Lattice Path Enumeration and Umbral Calculus

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1 Introduction

Twenty yeas ago, when I saw the "Finite Operator Calculus" [11] for the first time, I was captivated by its beauty and inspired by all the roads it opened up for further exploration. Sheffer polynomials became the magic tools for my thesis work on Ballot problems and Kolmogorov-Smirnov distributions, and I started to work on some generalizations, like piece-wise polynomial Sheffer functions ("Sheffer splines") and multi-indexed Sheffer sequences. However, none of the *generalizations* I have studied where as satisfying to me as the *specializations* that lead to real applications. There is of course a considerable amount of detail necessary before we can actually calculate a significance level, say, when we start at the Umbral Calculus.

All results in this paper have been published earlier, except Theorem 2 on geometric Sheffer sequences, and perhaps formula (8) on counting lattice path with weighted left turns staying *above* a parallel to the diagonal. However, this paper is not intended to be a survey on lattice path problems, but to show how the Umbral Calculus can serve as a tool in certain situations.

1.1 Notation

A *polynomial sequence* $\{p_n\}_{n\geq 0}$ is a sequence of polynomials such that $\deg p_n = n$ for all $n = 0, 1, \ldots$.

$$p(x,t) := \sum_{n\geq 0} p_n(x)t^n$$

is the generating function of this polynomial sequence. For convenience we will henceforth assume that $p_n \equiv 0$ for negative n.

$$p(x,t) = \sum_{n\geq 0} e_n(x)\phi_n(t)$$

is the generating function of the polynomial sequence $\{p_n\}$.

A *delta operator* B is a formal power series of order 1 in the derivative operator D_x,

$$B(D_x) = D_x + b_2 D_x^2 + \ldots$$

A *Sheffer sequence* $\{s_n\}$ (for B) is a polynomial sequence such that

$$Bs_n = s_{n-1}$$

for all $n = 0, 1, \ldots$. The basic sequence $\{b_n\}$ (for B) is the Sheffer sequence with initial values $b_n(0) = \delta_{0,n}$. Basic sequences and Sheffer sequences have generating functions of the form

$$b(x,t) = e^{x\beta(t)}, \quad s(x,t) = s(t)e^{x\beta(t)}$$

where β is the compositional inverse of $B(D_x)$, and $s(t) = \sum_{n\geq 0} s_n(0)t^n$ is a formal power series of order 0. A straightforward consequence of those generating functions is the *binomial theorem for Sheffer sequences*,

$$s_n(x+y) = \sum_{i=0}^{n} s_i(x)b_{n-i}(y) \tag{1}$$

2 Initial Value Problems

In lattice path enumeration, we frequently have to solve the system of difference equations

$$Br_n(x) = r_{n-1}(x)$$

for all $n = 0, 1, \ldots$ where B is a given delta operator and r_0 is a non-zero constant. This implies that $\{r_n\}$ is a Sheffer sequence for B. Finding a solution to this system usually means expanding $r_n(x)$ in terms of the corresponding basic sequence $\{b_n\}$ such that certain initial values are met (which are set by path boundaries),

$$r_n(x_n) = y_n$$

say, for $n = 0, 1, \ldots$. Such initial values uniquely determine $\{r_n\}$. The binomial theorem for Sheffer sequences (1) can be utilized for such an expansion if we know an initial value *at the same* input for all n, like $r_n(0)$ for example:

$$r_n(x) = \sum_{i=0}^{n} r_j(0)b_{n-j}(x). \tag{2}$$

The same theorem can help us now to recursively determine $r_n(0)$ from the given initial values, because of

$$y_n = r_n(x_n) = \sum_{j=0}^{n} r_j(0)b_{n-j}(x_n).$$

In other words, we must solve the matrix equation $Y = AR$ for R, where $Y = (y_i)_{i=0\ldots n}$, $R = (r_i(0))_{i=0\ldots n}$, and $A = (b_{i-j}(x_i))_{i=0\ldots n, j=0\ldots i}$ is lower triangular. Cramer's rule will easily express $r_n(0)$ as a determinant (note that $|A| = 1$), if necessary. However, in the context of this paper we do not consider the well known determinant as an *explicit* solution, because of its inherent recursive nature. To get $r_n(x)$ we need another triangular matrix, $C = (b_{i-j}(x))_{i=0\ldots n, j=0\ldots i}$, and find

$$(r_i(x))_{i=0\ldots n} = CR = CA^{-1}Y$$

We will see below that Umbral Calculus can find an "explicit" solution to the initial value problem if the inputs x_n are piecewise affine in n. The size of the initial values y_n is of minor importance; suppose we know a family of Sheffer sequences $\left\{t_n^{(i)}\right\}_{n\geq 0}$ for B with initial values $t_n^{(i)}(x_{n+i}) = \delta_{0,n}$ for all $i = 0, 1, \ldots$. It is straightforward to verify that

$$r_n(x) = \sum_{i=0}^{n} y_i t_{n-i}^{(i)}(x) \tag{3}$$

solves the original initial value problem.

It can be helpful to have a mental image of the solutions. In the context of initial value problems, I visualize a Sheffer sequence as rows of values:

...	*	*	*	*	*	*	$\tan(4) = 5$	*	*	...	$s_4(x)$	quartic
...	*	*	*	$y_3 = 4$	*	*	*	*	*	...	$s_3(x)$	cubic
...	$y_2 = 3$	*	*	*	*	*	*	*	*	...	$s_2(x)$	quadratic
...	*	*		*	*	*	*	$y_1 = 2$	*	...	$s_1(x)$	linear
...	1	1	1	1	$y_0 = 1$	1	1	1	1	...	$s_0(x)$	constant
...	$\tan 2$	-2	-1	$\tan 3$	$\tan 0$	1	$\tan 4$	$\tan 1$	2	...	$\leftarrow x$	

Example: $s_n(\tan n) = n + 1$

An important aspect of this example is that the recurrence need not to take place in an integer lattice. The difference operator and the derivative are both delta operators. In other words, we can simultaneously study lattice paths and empirical distribution functions, as in the Kolmogorov two-sample and one-sample tests.

2.1 The Role of e^x

The Finite Operator Calculus [11] is based on the reference sequence $\{x^n/n!\}$ and its generating function e^{xt}. The special analytical properties of this generating function stand behind many interesting results in the Finite Operator Calculus.

Suppose, $\{b_n\}$ is the basic sequence for the delta operator B, with generating function

$$\sum_{n\geq 0} b_n(x)t^n = e^{x\beta(t)}.$$

It is of great importance for our initial value problem that $D_t e^{x\beta(t)} = x\beta'(t)e^{x\beta(t)}$, because this implies that

$$p_n(x) := \frac{n+1}{x} b_{n+1}(x) \tag{4}$$

has the generating function $\beta'(t)e^{x\beta(t)}$, and therefore must be a Sheffer polynomial. The linear combination

$$s_n(x) := b_n(x-c) - ap_{n-1}(x-c) = \frac{x-an-c}{x-c} b_n(x-c)$$

of Sheffer polynomials is again a Sheffer polynomial for the same operator, and solves the initial value

$$Bs_n = s_{n-1}, \text{and } s_n(an + c) = \delta_{0,n}$$

for all $n = 0, 1, \ldots$, where a and c are given constants. This solution has already been given in [11]. In order to solve the problem

$$Br_n = r_{n-1}, \text{and } r_n(an + c) = y_n$$

for all $n = 0, 1, \ldots$ using the expansion (3) we must define $t_n^{(i)}(x) := s_n(x - ai)$ and get

$$r_n(x) = \sum_{i=0}^{n} y_i s_{n-i}(x - ai) = \sum_{i=0}^{n} y_i \frac{x - an - c}{x - ai - c} b_{n-i}(x - ai - c). \tag{5}$$

2.2 Piecewise affine boundaries

Suppose, we want to solve the system with initial values first along the line $x_n = an + c$

$$y_n = r_n(an + c)$$

for all $n = 0, \ldots, L - 1$, and thereafter on the line $x_n = \tilde{a}n + \tilde{c}$

$$y_n = r_n(\tilde{a}n + \tilde{c})$$

for all $n = L, \ldots,$ where $a, \tilde{a}, c, \tilde{c}$ and L are given constants. By (5), the beginning of the sequence can be calculated along the *second* line as

$$r_i(\tilde{a}i + \tilde{c}) = \sum_{j=0}^{i} y_j \frac{(\tilde{a} - a)i + \tilde{c} - c}{\tilde{a}i - aj + \tilde{c} - c} b_{i-j}(\tilde{a}i - aj + \tilde{c} - c)$$

for all $i = 0, \ldots, L - 1$, and applying (5) again gives for $n \geq L$ the expansion

$$r_n(x) = \sum_{i=0}^{L-1} r_i(\tilde{a}i + \tilde{c}) \frac{x - \tilde{a}n - \tilde{c}}{x - \tilde{a}i - \tilde{c}} b_{n-i}(x - \tilde{a}i - \tilde{c}) + \sum_{i=L}^{n} y_i \frac{x - \tilde{a}n - \tilde{c}}{x - \tilde{a}i - \tilde{c}} b_{n-i}(x - \tilde{a}i - \tilde{c}). \tag{6}$$

Substituting for $r_i(\tilde{a}i + \tilde{c})$ from beginning of the sequence into the first sum in (6) finishes the expansion, but makes it into a double sum.

This procedure can be repeated for initial values on more affine pieces. Obviously, the multiplicity of the summation will grow with the number of pieces. In the example in Section 2, $D_x s_n(x) = s_{n-1}(x)$ and $s_n(\tan(n)) = n + 1$, it suffices to use two pieces if we only want to find $s_4(x) = \frac{1}{4!}x^4 + .07\,3765 \times x^3 + .78994x^2 + 4.\,2212x - 1.\,1357$:

$$\begin{aligned} x_n &= (\tan 2 - \tan 1)\, n + 2\tan 1 - \tan 2 \text{ for } n = 0, 1, 2 \\ x_n &= (\tan 4 - \tan 3)\, n + 4\tan 3 - 3\tan 4 \text{ for } n \geq 3. \end{aligned}$$

2.3 Applications: Bounded paths

Some of the best known applications occur in the enumeration of lattice paths, sequences of horizontal $\rightarrow$ and vertical $\uparrow$ steps starting at the origin. Let $r_n(m)$ be the number of paths that reach the point (m, n) under some kind of restriction which respects the recurrence

$$r_n(m) = r_n(m-1) + r_{n-1}(m).$$

The (generalized) Ballot Problem requires the paths to remain below some boundary line; this translates into initial conditions of the form $r_n(-1) = \delta_{0,n}$ for all $n = 0, \ldots, L-1$, and $r_n(an + c) = 0$ for all $n = L, \ldots$.

...	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$*$	$\boxed{0}$	309	834	...	$r_5(x)$
...	$*$	$*$	$*$	$*$	$*$	$\boxed{0}$	52	132	248	309	525	...	$r_4(x)$
...	$*$	$*$	$\boxed{0}$	6	$\underset{\rightarrow}{16}$	$\underset{\rightarrow}{31}$	$\underset{\rightarrow}{52}$	$\underset{\rightarrow}{80}$	$\underset{\longrightarrow}{116}$	$\underset{\longrightarrow}{161}$	$\underset{\longrightarrow}{261}$	...	$r_L(x)$
...	$\boxed{0}$	$\underset{\rightarrow}{1}$	$\underset{\rightarrow}{3}$	$\underset{\rightarrow}{6}$	$\uparrow 10$	15	21	28	36	45	55	...	$r_2(x)$
...	$\boxed{0}$	$\uparrow 1$	2	3	4	5	6	7	8	9	10	...	$r_1(x)$
...	1	$\uparrow 1$	1	1	1	1	1	1	1	1	1	...	$r_0(x)$
		$\bullet$											
...	-1	0	1	2	3	4	5	6	7	8	9	...	x

Path boundary $3n - 8$, with sample path

Such initial value problems must be solved for calculating the exact distribution of the (one-sided) two-sample Kolmogorov-Smirnov test. The exact distribution of the *one*-sample Kolmogorov-Smirnov test derives from the Lebesque measure of certain empirical distribution functions instead of the counting measure of lattice paths; the same techniques apply, however the derivative operator D_x takes the place of the backwards difference operator. Sheffer sequences are also employed for the exact distribution of some multivariate generalizations of these tests. The two-sided distribution of the two-sample test has a "closed form"; why that is not the case for the *one*-sample case is explained in section 4.1. Details about these applications can be found in [4]. A general reference is S.G. Mohanty's book [2].

3 Systems of Operator Equations

In the previous section we investigated the rather simple system $Br_n = r_{n-1}$, for $n = 0, 1, \ldots$ with relatively general initial conditions. In this section we concentrate on finding basic solutions b_n, where $b_n(0) = \delta_{0,n}$, of more complicated systems of the form

$$Qb_n = Rb_{n-1} + Sb_{n-c} \tag{7}$$

for all $n = 0, 1, \ldots$, where c is a positive integer, R and S are translation invariant invertible operators (i.e. power series of order 0 in D_x), and Q is a delta operator. An example for R could be $Rb_n(x) = \sum_{i=1}^{r} \rho_i b_n(x - r_i)$ for some given constants $\rho_1, \ldots, \rho_k$ and $r_1, \ldots, r_k$.

With a *solution* of (7) we mean an expansion of $b_n(x)$ in terms of the basic sequence $\{q_n\}$ of Q.

Suppose, the unknown solution $\{b_n\}$ is the basic sequence for some delta operator B. If we can construct a solution under this hypothesis, then the assumption will be justified.

Because B and Q are both delta operators, there exists a translation invariant and invertible operator T such that $B = TQ$ (see [11, Corollary 4]). Any such T which solves the equation

$$I = RT + ST^c Q^{c-1}$$

also solves the equation

$$Q = RTQ + ST^c Q^c$$

which is equivalent to the system (7) (any two translation invariant operators commute). Equivalently, $I - RT = (RT)^c R^{-c} SQ^{c-1}$ and $(RT)^{-c} - (RT)^{1-c} = R^{-c} SQ^{c-1}$. Lagrange inversion gives

$$T^{-n} = \sum_{k \geq 0} \binom{n - k(c-1)}{k} \frac{n}{n - (c-1)k} R^{n-kc} S^k Q^{k(c-1)}$$

and the Transfer Formula [11, Section 4] says that

$$b_n(x) = xT^{-n} x^{-1} q_n(x) = x \sum_{k \geq 0} \binom{n - k(c-1)}{k} R^{n-kc} S^k q_{n-k(c-1)}(x)/x$$

(use that $\left\{ \frac{n+1}{x} q_{n+1}(x) \right\}$ is a Sheffer sequence; see (4)). It is easy to verify that this basic sequence really solves (7).

3.1 Applications: Lattice paths with Several Step Directions

Counting (weighted) lattice paths with several step directions leads to recurrence relations of the form

$$d_n(x) = d_n(x-1) + \sum_{i=1}^{r} \rho_i b_{n-1}(x - r_i) + \sum_{i=1}^{s} \sigma_i b_{n-c}(x - s_i)$$

if the step vectors are $(1, 0), (r_1, 1), \ldots, (r_r, 1), (s_1, c), \ldots, (s_s, c)$. In this case, $Q = \nabla$, the backwards difference operator. More details about the simple case $r = 1 = s$ are given in [3]. An application to a gamblers ruin problem and expected game duration can be found in [6].

4 Symmetric Sheffer Sequences

In section 2.3 we mentioned the general Ballot Problem as an application of formula (6). In the classical Ballot Problem the paths stay below the diagonal, or some line parallel to the diagonal. The initial values are therefore $r_n(-1) = 0$ for all $n = 1, \ldots, L-1$, and

$r_n(n-L) = 0$ for all $n \geq L$.

...	$*$	$*$	$*$	0	28	90	207	...	$r_5(m)$
...	$*$	$*$	0	9	28	62	117	...	$r_4(m)$
...	$*$	0	3	9	$\underrightarrow{19}$	$\underrightarrow{34}$	$\underrightarrow{55}$	...	$r_L(m)$
...	0	$\underrightarrow{1}$	$\underrightarrow{3}$	$\underrightarrow{6}$	$\uparrow 10$	15	21	...	$r_2(m)$
...	0	$\uparrow 1$	2	3	4	5	6	...	$r_1(m)$
...	1	$\uparrow 1$	1	1	1	1	1	...	$r_0(m)$
...	-1	0	1	2	3	4	5	...	m

The Ballot Problem ($L = 3$), with sample path

However, the solution to this initial value problem is much simpler than the sum in formula (6) indicates:

$$r_n(m) = \binom{n+m}{n} - \binom{n+m}{n-L}.$$

This well-known solution is easily identified as a difference of two Sheffer polynomials for the backwards difference operator ∇, and it is obviously zero at $m = -1$ for all $n = 1, \ldots, L-1$. However, for $n \geq L$ the initial values are attained because of a very special property:

$$r_n(n-L) = \binom{n+n-L}{n} - \binom{n+n-L}{n-L} = \binom{n+n-L}{n} - \binom{n+n-L}{n}.$$

In other words, for nonnegative integers n and m we can interchange the degree n with the argument m in the polynomial $s_n(m) := \binom{n+m}{n}$, and get again a polynomial $s_m(n) = \binom{m+n}{m} = s_n(m)$. We call such a Sheffer sequence *symmetric* – obviously, all symmetric Sheffer sequences can be used to construct the very simple solution $s_n(m) - s_{n-L}(L+m)$ to the above boundary problem. But are there any other symmetric Sheffer sequences besides $\left\{\binom{n+x}{n}\right\}$? In [9] we have shown that except for a scaling factor there is only one parameter that describes the whole class of symmetric Sheffer sequence:

Theorem 1 *All symmetric Sheffer sequences are of the form* $\left\{\alpha s_n^{(\mu)}(x)\right\}_{n\geq 0}$ *where* α *is a nonzero scaling factor, and*

$$s_n^{(\mu)}(x) = \sum_{l=0}^{n} \binom{n}{l}\binom{x}{l}\mu^l$$

($\mu \neq 0$*). The corresponding delta operator* $\Omega^{(\mu)}$ *has the expansion*

$$\Omega^{(\mu)} = \frac{\Delta}{\mu + \Delta}$$

in terms of the forward difference operator Δ*.*

4.1 Applications: Weighted Left Turns.

If $\mu = 1$, we obtain $s_n^{(1)}(x) = \binom{n+x}{n}$ and $\Omega^{(1)} = \nabla$, the backwards difference operator. In general, $s_n^{(\mu)}(m)$ equals the weighted sum of lattice paths from $(0,0)$ to (m,n), where every left turn $\rightarrow \overset{\uparrow}{\circ}$ gets the weight μ. Because of symmetry, the classical Ballot Problem has a simple solution for this kind of weighted paths:

$$s_n^{(\mu)}(m) - s_{n-L}^{(\mu)}(m+L)$$

is the weighted sum of lattice paths from $(0,0)$ to (m,n) that they below the line $m = n-L$. Switching to path above the line $m = n+K$ is no longer an equivalent problem (except if we also switch from weighted left turns to weighted right turns.)

1	$1+3\mu$	$1+5\mu+3\mu^2$	$1+6\mu+6\mu^2+\mu^3$	$1+6\mu+6\mu^2+\mu^3$	$0\ldots$	$w_3(m)$
1	$1+2\mu$	$1+3\mu+\mu^2$	$1+3\mu+\mu^2$	0		$w_2(m)$
1	$1+\mu$	$1+\mu$	0			$w_1(m)$
1	1	0				$w_0(m)$
0	1	K	3	4	$5\ldots$	m

Counts of paths with weighted left turns above $m = n+2$

However, it is easy to verify that the Sheffer polynomial

$$w_m(m) := s_n^{(\mu)}(m) - \sum_{i\geq 1} \mu^i \binom{m-K+2}{i+1}\binom{n+K-2}{i-1} \tag{8}$$

solves the problem for $m = 0, \ldots, n+K-1$, because it satisfies the necessary and sufficient condition $w_n(n+K-1) = w_n(n+K-2)$. Note that $w_n(n+K) \neq 0$.

More details and further references are given in [9] and [12]. Related topics are correlated random walks, and nonintersecting pairs of weighted lattice paths. The q-binomial coefficients are obviously also symmetric. How to use transforms of operators [1] to count lattice path with q-weighted left turns is explained in [8].

4.2 Paths Inside a Band

The exact distribution of the two-sided two-sample Kolmogorov-Smirnov test requires counting the number of lattice paths inside a band parallel to the diagonal. This number can be described by piecewise polynomial functions. The initial conditions are $t_n(-1) = 0$ for $n = 0, \ldots, L-1$, $t_n(n-L) = 0$ for all $n \geq L$, and $t_n(n+K) = 0$ for all n.

…	*	*	*	0	21	55	89	…	$t_5(m)$
…	*	*	0	8	21	↑ 34	34	…	$t_4(m)$
…	*	0	3	8	$\underrightarrow{13}$	↑ 13	0	…	$t_L(m)$
…	0	$\underrightarrow{1}$	$\underrightarrow{3}$	$\underrightarrow{5}$	↑ 5	0			$t_2(m)$
…	0	↑ 1	2	2	0				$t_1(m)$
…	1	↑ 1	1	0					$t_0(m)$
…	-1	0	1	K	3	4	5	…	m

Paths inside a band $(L = 3,\ K = 2)$, with sample path

Symmetry of the polynomials $s_n^{(1)}(x) = \binom{n+x}{n}$ is the reason why a (relatively simple) expansion of this function exists (in our view). We want to recall this expansion, because it is so often omitted in the literature. We saw that $r_n(m) = s_n^{(1)}(m) - s_{n-L}^{(1)}(m+L)$ is the number of lattice paths below the line $m = n - L$, and reaching (m, n); a ballot number. A sum gives the number of such paths that also stay above $m = n + K$:

$$t_n(m) = \sum_{i \geq 0} (r_{n-i(K+L)}(m + i(K+L)) - r_{m-i(K+L)-K}(n + i(K+L) + K))$$

It is amazing to watch how $t_n(m)$ satisfies the recurrence, and both types of boundary values. The telescoping nature of this sum becomes essential if we verify the value $t_n(n-L) = 0$ for $n \geq L$. For $K = L$ the formula was derived by Koroljuk [2] in 1955. See [1] for another proof of the general case.

In the two-sided one-sample case, the distribution must be expanded in terms of $x^n/n!$. Because of its lack of symmetry, no closed form is known.

5 Geometric Sheffer Sequences

A Sheffer sequence $\{s_n\}_{n \geq 0}$ is *geometric*, if $s_0 \equiv 1$ and if there exists a pair of constants a and $\hat{a}$ such that

$$s_n(an) = \hat{a} s_{n-1}(an)$$

for all $n = 1, 2, \ldots$.

...	5	15	35	2 × 35	126	...	$s_4(x)$
...	4	10	2 × 10	35	56	...	$s_3(x)$
...	3	2 × 3	10	15	21	...	$s_2(x)$
...	2 × 1	3	4	5	6	...	$s_1(x)$
...	1	1	1	1	1	...	$s_0(x)$
...	1	2	3	4	5	...	x

Example for a geometric Sheffer sequence ($a = 1, \hat{a} = 2$)

There exists a geometric Sheffer sequence for any delta operator B and for any pair $a, \hat{a}$ (with $\hat{a} \neq 0$) , because this initial value problem always has a solution; it can be recursively calculated from the expansion (5). The following theorem explains why they are called "geometric":

Theorem 2 *The Sheffer sequence $\{s_n\}_{n \geq 0}$ with generating function $\sum_{n \geq 0} s_n(x) t^n = s(t) e^{x\beta(t)}$ is geometric iff $\hat{a}^n$ is the coefficient of t^n in the expansion of the formal power series $\sum_{n \geq 0} s_n(an) t^n e^{-n\beta(t)}$.*

Proof. $E^{-a} = e^{-aD_x}$ denotes the translation operator by $-a$,

$$E^{-a} p(x) = p(x - a)$$

for any polynomial $p(x)$. $\{s_n(an + x)\}_{n \geq 0}$ is a Sheffer sequence for the delta operator BE^{-a}. Denote the compositional inverse of $B(t)e^{-at}$ by $\alpha(t)$. $\{s_n\}$ is geometric iff

$$\begin{aligned} \sum_{n \geq 0} s_n(an) t^n &= 1 + \sum_{n \geq 1} \hat{a} s_{n-1}(an) t^n = 1 + t\hat{a} \sum_{n \geq 0} s_n(an + a) t^n \\ &= 1 + t\hat{a} \left(\sum_{n \geq 0} s_n(an) t^n \right) e^{a\alpha(t)} \end{aligned}$$

Solving for $\sum_{n\geq 0} s_n(an)t^n$ gives $\sum_{n\geq 0} s_n(an)t^n = 1/\left(1-\hat{a}te^{a\alpha(t)}\right)$. Substitute $B(t)e^{-at}$ for t to get

$$\sum_{n\geq 0} s_n(an)\left(B(t)e^{-at}\right)^n = \frac{1}{1-\hat{a}B(t)} \tag{9}$$

Finally, substituting $\beta(t)$ for t shows that

$$\sum_{n\geq 0} s_n(an)t^n e^{-na\beta(t)} = \frac{1}{1-\hat{a}t} \tag{10}$$

■

The above identities are special cases of Lagrange inversion (c.f. Pólya and Szegö [10], Problems and Theorems in Analysis I). We check some examples:

1. $\{x^n/n!\}$ is a geometric Sheffer sequence for D_x with $a=\hat{a}$, and $\beta(t)=t=B(t)$. Both identities give $\sum_{n\geq 0}(nte^{-t})^n = 1/(1-t)$ [10, Part III, problem 214].
2. $\left\{\binom{n+x}{n}\right\}$ is a geometric Sheffer sequence for $\nabla = 1-E^{-1}$ with $a+1=\hat{a}$, and $\beta(t) = -\ln(1-t)$. Identity (10): $\sum_{n\geq 0}\binom{n+an}{n}t^n(1-t)^{an} = 1/\left(1-(a+1)t\right)$ [10, Part III, special case of problem 216].
3. $\left\{\frac{x-an+1}{x+1}\binom{n+x}{n}\right\}$ is a geometric Sheffer sequence for ∇ with $\hat{a}=1$. Identity (9): $\sum_{n\geq 0}\frac{1}{an+1}\binom{n+an}{n}(e^{-at}-e^{-(a+1)t})^n = e^t$ [10, Part III, problem 211].

5.1 Applications: Crossings

Denote by $D(n,m;l)$ the number of (restricted) lattice paths from $(0,0)$ to (m,n) with steps $\rightarrow$ and $\uparrow$ that go through at least l of some given nodes in the plane. It is usually not difficult to calculate $D(n,m;l)$ recursively. Closed forms are known if the nodes lie on a line, $(n, an+c)$ for $n>\bar{n}$, where a,c and $\bar{n}$ are given constants. Additional restrictions may be imposed on the path; for example, the path may be required

- to cross through the node coming from below,
    - and leave in a vertical step,
- to stay above a line (or some line segments),
- to walk in a higher dimensional lattice.

The surprisingly "simple" closed forms for $D(n,m;l)$ occur when $D(n,m;0)$ can be expressed by a *geometric* Sheffer polynomial $s_n(m)$. Only in this case we get for paths terminating on a node the recurrence

$$\begin{aligned} D(n, an+c; l) &= \sum_{i=\bar{n}+l-1}^{n-1}\left(D(ai+c,i;l-1)-D(ai+c,i;l)\right)s_{n-i}(a(n-i)) \\ &= \hat{a}D(n-1, an+c; l-1) \end{aligned}$$

which is essential for further simplifications in calculating $D(n, m; l)$, $m \geq an + c$. In statistics, tests based on the number of crossings are called Takács tests [7]. The above method applies to the one-sample Takács distribution [6] as well (empirical distribution functions instead of lattice paths), because $\{x^n/n!\}$ is geometric too. More details are given in Mohanty's work, [2], [4], [5], and in [5].

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