m)$ is positive, and the only additional numbers needed in the recursion (8) are the numbers

$p_{(r-2)m-j}(m-1) = t'_{m-2}((r-1)(m-1) - (r-2)m + j) = t'_{m-2}(m + j - r + 1) = 0$ for $j = 0, \dots, r-3$. We also add a row $p_n(0) = \delta_{n,0}$ for $n = 0, \dots, r-2$ to the table, so that the recursion (8) holds for $m = 1$. This part of the p -table, $p_n(m)$ for $0 \leq m$ and $0 \leq n \leq (r-2)(m+1)$, is shown below for $r = 4$. Note that $p_0(m) = t_{m-1}((r-1)m) = 1$ for all $m \geq 1$, and also $p_0(0) = 1$.

m	1	8	36	119	315	699	1338	2246	3344
7	1	7	28	83	197	391	667	991	1295
6	1	6	21	55	115	200	297	379	419
5	1	5	15	34	61	90	112	116	101
4	1	4	10	19	28	33	32	23	13
3	1	3	6	9	10	8	5	0	0
2	1	2	3	3	2	0	0	-2	2
1	1	1	1	0	0	-1	1	-2	4
0	1	0	0	-1	1	-1	2	-4	7
n	0	1	2	3	4	5	6	7	8

The rotated and shifted table $p_n(m)$ and its polynomial extension below the staircase, for $r = 4$. The bold numbers occur also on the second subdiagonal in the table below.

We obtained the recursion (7) as a discrete integral from (6) and (5). We can now take differences in recursion (8) and get $p_n(m) - p_{n-1}(m) = p_n(m-1) - p_{n-r}(m-1)$, or

$$p_n(m) - p_n(m-1) = p_{n-1}(m) - p_{n-r}(m-1)$$

for all $m \geq 1$ and $0 \leq n \leq (r-2)(m+1)$. The column $p_0(m)$ can be extended as a column of ones to all integers m ; hence $p_0(m)$ can be extended to the constant polynomial 1. The recursion shows by induction that the n -th column can be extended to a polynomial of degree n , and by interpolation we can assume that we have polynomials in a real variable. The extension of $p_n(m)$ is again denoted by $p_n(m)$. The above table shows some values of the polynomial expansion in cursive. The expansion follows the same recursion, hence

$$p_n(x) - p_n(x-1) = p_{n-1}(x) - p_{n-r}(x-1) \quad (9)$$

with initial values $p_{(r-2)m+j}(m) = 0$ for $j = 1, \dots, r-2$ and $m \geq 0$. These conditions, together with $p_0(0) = 1$, determine the solution uniquely.

Recursion (9) shows that $(p_n(x))$ is a Sheffer sequence for the same operator B as the sequences $(s_n(x))$ in recursion (2). Hence $p_n(x)$ can be written in terms of the same basis, the Eulerian coefficients, as $s_n(x)$. However, the initial values (zeroes) for $(p_n(x))$ are more difficult, because they are not on a line with positive slope. We introduce now a Sheffer sequence $(q_n(x; \alpha))_{n \geq 0}$ for the delta operator B that has roots on the parallel to the diagonal shifted by $\alpha + 1$, $q_n(n - \alpha - 1; \alpha) = 0$, and agrees with (p_n) at one position left of the roots, for each n .

Lemma 8 For the Sheffer sequence $(q_n(x; \alpha))$ for B with initial values $q_0(m; \alpha) = 1$, $q_n(0) = \delta_{n,0}$ for $0 \leq n \leq \alpha$ and $q_n(n - \alpha - 1; \alpha) = 0$ for $n > \alpha$, holds

$$q_{n+\alpha}(n; \alpha) = p_{(r-2)n-\alpha}(n)$$

for all $n \geq \lceil \alpha / (r - 2) \rceil$.

We will proof this Lemma in Subsection 3.2.

m	1	6	21	56	120	214	320	386	321
5	1	5	15	35	65	99	121	101	0
4	1	4	10	20	31	38	32	0	-70
3	1	3	6	10	12	10	0	-22	-58
2	1	2	3	4	3	0	-7	-18	-33
1	1	1	1	1	0	-2	-6	-10	-15
0	1	0	0	0	0	-2	-4	-4	-5
$n :$	0	1	2	3	4	5	6	7	8

The polynomials $q_n(m, 2)$ for $r = 4$

The sequence (q_n) agrees with the Euler coefficients $b_n(x) = \binom{x}{n}_r$ for the first degrees $n = 0, \dots, \alpha$. It follows from the Binomial Theorem for Sheffer sequences that

$$q_n(x; \alpha) = \sum_{i=0}^{\alpha} \binom{i - \alpha - 1}{i}_r \frac{x + \alpha + 1 - n}{x + \alpha + 1 - i} \binom{x + \alpha + 1 - i}{n - i}_r. \quad (10)$$

Corollary 9 The number of ballot paths avoiding $r \uparrow$ -steps equals

$$t_n(m; r) = \sum_{i=0}^{m-n-1} \frac{1}{m+1-i} \binom{i-m+n}{i}_r \binom{m+1-i}{m-i}_r$$

for $m > n \geq 0$. Furthermore, $t_n(n) = t_{n-1}(n) = \frac{1}{n+1} \binom{n+1}{n}_r$ for all $n > 0$.

Proof: $t_n(m; r) = p_{(r-1)(n+1)-m}(n+1) = q_m(n+1; m-1-n)$. ■

The Corollary shows that the number of Dyck paths to $(2n, 0)$ avoiding r up steps, $\frac{1}{n+1} \binom{n+1}{n}_r$, equals the number of Dyck paths to $(2n, 0)$ avoiding r down steps (see formula (4)).

3.1 A Conjecture for the Case $r = 4$.

A Motzkin path can take horizontal unit steps in addition to the up and down steps of a Dyck path. Suppose a Motzkin path is “peakless”, i.e., the pattern uu and ud does not occur in the path. Denote the number of peakless Motzkin paths to $(n, 0)$ by $M'(n)$. Starting at $n = 0$ we get the following sequence, 1, 1, 1, 2, 4, 7, 13, 26, 52, 104, 212, 438, 910, ... for $M'(n)$ (see <http://www.research.att.com/~njas/sequences/A023431>). It is easy to show that $M'(n) = \sum_{i=0}^{\lfloor n/3 \rfloor} \binom{n-i}{2i} \frac{1}{i+1} \binom{2i}{i}$.

For the case $r = 4$ we conjecture that $p_n(0) = (-1)^n M'(n-3)$ for all $n \geq 3$. That would imply $\sum_{n \geq 0} p_n(0) t^n = \left(3 + t - \sqrt{(1+t)^2 + 4t^3}\right) / 2$, and therefore

$$\sum_{n \geq 0} p_n(x) t^n = \frac{3 + t - \sqrt{(1+t)^2 + 4t^3}}{2} \left(\frac{1-t^4}{1-t}\right)^x.$$

For example, the coefficient of t^7 in this power series equals $(x-3)(x^6 + 24x^5 + 247x^4 + 426x^3 - 38x^2 - 2340x + 6720) / 7!$, which in turn equals $p_7(x)$, as can be checked using the table for $p_7(m)$.

3.2 Proof of Lemma 8

Because of recursion (8) we obtain the operator identity

$$I = E^{-1} (B^0 + B^1 + \dots + B^{r-1})$$

which holds for $(p_n(x))$ and $(q_n(x; \alpha))$, and shows that both polynomials enumerate lattice paths with steps $\langle 0, 1 \rangle, \langle 1, 1 \rangle, \langle 2, 1 \rangle, \dots, \langle r-1, 1 \rangle$ (above the respective boundaries). The number of such paths reaching (n, m) can also be seen as compositions of m into n terms taken from $\{0, 1, \dots, r-1\}$. In the case of $p_n(m)$ the terms a_1, \dots, a_n also have to respect the boundary, which means that $\sum_{i=1}^k a_i \leq (r-2)k$ for all $k = 1, \dots, n$. For $q_n(m; \alpha)$ we get

for the same reason that $\sum_{i=1}^k b_i \leq k + \alpha - 1$. Such restricted compositions have the following nice property.

Lemma 10 *Let $c \in \mathbb{N}_1$, $\alpha \in \mathbb{N}_0$, and n be a natural number such that $n - \alpha \geq 0$. Let P_n^α be the number of compositions of $cn - \alpha$ into n parts from $[0, c + 1]$ such that $a_1 + a_2 + \dots + a_n = cn - \alpha$, and $\sum_{i=1}^k a_i \leq ck$ for all $k = 1, \dots, n-1$. Let Q_n^α be the number of compositions of $n + \alpha$ into n*

parts from $[0, c+1]$ such that $b_1 + b_2 + \dots + b_n = n + \alpha$, and $\sum_{i=1}^k b_i \leq k + \alpha$ for all $k = 1, \dots, n-1$. Then $P_n^\alpha = Q_n^\alpha$.

Proof: Suppose, $b_1 + b_2 + \dots + b_n = n + \alpha$, and $\sum_{i=1}^k b_i \leq k + \alpha$. Define $a_i = c+1 - b_{n+1-i}$ for $i = 1, \dots, n$. Note that $a_i \in [0, c+1]$, and $n-k+\alpha \geq \sum_{i=1}^{n-k} b_i = n + \alpha - \sum_{i=1}^k b_{n+1-i}$. Hence

$$\sum_{i=1}^k a_i = (c+1)k - \sum_{i=1}^k b_{n+1-i} \leq (c+1)k - n + (n-k) = ck$$

and

$$\sum_{i=1}^n a_i = (c+1)n - \sum_{i=1}^n b_{n+1-i} = cn - \alpha.$$

■

We apply this Lemma with $c = r-2$ to obtain $p_{(r-2)n-\alpha}(n) = q_{n+\alpha}(n)$.

3.3 Abelization

Let (b_n) be the basic sequence for some arbitrary delta operator B , i.e., $Bb_n = b_{n-1}$ and $b_n(0) = \delta_{0,n}$. Every basic sequence is also a sequence of binomial type, which means that $\sum_{n \geq 0} b_n(x) t^n = e^{x\beta(t)}$, where $\beta(t) = t + a_2 t^2 + \dots$ is a formal power series. The compositional inverse of $\beta(t)$ is the power series that represents B ,

$$B = \beta^{-1}(D) = D + b_2 D^2 + \dots$$

where $D = \partial/\partial x$ is the x -derivative. The Abelization of (b_n) (by $a \in \mathbb{R}$) is the basic sequence $\left(\frac{x}{x+an} b_n(x+an)\right)_{n \geq 0}$ for the delta operator $E^{-a}B$ (see [4]). Note that with any Sheffer sequence (s_n) for B the sequence $(s_n(x+c-an))$ is a Sheffer sequence for $E^a B$. Hence $\left(\left(\frac{x+c-an}{x+c} b_n(x+c)\right)\right)_{n \geq 0}$ is a Sheffer sequence for $E^a E^{-a} B = B$ again. Choosing $c = a = 1$ shows (3).

Sheffer sequences and the basic sequence for the same delta operator are connected by the Binomial Theorem for Sheffer sequences,

$$s_n(y+x) = \sum_{i=0}^n s_i(y) b_{n-i}(x).$$

Applying this Theorem to $\left(\frac{x}{x+an}b_n(x+an)\right)$ and $(b_n(x+c+an))$ shows that

$$b_n(y+x+c+an) = \sum_{i=0}^n b_i(y+c+ai) \frac{x}{x+a(n-i)} b_{n-i}(x+a(n-i)).$$

Choosing x as $x+\alpha+1-an$, $a=1$, $c=0$, and $y=-\alpha-1$ gives $b_n(x) = \sum_{i=0}^n b_i(i-\alpha-1) \frac{x+\alpha+1-n}{x+\alpha+1-i} b_{n-i}(x+\alpha+1-i)$. This is not quite what we have in (10); there the summation stops at α . This effect in (10) is due to the ‘initial values’ $b_i(i-\alpha-1)$ which are 0 for $i > \alpha$.

The generating function of a Sheffer sequence (s_n) for B is of the form $\phi(t)e^{x\beta(t)}$, where $\phi(t) = \sum_{n \geq 0} s_n(0)t^n$. If $s_n(x) = \frac{x+c-an}{x+c}b_n(x+c) = b_n(x+c) - \frac{an}{x+c}b_n(x+c)$ then

$$\sum_{n \geq 1} \frac{n}{x+c} b_n(x+c)t^n = \frac{t}{x+c} \frac{\partial}{\partial t} e^{(x+c)\beta(t)} = t\beta'(t)e^{(x+c)\beta(t)}$$

and

$$\sum_{n \geq 0} s_n(x)t^n = e^{(x+c)\beta(t)}(1-at\beta'(t))$$

If $c=a=1$ and $e^{\beta(t)} = (1+t+\dots+t^{r-1})$, then $\beta'(t) = \frac{1-(r-tr+t)t^{r-1}}{(1-t^r)(1-t)}$, and therefore

$$\sum_{n \geq 0} s_n(m)t^n = \frac{(1-t^r)^m}{(1-t)^{m+2}} (rt^r(1-t) + (1-t^r)(1-2t)), \quad (11)$$

the generating function of the number of ballot paths to (n, m) , avoiding \rightarrow^r .

4 Euler Coefficients

The coefficients of the polynomial

$$(1+t+t^2+\dots+t^{r-1})^n$$

were considered by Euler in [2], where he gives the following recurrence:

$$\binom{n}{k}_{r+1} = \sum_{i=0}^{k/2} \binom{n}{k-i} \binom{k-i}{i}_r$$

To calculate the Euler coefficients in terms of only binomial coefficients, we rewrite the polynomial as follows:

$$\begin{aligned} (1 + t + t^2 + \dots + t^{r-1})^n &= \left(\frac{1 - t^r}{1 - t} \right)^n \\ &= \sum_{i=0}^n \binom{n}{i} t^{ri} (-1)^i \sum_{j \geq 0} \binom{n+j-1}{j} t^j. \end{aligned}$$

Thus we have proven

$$\binom{n}{k}_r = \sum_{i=0}^{\lfloor k/r \rfloor} (-1)^i \binom{n}{i} \binom{n+k-ri-1}{k-ri}. \quad (12)$$

Note that this identity implies $\lim_{r \rightarrow \infty} \binom{n}{k}_r = \binom{n+k-1}{k}$. Combinatorially, these are *all* $\{\uparrow, \rightarrow\}$ paths avoiding \rightarrow^r . This problem occurs in Wilf's *generatingfunctionology* [6], Section 4.12. Note that identity (12) implies $\lim_{r \rightarrow \infty} \binom{n}{k}_r = \binom{n+k-1}{k}$.

x	1	8	36	120	322	728	1428	2472	3823
7	1	7	28	84	203	413	728	1128	1554
6	1	6	21	56	120	216	336	456	546
5	1	5	15	35	65	101	135	155	155
4	1	4	10	20	31	40	31	20	10
3	1	3	6	10	12	12	10	6	3
2	1	2	3	4	3	2	1	0	0
1	1	1	1	1	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
$n :$	0	1	2	3	4	5	6	7	8

A table of Euler coefficients $\binom{x}{n}_4$ for $r = 4$

We now show some properties about Euler Coefficients similar to the basic properties of binomial coefficients.

1. For binomial coefficients, this property is usually called Pascal's Identity:

$$\binom{n}{k}_r = \sum_{i=0}^{r-1} \binom{n-1}{k-i}_r$$

Proof

$$\begin{aligned} (1+t+\dots+t^{r-1})^n &= (1+t+\dots+t^{r-1})^{n-1}(1+t+\dots+t^{r-1}) \\ &= \sum_{i=0}^{r-1} t^i (1+t+\dots+t^{r-1})^{n-1} \end{aligned}$$

so

$$\binom{n}{k}_r = \sum_{i=0}^{r-1} [t^{k-i}] (1+t+\dots+t^{r-1})^{n-1} = \sum_{i=0}^{r-1} \binom{n-1}{k-i}_r$$

■

2. The table of Euler Coefficients is symmetric similar to Pascal's Triangle:

$$\binom{n}{k}_r = \binom{n}{n(r-1)-k}_r$$

Proof We proceed by induction on n , fixing $r \geq 2$. For $n = 2$ we have the well known symmetry for binomial coefficients. Suppose true for some $l > 2$. From the above recurrence we have

$$\begin{aligned} \binom{l+1}{k}_r &= \sum_{i=0}^{r-1} \binom{l}{k-i}_r = \sum_{i=0}^{r-1} \binom{l}{l(r-1)-(k-i)}_r \\ &= \sum_{i=0}^{r-1} \binom{l}{(l+1)(r-1)-k-i}_r = \binom{l+1}{(l+1)(r-1)-k}_r \end{aligned}$$

and the induction follows. ■

3. This property is similar to Vandermondt Convolution for binomial coefficients:

$$\binom{n+m}{k}_r = \sum_{i=0}^k \binom{n}{i}_r \binom{m}{k-i}_r.$$

It follows because the Euler coefficients are of binomial type [4].

4. Here we have an identity that is trivial for binomial coefficients, i.e. $r = 2$, and gives an identity for the Catalan numbers as $r \rightarrow \infty$.

$$\frac{1}{n+1} \binom{n+1}{n}_r = \binom{n}{n}_r - \sum_{i=1}^{r-2} i \binom{n}{n-i-1}_r$$

Proof Let $s_n(x) = \frac{x-n+1}{x+1} \binom{x+1}{n}_r$, as in (3). The binomial theorem for Sheffer sequences states that $s_n(x+y) = \sum_{i=0}^n s_i(y) b_{n-i}(x)$.

Let $x = n$, $y = 0$ and noting that $s_n(0) = (1 - n) \binom{1}{n}_r = 1 - n$ for $0 < n < r$ and 0 otherwise, we have $s_n(n) = \sum_{i=0}^n s_i(0) b_{n-i}(n)$, hence

$$\begin{aligned} \frac{1}{n+1} \binom{n+1}{n}_r &= \sum_{i=0}^{r-1} (1-i) \binom{n}{n-i}_r \\ &= \binom{n}{n}_r - \sum_{i=1}^{r-2} i \binom{n}{n-i-1}_r. \end{aligned}$$

We have already noted that $\binom{n}{k}_r \rightarrow \binom{n+k-1}{k}$ as $r \rightarrow \infty$, so

$$\lim_{r \rightarrow \infty} \frac{1}{n+1} \binom{n+1}{n}_r = \binom{2n-1}{n} - \sum_{i=1}^{n-1} i \binom{2n-i-2}{n-1} = C_n.$$

■

References

- [1] E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999) 167 – 202.
- [2] L. Euler, De evolutione potestatis polynomialis cuiuscunque $(1+x+x^2+x^3+x^4+etc.)^n$, *Nova Acta Academiae Scientiarum Imperialis Petropolitinae* **12** (1801) 47 – 57.
- [3] H. Niederhausen, Rota’s umbral calculus and recursions, *Algebra Univers.* **49** (2003) 435 – 457.
- [4] G.-C. Rota, D. Kahaner, and A. Odlyzko, On the Foundations of Combinatorial Theory VIII: Finite operator calculus, *J. Math. Anal. Appl.* **42** (1973) 684 – 760.
- [5] A. Sapounakis, L. Tasoulas, and P. Tsikouras, Counting strings in Dyck paths, *Discrete Math.*, to appear 2007.
- [6] H. S. Wilf, *generatingfunctionology*, third edition, A. K. Peters, Ltd. 2006.