PROBLEMS

11558. Proposed by Andrew McFarland, Płock, Poland. Given four concentric circles, find a necessary and sufficient condition that there be a rectangle with one corner on each circle.

11559. Proposed by Michel Bataille, Rouen, France. For positive $p$ and $x \in (0, 1)$, define the sequence $(x_n)$ by $x_0 = 1$, $x_1 = x$, and, for $n \geq 1$,

$$x_{n+1} = \frac{px_n x_{n-1} + (1 - p)x_n^2}{(1 + p)x_{n-1} - px_n}.$$ 

Find positive real numbers $\alpha$, $\beta$ such that $\lim_{n \to \infty} n^\alpha x_n = \beta$.

11560. Proposed by Gregory Galperin, Eastern Illinois University, Charleston, IL, and Yury Ionin, Central Michigan University, Mount Pleasant, MI.

(a) The diagonals of a convex pentagon $P_0 P_1 P_2 P_3 P_4$ divide it into 11 regions, of which 10 are triangular. Of these 10, five have two vertices on the diagonal $P_0 P_2$. Prove that if each of these has rational area, then the other five triangles, and the original pentagon, all have rational areas.

(b) Let $P_0, P_1, \ldots, P_{n-1}, n \geq 5$ be points in the plane. Suppose no three are collinear, and, interpreting indices on $P_k$ as periodic modulo $n$, suppose that for all $k$, $P_{k-1} P_{k+1}$ is not parallel to $P_k P_{k+2}$. Let $Q_k$ be the intersection of $P_{k-1} P_{k+1}$ with $P_k P_{k+2}$. Let $\alpha_k$ be the area of triangle $P_k Q_k P_{k+1}$, and let $\beta_k$ be the area of triangle $P_{k+1} Q_k Q_{k+1}$. For $0 \leq j \leq 2n - 1$, let

$$\gamma_j = \begin{cases}
\alpha_{j/2}, & \text{if } j \text{ is even}; \\
\beta_{(j-1)/2}, & \text{if } j \text{ is odd}.
\end{cases}$$

Interpreting indices on $\gamma_j$ as periodic modulo $2n$, find the least $m$ such that if $m$ consecutive $\gamma_j$ are rational, then all are rational.

doi:10.4169/amer.math.monthly.118.03.275
11561. Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania. Let \( f_1, \ldots, f_n \) be continuous real valued functions on \([0, 1]\), none identically zero, such that \( \int_0^1 f_i(x) f_j(x) \, dx = 0 \) if \( i \neq j \). Prove that

\[
\prod_{k=1}^n \int_0^1 f_k^2(x) \, dx \geq n^n \left( \prod_{k=1}^n \int_0^1 f_k(x) \, dx \right)^2,
\]

\[
\sum_{k=1}^n \int_0^1 f_k^2(x) \, dx \geq \left( \sum_{k=1}^n \int_0^1 f_k(x) \, dx \right)^2, \quad \text{and}
\]

\[
\frac{\sum_{k=1}^n \int_0^1 f_k^2(x) \, dx}{\left( \sum_{k=1}^n \int_0^1 f_k(x) \, dx \right)^2} \geq n^2.
\]

11562. Proposed by Pál Péter Dályay, Szeged, Hungary. For positive \( a, b, c, \) and \( z \), let \( \Psi_{a,b,c}(z) = \Gamma((za + b + c)/(z + 2)) \), where \( \Gamma \) denotes the gamma function. Show that \( \Psi_{a,b,c}(z) \Psi_{b,c,a}(z) \Psi_{c,a,b}(z) \) is increasing in \( z \) for \( z \geq 1 \).

11563. Proposed by Vlad Matei (student), University of Bucharest, Bucharest, Romania. For each integer \( k \geq 2 \), find all nonconstant \( f \) in \( \mathbb{Z}[x] \) such that for every prime \( p \), \( f(p) \) has no nontrivial \( k \)-th-power divisor.

**SOLUTIONS**

**Explaining a Polynomial**

11403 [2008, 949]. Proposed by Yaming Yu, University of California–Irvine, Irvine, CA. Let \( n \) be an integer greater than 1, and let \( f_n \) be the polynomial given by

\[
f_n(x) = \sum_{i=0}^n \binom{n}{i} (-x)^{n-i} \prod_{j=0}^{i-1} (x + j).
\]

Find the degree of \( f_n \).

Solution by Nicolás Caro, IMPA, Rio de Janeiro, Brazil, and independently by Cosmin Pohoata, Tudor Vianu National College, Bucharest, Romania. The degree of \( f_n \) is \( \lfloor n/2 \rfloor \). This follows immediately from the stronger statement that the coefficient of \( x^r \) in \( f_n(x) \) is the number of derangements of \([n]\) with \( r \) cycles, since each cycle must have at least two elements. Here \([n] = \{1, \ldots, n\} \), and a derangement is a permutation with no fixed points.

Let \( c(n, k) \) be the number of permutations of \([n]\) with \( k \) cycles (the unsigned Stirling number of the first kind). The well-known generating function for these numbers is given by \( \sum_{k=1}^n c(n, k) x^k = \prod_{j=0}^{n-1} (x + j) \) (provable in many ways, including induction on \( n \)). Thus

\[
f_n(x) = \sum_{i=0}^n \binom{n}{i} (-x)^{n-i} \sum_{k=1}^i c(i, k) x^k = \sum_{\ell=0}^n \binom{n}{\ell} (-x)^{\ell} \sum_{k=0}^{n-\ell} c(n-\ell, k) x^k
\]

\[
= \sum_{\ell=0}^n \binom{n}{\ell} (-1)^\ell \sum_{r=\ell}^n c(n-\ell, r-\ell) x^r = \sum_{r=0}^n \sum_{\ell=0}^{r} \binom{n}{\ell} (-1)^\ell c(n-\ell, r-\ell) x^r.
\]
The coefficient of $x^r$ in this expression is precisely the inclusion-exclusion formula to count permutations with $r$ orbits that have no fixed points. The universe is the set of permutations with $r$ orbits, and the $i$th of the $n$ sets to be avoided is the set of permutations in which element $i$ is a fixed point.

*Editorial comment.* Let $d(n, r)$ be the number of derangements of $[n]$ with $r$ cycles. The fact that the coefficient of $x^r$ in $f_n(x)$ is $d(n, r)$ can also be proved by induction on $n$ using Pascal’s formula and the recurrence $d(n, r) = (n - 1)[d(n - 2, r - 1) + d(n - 1, r)].$

O. P. Lossers (and others) gave a short proof of the degree statement by observing that $f_n(x)$ is the $r$th derivative (with respect to $t$) of the product $e^{-tx}(1 - t)^{-x}$, evaluated at $t = 0$.

This polynomial appears explicitly in *An Introduction to Combinatorial Analysis*, chapter 4 section 4, pp. 72–74 by Riordan (Wiley, 1958). It is also mentioned in *Advanced Combinatorics*, chapter VI, section 6.7, p. 256, by Comtet (Reidel, 1974) with some references to previous non-combinatorial appearances in articles by Tricomi and Carlitz.

Also solved by T. Amdeberhan & S. B. Ekhad, R. Bagby, D. Beckwith, R. Chapman (U.K.), P. Corn, P. P. Dályay (Hungary), O. Geupel (Germany), J. Grivaux (France), S. J. Herschkorn, P. J. Fitzsimmons, D. Grinberg, J. Grivaux (France), S. J. Herschkorn, E. A. Herman, T. Konstantopoulos (U.K.), O. Kouba (Syria), J. H. Lindsey II, O. P. Lossers (Netherlands), ´A. Plaza & S. Falc ´on (Spain), M. A. Prasad (India), R. Pratt, O. G. Ruehr, B. Schmuland (Canada), A. Studler (Switzerland), J. H. Steelman, R. Stong, R. Tauraso (Italy), M. Tetiva (Romania), BSI Problems Group (Germany), GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, NSA Problems Group, and the proposer.

### Mean Inequalities

**11434** [2009, 463]. Proposed by Slavko Simic, Mathematical Institute SANU, Belgrade, Serbia. Fix $n \in \mathbb{N}$ with $n \geq 2$. Let $x_1, \ldots, x_n$ be distinct real numbers, and let $p_1, \ldots, p_n$ be positive numbers summing to 1. Let

$$S = \frac{\sum_{k=1}^{n} p_k x_k^3 - \left(\sum_{k=1}^{n} p_k x_k\right)^3}{3 \left(\sum_{k=1}^{n} p_k x_k^2 - \left(\sum_{k=1}^{n} p_k x_k\right)^2\right)}.$$

Show that $\min\{x_1, \ldots, x_n\} \leq S \leq \max\{x_1, \ldots, x_n\}$.

*Solution by Jim Simons, Cheltenham, U.K.* Consider a probability distribution on the real line that takes value $x_j$ with probability $p_j$ for $1 \leq j \leq n$. Write $\mu'_j$ for the $i$th moment about 0 and $\mu_i$ for the $i$th moment about the mean $\mu'_1$. Now

$$S = \frac{\mu'_3 - \mu'_1^3}{3(\mu'_2 - \mu'_1^2)} = \frac{\mu_3 + 3\mu'_1 \mu_2}{3\mu_2} = \mu'_1 + \frac{\mu_3}{3\mu_2}.$$

From this we obtain inequalities stronger than those proposed:

$$\frac{1}{3} \min\{x_1, \ldots, x_n\} + \frac{2}{3} \mu'_1 \leq S \leq \frac{1}{3} \max\{x_1, \ldots, x_n\} + \frac{2}{3} \mu'_1.$$

A Circumradius Equation

11443 [2009, 548]. Proposed by Eugen Ionascu, Columbus State University, Columbus, GA. Consider a triangle $ABC$ with circumcenter $O$ and circumradius $R$. Denote the distances from $O$ to the sides $AB$, $BC$, $CA$, respectively, by $x$, $y$, $z$. Show that if $ABC$ is acute then $R^3 - (x^2 + y^2 + z^2)R = 2xyz$, and $(x^2 + y^2 + z^2)R - R^3 = 2xyz$ otherwise.

Solution by Philip Benjamin, Berkeley College, Woodland Park, NJ. We first prove the identity

$$1 - (\cos^2 A + \cos^2 B + \cos^2 C) = 2 \cos A \cos B \cos C.$$

Indeed, $C = \pi - (A + B)$, so $\cos C = -\cos(A + B) = \sin A \sin B - \cos A \cos B$. Isolating $\sin A \sin B$ and squaring yields $\cos^2 A \cos^2 B + 2 \cos A \cos B \cos C + \cos^2 C = \sin^2 A \sin^2 B = 1 - \cos^2 A - \cos^2 B + \cos^2 A \cos^2 B$. This simplifies to ($\ast$).

Let the side lengths $a$, $b$, and $c$ be opposite angles $A$, $B$, and $C$, respectively, so $a / \sin A = b / \sin B = c / \sin C = 2R$. The perpendicular from $O$ to side $AB$ bisects $AB$, so we have a right triangle with side lengths $x$, $c/2$, and $R$. Since $c = 2R \sin C$, we conclude that $x = R |\cos C|$. Similarly $y = R |\cos A|$ and $z = R |\cos B|$. If $\triangle ABC$ is acute, then the three cosines are positive, so multiplying ($\ast$) by $R^3$ produces the desired result. Otherwise, say angle $C$ is right or obtuse. Now $x = -R \cos C$ and the other two cosines are positive. Again, multiplying ($\ast$) by $R^3$ produces the desired result.

Editorial comment. A similar problem was proposed in Crux Mathematicorum with Mathematical Mayhem, December, 2008, Problem 3395.


Extrema

11449 [2009, 647]. Proposed by Michel Bataille, Rouen, France. (corrected) Find the maximum and minimum values of

$$\frac{(a^3 + b^3 + c^3)^2}{(b^2 + c^2)(c^2 + a^2)(a^2 + b^2)}$$

given that $a + b \geq c > 0$, $b + c \geq a > 0$, and $c + a \geq b > 0$.

Solution by Jim Simons, Cheltenham, U.K. Call this big expression $X$. Since $X$ is homogeneous, we may assume $a^2 + b^2 + c^2 = 1$. The feasible region then consists of a triangular patch on the positive octant of the unit sphere, excluding the vertices (where one of $a$, $b$, $c$ is zero), but including the interiors of the sides (where two of $a$, $b$, $c$ are equal). Using spherical polar coordinates, we may set $(a, b, c) =$
(cos α, sin α cos θ, sin α sin θ), where, since a, b, c are positive, θ is uniquely determined and 0 < θ < π/2. Now

\[
X = \frac{(\cos^3 \alpha + \sin^3 \alpha (\cos^3 \theta + \sin^3 \theta))^2}{\sin^2 \alpha (\cos^2 \alpha + \sin^2 \alpha \sin^2 \theta) (\cos^2 \alpha + \sin^2 \alpha \cos^2 \theta)}
\]

\[
= \frac{(\cos^3 \alpha + \sin^3 \alpha (\cos^3 \theta + \sin^3 \theta))^2}{\sin^2 \alpha (\cos^4 \alpha + \cos^2 \alpha \sin^2 \alpha + \sin^4 \alpha \sin^2 \theta \cos^2 \theta)}.
\]

If \( f(\theta) = \cos^3 \theta + \sin^3 \theta \), then \( f'(\theta) = 3 \cos \theta \sin \theta (\sin \theta - \cos \theta) \). In the feasible region for \( \theta \), this is positive for \( \theta > \pi/4 \) and negative for \( \theta < \pi/4 \). Thus \( f \), and with it, the numerator of \( X \) for fixed \( \alpha \), is less at \( \theta = \pi/4 \) than at any other feasible \( \theta \). Similarly, if \( g(\theta) = \sin^2 \theta \cos^2 \theta \), \( g \), and with it, the denominator of \( X \) for fixed \( \alpha \), is increasing in \( \theta \) for \( \theta < \pi/4 \) and decreasing in \( \theta \) for \( \theta > \pi/4 \). Thus \( X \) is, for fixed \( \alpha \), smallest at \( \theta = \pi/4 \), and greatest at an edge of the feasible region. By symmetry, the minimum value of \( X \) is 9/8, attained when \( a = b = c \).

From the foregoing, the maximum value of \( X \) on the closure of the feasible region occurs at a point where, with respect to any translation into spherical coordinates, \( \theta \) is extremal. The only such points are the corners of the region. At \((a, b, c) = (2^{-1/2}, 2^{-1/2}, 0)\), \( X = 2 \). However, this maximum is not attained because these corners are not in the feasible region.


**Max Min Coordinate Difference**

**11450** [2009, 647]. Proposed by Cosmin Pohoata (student), National College “Tudor Vianu,” Bucharest, Romania. Let \( A \) be the unit ball in \( \mathbb{R}^n \). Find

\[
\max_{a \in A} \left\{ \min_{1 \leq i < j \leq n} |a_i - a_j| \right\}.
\]

**Solution by Omran Kouba,** Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let \( M_n \) denote the desired maximum. It is implicit in the statement of the problem that \( n \geq 2 \). We show that \( M_n = \sqrt{12/(n(n^2 - 1))} \).

Let \((a_1, \ldots, a_n)\) be an element of \( A \) at which the maximum is achieved, and let \( M_n = \min(|a_i - a_j|): 1 \leq i < j \leq n \). There is a permutation \( \sigma \) of the set \( \{1, 2, \ldots, n\} \) such that \( a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)} \). Write for simplicity \( b_j = a_{\sigma(j)} \). For \( j > i \), we then have

\[
b_j - b_i = \sum_{k=i+1}^{j} (b_k - b_{k-1}) \geq (j - i) M_n.
\]
From this we conclude that $|b_j - b_i| \geq |j - i| M_n$ for $1 \leq i, j \leq n$. Therefore

$$M_n^2 \sum_{1 \leq i, j \leq n} (j - i)^2 \leq \sum_{1 \leq i, j \leq n} (b_j - b_i)^2 = \sum_{1 \leq i, j \leq n} (a_j - a_i)^2$$

$$= \sum_{1 \leq i, j \leq n} (a_j^2 + a_i^2 - 2a_i a_j)$$

$$\leq 2n \sum_{k=1}^{n} a_k^2 - 2 \left( \sum_{k=1}^{n} a_k \right)^2 \leq 2n,$$

since $\sum_{k=1}^{n} a_k^2 \leq 1$ when $(a_1, \ldots, a_n) \in A$. On the other hand,

$$\sum_{1 \leq i, j \leq n} (j - i)^2 = 2n \sum_{k=1}^{n} k^2 - 2 \left( \sum_{k=1}^{n} k \right)^2 = \frac{n^2(n^2 - 1)}{6}.$$ 

It follows that $M_n^2 \leq 12/(n(n^2 - 1))$, so $M_n \leq \sqrt{12/(n(n^2 - 1))}$.

Conversely, if we consider $(a_1^{(0)}, a_2^{(0)}, \ldots, a_n^{(0)})$ defined by

$$a_k^{(0)} = \sqrt{\frac{12}{n(n^2 - 1)}} \left( k - \frac{n + 1}{2} \right), \quad k = 1, 2, \ldots, n,$$

then $(a_1^{(0)}, \ldots, a_n^{(0)}) \in A$ and

$$\min_{1 \leq i < j \leq n} |a_i^{(0)} - a_j^{(0)}| = \sqrt{\frac{12}{n(n^2 - 1)}}.$$

Thus $M_n \geq \sqrt{12/(n(n^2 - 1))}$.

**Editorial comment.** Marian Tetiva (Romania) notes that a stronger form of this problem appeared as Problem E2032, this *Monthly* 76 (1969) 691–692, proposed by D. S. Mitrinović. See also Problem 3.9.9 in Mitrinović, *Analytic Inequalities* (Springer-Verlag, 1970).


**A Cauchy–Schwarz Puzzle**

11458 [2009, 747]. Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania, and Vicențiu Rădulescu, Institute of Mathematics “Simion Stoilow” of the Romanian Academy, Bucharest, Romania. Let $a_1, \ldots, a_n$ be nonnegative and let $r$ be a positive integer. Show that

$$\left( \sum_{1 \leq i, j \leq n} \frac{i^r j^r a_i a_j}{i + j - 1} \right)^2 \leq \sum_{m=1}^{n} m^{r-1} a_m \sum_{1 \leq i, j, k \leq n} \frac{i^r j^r k^r a_i a_j a_k}{i + j + k - 2}.$$
Solution by Francisco Vial, student, Pontificia Universidad Católica de Chile, Santiago, Chile. Let $f(x) := \sum_{i=1}^{n} i^r a_i x^{i-1}$, so

\[
\int_{0}^{1} f(x) \, dx = \sum_{m=1}^{n} m^{r-1} a_m,
\]

\[
\int_{0}^{1} f^2(x) \, dx = \int_{0}^{1} \left( \sum_{1 \leq i,j \leq n} i^r j^r a_i a_j x^{i+j-2} \right) \, dx = \sum_{1 \leq i,j \leq n} \frac{i^r j^r a_i a_j}{i + j - 1}, \quad \text{and}
\]

\[
\int_{0}^{1} f^3(x) \, dx = \int_{0}^{1} \left( \sum_{1 \leq i,j,k \leq n} i^r j^r k^r a_i a_j a_k x^{i+j+k-3} \right) \, dx = \sum_{1 \leq i,j,k \leq n} \frac{i^r j^r k^r a_i a_j a_k}{i + j + k - 2}.
\]

The stated inequality is equivalent to

\[
\left( \int_{0}^{1} f^2(x) \, dx \right)^2 \leq \left( \int_{0}^{1} f(x) \, dx \right) \left( \int_{0}^{1} f^3(x) \, dx \right),
\]

which follows by applying the Cauchy–Schwarz inequality to $f(x)^{1/2}$ and $f(x)^{3/2}$.

Remarks. Because $a_1, \ldots, a_n$ are nonnegative, $f(x)$ is nonnegative and continuous on $[0, 1]$, so $f(x)^{1/2}$ and $f(x)^{3/2}$ are real and well defined. The parameter $r$ need not be an integer.

Also solved by M. R. Avidon, R. Chapman (U.K.), P. P. Dályay (Hungary), D. Grinberg, O. Kouba (Syria), O. P. Lossers (Netherlands), J. Simons (U.K.), R. Stong, GCHQ Problem Solving Group (U.K.), and the proposers.

An Orthocenter Inequality

11461 [2009, 844]. Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy. Let $a$, $b$, and $c$ be the lengths of the sides opposite vertices $A$, $B$, and $C$ of an acute triangle. Let $H$ be the orthocenter. Let $d_a$ be the distance from $H$ to side $BC$, and similarly for $d_b$ and $d_c$. Show that

\[
\frac{1}{d_a} \cdot \frac{1}{d_b} \cdot \frac{1}{d_c} \geq \frac{2}{3} \left( \frac{3}{abc} \left( \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} + \frac{1}{\sqrt{ab}} \right) \right)^{1/4}.
\]

Solution by Michael Vowe, Fachhochschule Nordwestschweiz, Muttenz, Switzerland.

Let $R$ be the circumradius, $r$ the inradius, $F$ the area, and $s$ the semiperimeter. From $d_a = 2R \cos B \cos C$, $d_b = 2R \cos C \cos A$, $d_c = 2R \cos A \cos B$, we obtain

\[
d_a + d_b + d_c = 2R \left( \cos A \cos B + \cos B \cos C + \cos C \cos A \right) \leq 2r \left( 1 + \frac{r}{R} \right)
\]

(see 6.10, p. 181, in D. Mitrinovic et al., Recent Advances in Geometric Inequalities, Dordrecht, 1989). From Jensen’s inequality for concave functions (here, the square root), we have

\[
\frac{1}{\sqrt{ab}} + \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} \leq 3 \cdot \sqrt{\frac{1}{3} \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right)} = \sqrt{\frac{6s}{abc}}.
\]
From \( abc = 4RF = 4Rrs \) and \( s^2 \geq 27r^2 \) (6.1, p. 180, ibid.) we get
\[
\frac{2}{3} \left( \frac{3}{abc} \left( \frac{1}{\sqrt{bc}} + \frac{1}{\sqrt{ca}} + \frac{1}{\sqrt{ab}} \right) \right)^{1/4} \leq \frac{2}{3} \left( \frac{3\sqrt{6}s}{(abc)^{3/2}} \right)^{1/4} 
\]
\[
\leq \frac{2}{3} \left( \frac{3\sqrt{6}}{(4Rr)^{3/2}} \cdot \frac{1}{3\sqrt{3}r} \right)^{1/4} = \frac{2^{3/8}}{3} \cdot \frac{1}{R^{3/8}} \cdot \frac{1}{r^{5/8}}.
\]
Thus it suffices to prove
\[
\frac{1}{2r \left( 1 + \frac{R}{r} \right)} \geq \frac{2^{3/8}}{3} \cdot \frac{1}{R^{3/8}} \cdot \frac{1}{r^{5/8}}.
\]
Writing \( x = r/R \), this means we must prove \( x^{3/8}(1 + x) \leq 3/(2 \cdot 2^{3/8}) \) for \( 0 < x \leq 1/2 \). The function \( f(x) = x^{3/8}(1 + x) \) is increasing on \([0, 1/2]\), though, and we are done.

Equality holds only if \( x = 1/2 \), or equivalently, \( R = 2r \), which makes the triangle equilateral.

Also solved by P. P. Dályay (Hungary), O. Faynshteyn (Germany), K.-W. Lau (China), C. R. Pranesachar (India), R. Stong, GCHQ Problem Solving Group (U.K.), Microsoft Research Problems Group, and the proposer.

**An Erroneous Claim**

11465 [2009, 845]. Proposed by Pantelimon George Popescu, Polytechnic University of Bucharest, Bucharest, Romania, and José Luis Díaz-Barrero, Polytechnic University of Catalonia, Barcelona, Spain. Consider three simple closed curves in the plane, of lengths \( p_1, p_2, \) and \( p_3 \), enclosing areas \( A_1, A_2, \) and \( A_3 \), respectively. Show that if \( p_3 = p_1 + p_2 \) and \( A_3 = A_1 + A_2 \), then \( 8\pi A_3 \leq p_3^2 \).

Solution by the Texas State University Problem Solvers Group, San Marcos, TX. The problem as stated is false. Consider the following counterexample. Let the first curve be a square with \( p_2 = 40 \) and \( A_2 = 100 \). Let the third curve be a rectangle with sides \( 11 + 2\sqrt{5} \) and \( 101/(11 + 2\sqrt{5}) \) so that \( p_3 = 44 \) and \( A_3 = 101 \). These three curves fulfill the requirements of the problem, and yet \( 8\pi A_3 > p_3^2 \).

Let us incorporate the additional requirement that \( p_1^2 + p_2^2 = 2p_1 p_2 \). Then the required inequality can be proved as follows. The isoperimetric inequality applied to any of the curves is
\[
A_i \leq \pi \left( \frac{p_i}{2\pi} \right)^2,
\]
and thus \( 4\pi A_i \leq p_i^2 \). Therefore
\[
4\pi A_3 = 4\pi A_1 + 4\pi A_2 \leq p_1^2 + p_2^2.
\]
With the newly-added condition we get
\[
8\pi A_3 = 8\pi A_1 + 8\pi A_2 \leq 2p_1^2 + 2p_2^2 = p_1^2 + p_2^2 + 2p_1 p_2 = (p_1 + p_2)^2 = p_3^2.
\]