AN ENUMERATION PROBLEM AND SOME CONSEQUENCES

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Introduction: The general problem. Let $\mathcal{C}(p,q)$ be the set of all lattice paths, starting at (0,0) and ending at (p,q). We allow only steps going one unit either upwards or to the right. A path $C \in \mathcal{C}(p,q)$ dominates $D \in \mathcal{C}(p,q)$, iff D lies below C in the weak sense ($C \geq D$).

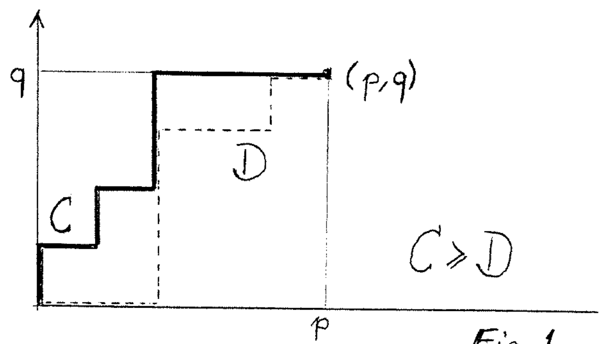

Fig. 1

Denote by $\omega(C)$ the number of all paths in $\mathcal{C}(p,q)$ which are dominated by C:

$$\omega(C) = \# \{D \in \mathcal{C}(p,q) \mid C \geq D\}.$$

Problem: Find $\sum_{C \in \mathcal{C}(p,q)} \omega(C)^n$ for n = 0, 1, ...

I learned about this problem from G. Kreweras, who had already solved the problem for n = 1 in [3]. The case n = 0 is trivial. He also had a conjecture (see (2.5)) for the quadratic case n = 2. It is natural to attack first the linear case, hoping the method will somehow generalize to the quadratic problem. The only proof (until today!) of Kreweras' conjecture in [4] may be seen this way, even if

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it was not obtained by such a generalization. And a further generalization to n = 3 seems not to be possible. Several other approaches have been tried, by J. Deken, E. Ihrig and myself. Each approach solves the linear problem but could not be generalized to the quadratic case. Still, these proofs have a "beauty of their own" and should not be completely neglected. Perhaps, the reader has the missing idea to push them further to the quadratic or even to the cubic case!

1. <u>The linear case</u>. There are many notations for a path $C \in \mathcal{C}(p,q)$. For instance, we can characterize C by the sequence of the p + 1 <u>largest</u> y-coordinates such that $(0,y_0), \ldots, (p,y_p)$ lie on C. In our example (Fig. 1) we get C = (1,2,4,4,4,4) and D = (0,0,3,3,4,4). Of course, y_p always equals q. G. Kreweras [3] proved 1965:

$$\omega(C) = \det \binom{y_{j-1}+1}{i+1-j}_{i,j=1,\ldots,p}$$

Therefore, the linear problem can be stated as

$$\sum_{0\leq y_0\leq\cdots\leq y_{p-1}\leq q} \det \binom{y_{j-1}+1}{i+1-j}^{n}_{i,j=1,\ldots,p}$$

Another way to describe C is by its sequence of p + q steps $(\rightarrow,\uparrow)$. In our example we have

(1.1) $C = (\uparrow\ \rightarrow\ \uparrow\ \rightarrow\ \uparrow\ \uparrow\ \rightarrow\ \rightarrow\ \rightarrow)$

(1.2) $D = (\rightarrow\ \rightarrow\ \uparrow\ \uparrow\ \uparrow\ \rightarrow\ \rightarrow\ \uparrow\ \rightarrow)$

$P = (\uparrow\ \circlearrowright\ \nearrow\ \rightarrow\ \nearrow\ \uparrow\ \circlearrowright\ \rightarrow\ \circlearrowright)$

The path P is obtained from the pair (C,D) by an obvious one to-one mapping. The symbol $\circlearrowright$ means a "circle step", i.e., remaining on the spot. J. Deken [1] suggested this mapping, and the following proof is due to him:

(a) P ends at the point (q,q)
Proof: P goes upwards one unit iff C goes upwards one unit.
P goes to the right one unit iff D goes upwards one unit.

(b) It is easy to check that P stays (weakly) over the diagonal iff $C \geq D$. P crosses the diagonal iff C does not dominate D. Such a P is drawn in figure 2.

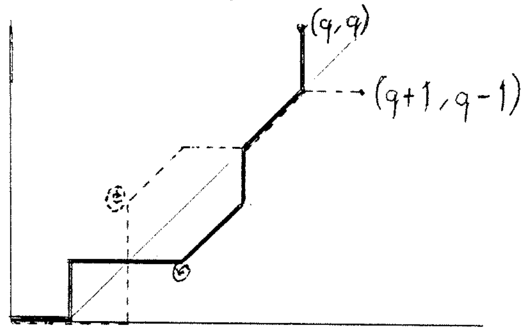


Fig. 2

Reflecting this path at the line $y = x - 1$ after its first contact with this line yields an unrestricted path with the same type of step directions. The reflected path ends at $(q + 1, q - 1)$. By elementary methods it can be shown that the number of all such unrestricted paths with $p + q$ steps of the type ↑, →, ↗ or Ⓑ equals $\binom{p+q}{q+1} \cdot \binom{p+q}{q-1}$.

(c) In (b) we found that there are $\binom{p+q}{q+1} \cdot \binom{p+q}{q-1}$ pairs $(C,D) \in \mathcal{C}(p,q)^2$ where C does not dominate D. Therefore,

$$\sum_{C \in \mathcal{C}(p,q)} \omega(C) = \binom{p+q}{p}^2 - \binom{p+q}{q+1}\binom{p+q}{q-1} = \frac{(p+q)!(p+q+1)!}{p!q!(p+1)!(q+1)!} \tag{1.3}$$

A proof for this result in terms of generating functions can be found already in MacMahon's work [5, Vol. 2, No. 242]. J. Deken's proof above is perhaps the most elementary and beautiful one, using only elementary path counting techniques. But a generalization of this proof to the case $n = 2$ is still an open problem. The same is true for the following proof of E. Ihrig [2]. The previous proof made use of the reflection principle, what can be characterized as "count" something else (easier) and use a one-to-one correspondence". We will do that again, but now we count equivalent classes. If we write a 1 for each ↑ and a 0 for each →, the paths C and D in (1.1) and (1.2) can be written as

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

where the dominance of C over D forces the partial sums in the top row to be larger or equal to the corresponding sums in the bottom row. Let $\mathcal{C}(p,q)^2_{\geq}$ be the set of all such restricted matrices. We want to find the number of elements of $\mathcal{C}(p,q)^2_{\geq}$. Denote by $\mathcal{L}(p,q)$ the set of unrestricted $2 \times (p + q + 1)$ matrices with p zeros and $(q + 1)$ ones in the top row, and $(p + 1)$ zeros and q ones in the bottom row. If K and L are two matrices in $\mathcal{L}(p,q)$ we call $K \sim L$ iff L can be obtained from K by a cyclic permutation of its columns.

Lemma: $\mathcal{L}$ has $\binom{p+q+1}{p}\binom{p+q+1}{q}\frac{1}{p+q+1}$ equivalence classes.

Proof (E. Ihrig [2]): $\mathcal{L}(p,q)$ has $\binom{p+q+1}{p}\binom{p+q+1}{q}$ elements. Therefore, it is sufficient to show that each equivalence class has exactly $p + q + 1$ elements. Denote by σ the cyclic permutation $\begin{pmatrix} 1 & 2 & 3 & \ldots & p+q & p+q+1 \\ 2 & 3 & 4 & & p+q+1 & 1 \end{pmatrix}$. We have $K \sim L$ iff

$$\sigma^i(K) = \sigma^j(L) \tag{1.4}$$

for certain integers i and j, where $\sigma(K)$ is the matrix in $\mathcal{L}(p,q)$ which is obtained from the permutation σ of the columns of K. Each equivalence class has exactly $p + q + 1$ elements iff $\sigma^j(K) = K$ <u>only</u> for $j \equiv 0 \mod(p + q + 1)$ (we have to count the orbits of σ acting on $\mathcal{L}(p,q)$). This last assertion can be proved in two steps:

(1) Let j_0 be the smallest positive integer such that $\sigma^{j_0}(K) = K$. Then $j_0 \mid p + q + 1$. (Details omitted.)

(2) From (1) follows that K consists of $r_0 = \frac{p+q+1}{j_0}$ equal $2 \times j_0$ - submatrices K_0: $K = (\underbrace{K_0, K_0, \ldots, K_0}_{r_0 \text{ matrices}})$.

The sum of elements in the top row of K is $q + 1$, in the bottom row is q. Denote the top sum in K_0 by t and the bottom sum by s. Then $q + 1 = r_0 t$ and $q = r_0 s$. Hence, $r_0 = 1$ because it divides q and $q + 1$. In other words,

$j_0 = p + q + 1$. The lemma above tells us that $\mathcal{L}/\sim$ has the desired number of elements. Therefore it remains to construct a one-to-one mapping between the matrices in $\mathcal{C}(p,q)^2_{\geq}$ and $\mathcal{L}/\sim$. This will be done by the mapping ϕ which places the additional column $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ before any matrix $A \in \mathcal{C}(p,q)^2_{\geq}$.

Theorem: ϕ is a bijection from $\mathcal{C}(p,q)^2_{\geq}$ onto $\mathcal{L}/\sim$.

Proof (E. Ihrig [2]):

(a) First we prove that ϕ is a bijection. If is not bijective then there exist $A, B \in \mathcal{C}(p,q)^2_{\geq}$ such that $\phi(A) \sim \phi(B)$ and $A \neq B$. Hence $\sigma^{-t}(\phi(A)) = \phi(B)$ for a certain t (see (1.4)). From $A \neq B$ we get $t \neq 0 \bmod (p + q + 1)$. The idea of the proof can be seen in the following scheme:

$$\phi(B) = \begin{pmatrix} 1 & b_{11} & \cdots & b_{1,t-1} & b_{1,t} & \cdots & b_{1,p+q} \\ 0 & b_{21} & \cdots & b_{2,t-1} & b_{2,t} & \cdots & b_{2,p+q} \end{pmatrix} = \sigma^{-t}(\phi(A))$$

$$= \begin{pmatrix} a_{1t} & a_{1,t+1} & \cdots & a_{1,p+q} & 1 & a_{11} & a_{12} & \cdots & a_{1,t-1} \\ a_{2t} & a_{2,t+1} & \cdots & a_{2,p+q} & 0 & a_{21} & a_{12} & \cdots & a_{2,t-1} \end{pmatrix}.$$

Each partial sum in the top row of $\phi(B)$ is larger than the corresponding sum in the bottom row. Therefore,

$a_{1t} + a_{1,t+1} + \cdots + a_{1,p+q+1} > a_{2t} + a_{2,t+1} + \cdots + a_{2,p+q}$.

But A lies also in $\mathcal{C}(p,q)^2_{\geq}$. Hence,

$a_{11} + a_{12} + \cdots a_{1,t-1} \geq a_{21} + a_{22} + \cdots + a_{2,t-1}$.

Combining both inequalities gives that the top sum in A is strictly larger than the bottom sum. But both sums are equal to q, a contradiction.

(b) For proving the onto part we will show that for each $L \in \mathcal{L}$ we can find a j such that $\overline{L} := \sigma^{j}(L)$ satisfies the inequality

(1.5) $$\sum_{k=1}^{i}(\overline{\ell}_{1,k} - \overline{\ell}_{2,k}) \geq 1, \quad \text{for all } i = 1, \ldots, p + q + 1.$$

For $i = 1$ this means that $\binom{1}{0}$ is the first column in $\overline{L}$. Removing this first colum gives a matrix in $\mathcal{L}(p,q)^2_{\geq}$ (use (1.5) again). Therefore, ϕ is onto if we can prove the existence of a j such that $\sigma^j(L) = L$. We will do that by showing that the set

(1.6) $$S := \{\ell \mid \text{there is a } j \text{ such that } \sum_{k=1}^{i} (\sigma^j(L)_{1,k} - \sigma^j(L)_{2,k}) \geq 1 \text{ for all } i = 1, \ldots, \ell\}$$

contains the number $p + q + 1$. It is easy to see that $1 \in S$. $p + q + 1$ lies in S because we can prove that from $\ell \in S$ and $\ell < p + q + 1$ follows that there must be a number larger than ℓ in S. We sketch the basic steps:

(1) Let t be the largest integer such that
$$\sum_{k=t}^{p+q+1} (\sigma^j(L)_{1,k} - \sigma^j(L)_{2,k}) \geq 1$$
where σ^j corresponds to ℓ (see (1.6)). t is larger than ℓ.

(2) $\sum_{k=t}^{i} (\sigma^j(L)_{1,k} - \sigma^j(L)_{2,k}) \geq 1$ for all $i = t, \ldots, p + q + 1$.

(3) $\ell + p + q + 2 - t \in S$ (use $\sigma^{j+p+q+2-t}$)

The result (2.5) for the quadratic case suggests that there may be generalization of this proof to $n = 2$. Again, this generalization has not been done until today. There is also a recursive approach to the linear case, which solves even a slightly more general problem. For $0 \leq j \leq q$ let

$$a(p,j,q) = \# \{(C,D) \mid C \in \mathcal{L}(p,q),\ D \in \mathcal{L}(p,j),\ C \geq D\}.$$

Obviously, $a(p,q,q) = \# \mathcal{L}(p,q)^2_{\geq}$, the number in question.

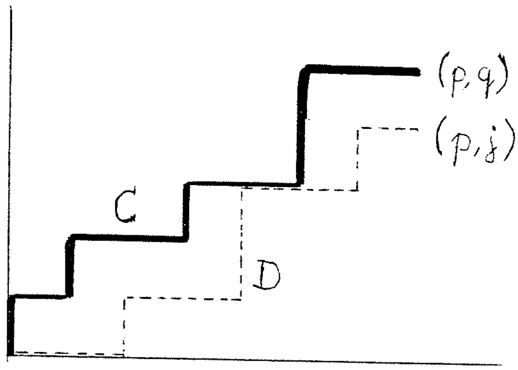


Fig. 3

a(p,j,q) is determined by the following recursion and initial conditions:

(a) $a(p,j,q) = a(p,j,q-1) + a(p,j-1,q) - a(p,j-1,q-1) + a(p-1,j,q)$

(b) $a(0,j,q) = 1$ for all $0 \leq j \leq q$

(c) $a(p,-1,q) = 0$ for all $p \geq 0$ and $q \geq 0$

(d) $a(p,j,j-1) = a(p,j-1,j-1)$ for all $p \geq 0$ and $j \geq 0$.

Condition (d) allows us to use recursion (a) also for the case $j = q$. The definition of the numbers a(p,j,q) may be extended to <u>all</u> $j \geq -1$ and integers q without changing a(p,j,q) for $0 \leq j \leq q$. We use now the notation ∇ for the backwards difference operator $\nabla f(x) = f(x) - f(x-1)$. In addition, $\nabla_i\, a(p,j,q)$ means that ∇ acts on the i-th variable of the function a(p,j,q) (i = 1,2,3,). The extended definition of a(p,j,q) can now be written as

$$\nabla_2\nabla_3\, a(p,j,q) = a(p-1,j,q) \text{ for all } q \text{ and } j \geq 0$$

$$a(0,j,q) = 1 \text{ for all } q \text{ and } j \geq -1$$

$$a(p,-1,q) = 0 \text{ for all } q \text{ and } p \geq 0$$

$$\nabla_2\, a(p,j,j-1) = 0 \text{ for all } p \geq 0 \text{ and } j \geq 0.$$

The number $a(p,q,q)$, our number in question, can be expressed as

$$a(p,q,q) = \nabla_2\, a(p+1,q,q) \quad \text{(see (a) and (d))}. \tag{1.7}$$

Let $a'(p,j,q) = \nabla_2\, a(p,j,q)$. For technical reasons working with $a'(p,j,q)$ is easier. We have

(a') $\nabla_2\nabla_3\, a'(p,j,q) = a'(p-1,j,q)$ for all q and $j \geq 0$
(b') $a'(0,j,q) = \delta_{0,j}$ for all q and $j \geq -1$
(c') $a'(p,-1,q) = 0$ for all q and $p \geq 0$
(d') $a'(p,j,j-1) = 0$ for all $p \geq 1$ and $j \geq 0$.

Properties (a') - (d') uniquely define $a'(p,j,q)$. It is easy to check that

$$\binom{p+j-1}{j}\binom{p+q}{p}\frac{q-j+1}{q+1} \tag{1.8}$$

satisfies all those conditions. Therefore,

$$\#\,\mathcal{C}(p,q)^2_{\geq} = a(p,q,q) = a'(p+1,q,q) = \binom{p+q}{q}\binom{p+q+1}{p+1}\frac{1}{q+1}$$

(see (1.7)).

2. The quadratic case. The recursive method for the linear case can be easily extended to the quadratic case: Let

$$A(p,j,k,q) = \#\{(C,D,E) \mid C\in\mathcal{C}(p,q), D\in\mathcal{C}(p,j), E\in\mathcal{C}(p,k);\ C \geq D \text{ and } C \geq E\}$$

for all $0 \leq j \leq q$ and $0 \leq k \leq q$.

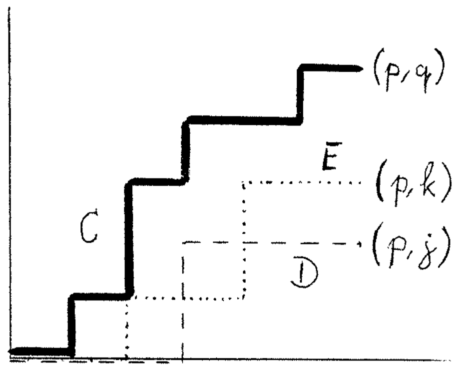


Fig. 4

We get $A(p,j,0,q) = a(p,j,q)$ and $A(p,q,q,q) = \sum_{C\in\mathcal{C}(p,q)} \omega(C)^2$.

Again, the numbers $A(p,j,k,q)$ count essentially more general objects than $A(p,q,q,q)$. Therefore, the following (extended) recursion for $A'(p,k,j,q) = \nabla_2\nabla_3 A(p,i,j,q)$ has not been solved (to my knowledge).

(A') $\nabla_2\nabla_3\nabla_4 A'(p,j,k,q) = A'(p-1,j,b,q)$ for all q and $p \geq 1$, $j \geq 0$, $k \geq 0$

(B') $A'(0,j,k,q) = \delta_{0,j+k}$ for all q and $j \geq 0$, $k \geq 0$.

(C') $A'(p,-1,k,q) = A'(p,j,-1,q) = 0$ for all q and $p \geq 0$, $j \geq -1$, $k \geq -1$.

(D') $A'(p,j,k,\max\{j,k\}-1) = 0$ for all $p \geq 1$ and $j \geq 0$, $k \geq 0$.

It is this last condition that gives rise to all the difficulties. It is possible to reduce the number of parameters by one and to consider $b(p,j,q) = A'(p,j,j,q)$ only. Now the initial conditions are

$b(0,j,q) = \delta_{0,j}$ for all q and $j \geq 0$
$b(p,-1,q) = 0$ for all q and $p \geq 0$
$b(p,j,j-1) = 0$ for all $p \geq 1$ and $j \geq 0$.

But the recursion for the b(p,j,q) seems to be untractable:

$$b(p,j,q) = \binom{p+j-1}{j} a'(p,j,q) - \sum_{i=1}^{p-1} \sum_{\ell=0}^{j-1} \sum_{k=\ell}^{j-1} b(i,\ell,k)a'(p-i,j-\ell,q-\ell) \times$$
$$\times \binom{p-i+j-k-2}{j-1-k} \quad \text{(see (1.8) for } a'(p,j,q)\text{)}.$$

G. Kreweras and myself proved the quadratic case in [4] by using a combination of the two priciple methods of the previous chapter:

(a) Find a bijection onto a "simpler" set

(b) Find a recursion.

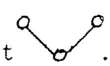
Method (a) is useful only if the simpler set can be counted. Method (b) is useful only if the recursion can be solved. In the following combination of (a) and (b) the recursion can be solved if certain side conditions hold. By method (a) we find that these side conditions are true in the "simpler" set (without counting it). To be more specific, let (D_p^q) be a double sequence of polynomials of degree p such that

(2.1) $\quad D_p^q(t) - D_p^q(t+1) = D_{p-1}^q(t)$ for all t and $p \geq 1$, $q \geq 0$.

The following can be shown: if we initialize the polynomials D_p^q by our unknown numbers $D_p^q(0) = \sum_{C \in \mathcal{C}(p,q)} \omega(C)^2$ for all $p \geq 0$ and $q \geq 0$, then

(2.2) $\quad D_p^q(0) = D_q^p(0)$ for all $p \geq 0$ and $q \geq 0$

(2.3) $\quad D_0^q(t) \equiv 1$ for all $q \geq 0$ (see [4,(A"),(B")])

But the initial conditions (2.2) and (2.3) are not sufficient to define the polynomials D_p^q uniquely. Let M_q be the direct product of a q-element chain (like {0,1, ..., q-1}) with the three element poset .
For brevity, the reader is referred to [4] or to R.P. Stanley's "Ordered Structures and Partitions" [6] for technical details and definitions. In [4] it is proved that $D_p^q(3q+1)$ = number of linear extensions of

M_q with exactly p descents. Each maximal chain in M_q consits of exactly q + 1 elements. Therefore, proposition 18.6 of R.P. Stanley [6, p. 70] tells us that

(2.4) $$D_p^q(3q+1) = D_{2q-1-p}^q(3q+1)$$

(observe that $D_n^q \equiv 0$ for all $n < 0$). Now, the conditions (2.2), (2.3), (2.4) and the recursion (2.1) define the polynomials D_p^q uniquely. It remains to show that Kreweras' conjecture

(2.5) $$D_p^q(0) = \sum_{C\in\mathcal{C}(p,q)} \omega(C)^2 = \frac{(p+q+1)!(2p+2q+1)!}{(p+1)!(q+1)!(2p+1)!(2q+1)!}$$

leads to polynomials which satisfy (2.2), (2.3) and (2.4). This is done in [4,6)]. The polynomials D_p^q count several interesting objects in the poset M_q. In Stanley's notation and terminology we get

(2.6) $$D_p^q(3q+1) = W_p(M_q;1) = w_p \quad [6, \text{p. } 24 \text{ and p. } 43]$$

$$= \text{number of linear extensions of } M_q \text{ with p descents.}$$

$$= M_q \text{ - Eulerian number.}$$

$e(P)$ = number of all linear extensions of M_q [6, p. 10]

$$= \sum_{s\geq 0} w_s = \sum_{2\geq 0} D_s^q(3q+1)$$

$$= D_p^q(3q) \text{ for all } s \geq 2q-1$$

(observe that $D_s^q(3q+1) = 0$ for all $s \geq 2q$).

$$= \frac{2^{2q}\,(3q)!}{(q+1)!(2q+1)!} \quad (\text{see } [4,8]).$$

Even the numbers $D_p^q(0)$ themselves find an interpretation in M_q:

$$D_p^q(0) = \sum_{k=0}^{p} D_s^q(3q+1)\binom{p-s+3q}{p-s} \qquad \text{(follows from (2.1))}$$

$$= \sum_{k=0}^{p} w_s \binom{3q+p-s}{3q} \qquad (2.6)$$

$$= \Omega(M_q;p+1) \qquad [6,p.\ 26]$$

$$= \text{the } M_q \text{ - order polynomial.}$$

The number of surjective M_q-partitions $P \to \{1,2,\ldots,s\}$ is denoted by e_s in [6, p. 45].

$$e_s = \sum_{k=0}^{3q-1} w_k \binom{3q-1-1}{3q-s} \qquad [6,\ p.\ 47]$$

$$= \sum_{k=0}^{s-1} D_k^q(3q+1)\binom{3q-k-1}{s-1-k} \quad \text{for all } 1 \leq s \leq p$$

$$= D_{s-1}^q(s) \text{ for all } 1 \leq s \leq p.$$

The w_p can be expressed as hypergeometric functions (nearly well poised of the second kind and Saalschützian). From (2.4) one obtains the identity

$$w_p = (-1)^p \binom{3q+1}{p} {}_4F_3\left[\begin{matrix} q+2, q+3/2, q+1, -p; \\ 3/2,\ 2,\ 3q+2-p \end{matrix}\right]$$

$$= (-1)^{p+1} \binom{3q+1}{q+p+2} {}_4F_3\left[\begin{matrix} q+2, q+3/2, q+1,\ -2q+1+p; \\ 3/2,\ 2,\ q+3+p \end{matrix}\right]$$

$$= w_{2q-1-p}.$$

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