A Finite Operator Approach to the Tennis Ball Problem

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Abstract

The tennis ball problem can be stated as follows: Given integers r, s, n, with $0 < r < s$, label sn balls $1, 2, \ldots, sn$. Place the first s balls, labeled $1, 2, \ldots, s$ into a bin, then remove r balls from the bin. Repeat this process n times, each time inserting the next s balls in sequence and removing r balls. The question we seek to answer is, "How many different sets of rn balls lie outside the bin after n turns?" We state a conjecture for the case $s = 2r$, and derive a generating function for $r = 2$ and $r = 3$.

1 Introduction

The tennis ball problem can be viewed as a lattice path enumeration. Consider lattice walks in the plane with East $\langle 1, 0\rangle$ and North $\langle 0, 1\rangle$ steps. We count the number of paths from $(0, 1)$ to $((s-r)n+1, rn+1)$ that stay weakly above the boundary $(E^{s-r}N^r)^n$. It is easy to see that, for each $1 \leq i \leq n$, at least ri of the first si steps must be N steps, and for $i = n$ this is an equality. We associate with each of the N steps one of the labeled balls, namely the ball with the label matching the number, in sequence, of the N step as a member of the path to $((s-r)n+1, rn+1)$. Note that this method of counting does not take into consideration the order in which the balls are removed, only the set of labels.

Example 1 $r = 2$, $s = 4$, $n = 3$.
Case 1: Insert balls 1-4, and remove balls 1 and 2. Then insert balls 5-8 and remove balls 3 and 5. Then insert balls 9-12 and remove balls 7 and 9. The set of balls outside the bin is $\{1, 2, 3, 5, 7, 9\}$*, corresponding to the lattice walk* $NNNENENENEEE$.
Case 2: Insert balls 1-4, and remove balls 1 and 3. Then insert balls 5-8 and remove balls 5 and 7. Then insert balls 9-12 and remove balls 2 and 9. The set of balls outside the bin is $\{1, 2, 3, 5, 7, 9\}$*, corresponding to the lattice walk* $NNNENENENEEE$.
Although the balls removed on each turn differ in the two cases, both the sets of labels and lattice walks were the same. We see, then, that this method of counting avoids redundancy.

Given a boundary $(E^{s-r}N^r)^n$, let $t_n(i)$ represent the number of paths to (n, i) for points above the boundary. The number of walks to (n, i) follows the recursion $t_n(i) = t_{n-1}(i) + t_n(i-1)$. Because $t_0(i) = 1$ for all $i \geq 1$, and $t_n(i) = 0$ at all points (n, i) directly below the boundary $(n > 0)$, we can uniquely extend the values of $t_n(i)$ to polynomials of degree n on points below the boundary. We call the polynomials again $t_n(i)$. Note that $t_0(x) = 1$ for all x, r, s.

i	1	7	28	84	195	381	662	*662*
6	1	6	21	56	111	186	281	0
5	1	5	15	35	55	75	95	-281
4	1	4	10	20	*20*	20	20	-376
3	1	3	6	10	0	0	0	-396
2	1	2	3	4	-10	0	0	-396
1	1	*1*	1	1	-14	-10	0	-396
0	1	0	0	0	-15	4	-10	-396
-1	1	-1	0	0	-15	19	-14	-386
$n:$	0	1	2	3	4	5	6	7

The polynomials $t_n(i)$ for the case $s = 6$, $r = 3$

In [2], Anna de Mier and Marc Noir showed that the generating function

$$f(z) = \sum_{n \geq 0} t_{(s-r)n+1}(rn+1) z^n$$

satisfies $f(z) = -(1-w_1)\cdots(1-w_{s-r})/z$, where $w_1, \ldots, w_{s-r}$ are the unique fractional power series solutions of $(w-1)^{s-r} - zw^s = 0$. Explicit solutions are hard to get from this relationship when $s - r \nmid r$. The case $r = 2$, $s = 4$ has been solved in [1] by different methods. In the following, we present a conjecture for the case $s - r = r$, prove it for $r = 2$, and show the resulting generating function for $r = 3$ based on this conjecture.

2 The conjecture

As in the Introduction, we denote by $t_j(k)$ the number of paths from $(0,1)$ to (j,k) with steps $E = \langle 1, 0 \rangle$ and $N = \langle 0, 1 \rangle$, staying weakly above the boundary $(E^r N^r)^n$.

Conjecture 2

$$\frac{1}{r} \sum_{i=0}^{r-1} t_{rn}(rn-i) = C_{rn},$$

where C_n is the n^{th} Catalan number, $C_n = \frac{1}{n+1}\binom{2n}{n}$.

The conjecture is trivial for $r = 1$. We will show it now for $r = 2$.

Lemma 3 *If $r = 2$, then $t_{2n}(2n) + t_{2n}(2n-1) = 2C_{2n}$*

Proof. $t_{2n}(2n) - C_{2n}$ is the number of paths that do not stay above the boundary $(EN)^n$, but stay above $(E^2N^2)^n$. These paths stay above the boundary $(EN)^n$ until they reach some point $(2i, 2i-1)$ crossing under this boundary, then stay above the boundary $(E^2N^2)^n$ from then on, so $t_{2n}(2n) - C_{2n} = \sum_{i=1}^{n} C_{2i-1} t_{2(n-i)}(2(n-i))$. In the same way, we see that $t_{2n}(2n-1) = \sum_{i=1}^{n} C_{2i-1} t_{2(n-i)}(2(n-i)-1)$. We now proceed by induction over n. By observation, the lemma holds for $n=1$. Now assume that it holds for $n-1$. Then $t_{2n}(2n) + t_{2n}(2n-1)$

$$
\begin{aligned}
&= C_{2n} + \sum_{i=1}^{n} C_{2i-1}\left(t_{2(n-i)}(2(n-i)) + t_{2(n-i)}(2(n-i)-1)\right) \\
&= C_{2n} + 2\sum_{i=1}^{n} C_{2i-1} C_{2(n-i)}
\end{aligned}
$$

by our induction hypothesis. Expanding this sum, we see that

$$
\begin{aligned}
2\sum_{i=1}^{n} C_{2i-1} C_{2(n-i)} &= \sum_{i=1}^{n} C_{2i-1} C_{2(n-i)} + \sum_{i=0}^{n-i} C_{2(n-i)-1} C_{2i} \\
&= \sum_{i=0}^{2n-1} C_i C_{2n-1-i} = C_{2n}.
\end{aligned}
$$

Thus $t_{2n}(2n) + t_{2n}(2n-1) = 2C_{2n}$. ■

The numbers $t_{2n}(2n-1) = t_{2n-1}(2n-1)$ are the numbers we desire for the tennis ball problem in the case $s = 4, r = 2$.

3 The tennis ball numbers for $r = 2$ and $s = 4$.

For the context of the following definition see "The Finite Operator Calculus" by Rota, Kahaner, and Odlyzko [3].

Definition 4 *A B-Sheffer sequence is a sequence of polynomials* $(p_i)_{i\in\mathbb{N}_0}$ *such that* $\deg p_i = i$, $Bp_i = p_{i-1}$, *and* $p_0 \neq 0$, *associated with an operator* B *that can be written as a power series of order* 1 *in the derivative operator,* $\mathcal{D}$. *If a B-Sheffer sequence has the initial value* $p_n(0) = \delta_{0,n}$, *then it is a B-basic sequence.*

Lattice paths with steps N and E describe a Sheffer sequence $(t_i(x))$ for the backwards difference operator ∇, because $\nabla t_n(x) = t_{n-1}(x)$. Since we want to know the values $t_{2n+1}(2n+1)$, we need an operator for the sequence $(t_n(n+x))$, and so we compose the operators E^{-1} and ∇, where

$E^{-1}p_i(x) = p_i(x-1)$, so that $E^{-1}\nabla t_n(n+x) = t_{n-1}(n-1+x)$. The operator $E^{-1}\nabla$ has basic polynomials $b_n(x) = \frac{x}{x+n}\binom{2n+x-1}{n}$.

i	1	7	28	80	185	343	554	*554*
6	1	6	21	52	105	158	211	0
5	1	5	15	31	53	*53*	53	-211
4	1	4	10	16	22	0	0	-264
3	1	3	6	*6*	6	-22	0	-264
2	1	2	3	0	0	-28	-22	-264
1	1	*1*	1	-3	0	-28	6	-244
0	1	0	0	-4	-3	-28	34	-250
-1	1	-1	0	-4	1	-25	62	-284
$n:$	0	1	2	3	4	5	6	7

The polynomials $t_n(i)$ for the case $s = 4$, $r = 2$

We consider now the tennis ball problem with $s = 4, r = 2$. Because of the boundary given, we get $t_{2n}(2n-3) = 0$ for all $n > 0$, and $t_{2n+1}(2n-2) = -t_{2n}(2n) - t_{2n}(2n-1) = -2C_{2n}$ by Lemma 3 for all $n \geq 0$. By the Binomial Theorem for Sheffer sequences [3],

$$\begin{aligned} t_n(n) &= \sum_{i=0}^{n} t_i(i-3)\frac{3}{n+3-i}\binom{2n-2i+2}{n-i} \\ &= \frac{3}{n+3}\binom{2n+2}{n} + \sum_{i=0}^{(n-1)/2} \frac{3t_{2i+1}(2i-2)}{n+2-2i}\binom{2n-4i}{n-2i-1} \\ &= \frac{3}{n+3}\binom{2n+2}{n} - 2\sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{3C_{2i}}{n+2-2i}\binom{2n-4i}{n-2i-1} \end{aligned}$$

The tennis ball numbers, $t_{2n+1}(2n+1)$, and their generating function are easily obtained from the above closed form.

4 The generating function for $r = 3$ and $s = 6$

We consider now the case $s = 6, r = 3$. In this case, the tennis ball numbers are $t_{3n+1}(3n+1)$. Note that the t_i are different functions from those used in the $(4, 2)$ case. Using the same operator and basic sequence as before, by the binomial theorem for Sheffer sequences

$$t_n(n) = \sum_{i=0}^{n} t_i(i-3)\frac{3}{3+n-i}\binom{2(n-i+1)}{n-i}.$$

The zeroes below the given boundary enforce the condition $t_i\,(i-3)=0$ unless $i\equiv 1 \bmod 3$ (see the Table in the Introduction). Therefore this sum simplifies to

$$t_n\,(n)=\frac{3}{3+n}\binom{2n+2}{n}-\frac{6}{n+2}\binom{2n}{n-1}+\sum_{i=1}^{\frac{n-1}{3}}\frac{t_{3i+1}\,(3i-2)}{n-3i+2}\binom{2n-6i}{n-3i+1}.$$

Let the function $f_{6,3}(z):=\sum_{n\geq 0}t_{3n+1}(3n+1)z^n$ describe the generating function for the tennis ball numbers. The zeroes below the boundary also imply that $t_{3n}(3n-2)+t_{3n}(3n-1)+t_{3n}(3n)=-t_{3n+1}(3n-3)$. Applying the conjecture we get $f_{6,3}(z)$

$$\begin{aligned}
&= \sum_{n\geq 0}\Bigg(\frac{3}{3n+4}\binom{6n+4}{3n+1}-\frac{2}{n+1}\binom{6n+2}{3n}\\
&\quad +\sum_{i=1}^{n}\frac{t_{3i+1}(3i-2)}{n-i+1}\binom{6n-6i+2}{3n-3i}\Bigg)z^n\\
&= \sum_{n\geq 0}\left(\frac{3}{3n+4}\binom{6n+4}{3n+1}-\frac{2}{n+1}\binom{6n+2}{3n}\right)z^n\\
&\quad +\left(\sum_{n\geq 1}t_{3n+1}(3n-2)z^n\right)\left(\sum_{n\geq 0}\frac{1}{n+1}\binom{6n+2}{3n}z^n\right)\\
&= \sum_{n\geq 0}\left(\frac{3}{3n+4}\binom{6n+4}{3n+1}-\frac{2}{n+1}\binom{6n+2}{3n}\right)z^n+\sum_{n\geq 0}\frac{z^n}{n+1}\binom{6n+2}{3n}\\
&\quad \times\left(\sum_{n\geq 0}t_{3n+1}(3n+1)z^{n+1}-\sum_{n\geq 0}\frac{6}{3n+4}\binom{6n+5}{3(n+1)}z^{n+1}\right).
\end{aligned}$$

Solving for $f_{6,3}(z)$, we obtain the "explicit" generating function for the $(6,3)$-tennis ball numbers: $f_{6,3}(z)=$

$$\frac{\sum_{n\geq 0}\frac{3z^n}{3n+4}\binom{6n+4}{3n+1}z^n+\left(1-3\sum_{n\geq 0}C_{3(n+1)}z^{n+1}\right)\left(\sum_{n\geq 0}\frac{z^n}{n+1}\binom{6n+2}{3n}\right)}{1-\sum_{n\geq 0}\frac{1}{n+1}\binom{6n+2}{3n}z^{n+1}}.$$

Note that the terms $\frac{1}{n+1}\binom{6n+2}{3n}$ and $\frac{3}{3n+4}\binom{6n+4}{3n+1}$ equal the coefficients of z^m in $\left(2/\left(1+\sqrt{1-4z}\right)\right)^3$ where $m=0 \bmod 3$, and $m=1 \bmod 3$, respectively. Therefore, the above generating function can be expressed in term of $2/\left(1+\sqrt{1-4z}\right)$.

References

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[3] Rota, G.-C., Kahaner, D. and Odlyzko, A., On the Foundations of Combinatorial Theory VIII: Finite operator calculus. *J. Math. Anal. Appl.* **42** (1973) 684 – 760.