Continued Fractions and Pell equations

This is not intended to be a complete set of notes on the subject. You can find more details (and proofs) either in texts or on the web.

Let’s use the Pell equation $x^2 - 7y^2 = 1$, as our example. We begin with a diversion into the topic of continued fractions.

Consider the sequence which begins

\[
\begin{align*}
z_0 &= 2, \\
z_1 &= 2 + \frac{1}{1} = 3, \\
z_2 &= 2 + \frac{1}{1 + \frac{1}{1}} = \frac{5}{2}, \\
z_3 &= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}} = \frac{8}{3}, \\
z_4 &= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}} = \frac{37}{14}, \\
z_5 &= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}} = \frac{45}{17}, \\
z_6 &= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}}} = \frac{82}{31}, \\
z_7 &= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}}}} = \frac{127}{48}, \\
z_8 &= 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}}}}} = \frac{590}{223},
\end{align*}
\]

In the limit case, assuming the pattern $[1,1,1,4]$ continues, we set

\[
\beta = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1}}}}}}}},
\]

and note that
\[ \beta = \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}} = \frac{2\beta + 9}{3\beta + 14}. \]

Solving
\[ \beta = \frac{2\beta + 9}{3\beta + 14}, \]
yields \( \beta \in \{-\sqrt{7} - 2, \sqrt{7} - 2\} \). Since one of these is negative, and \( \beta \) is obviously positive, we have \( \beta = \sqrt{7} - 2 \), and
\[ \sqrt{7} = 2 + \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \frac{1}{4 + \frac{1}{1 + \frac{1}{1 + \frac{1}{2}}}}}}}. \]

This is the continued fraction representation of \( \sqrt{7} \). A more compact notation is \( \sqrt{7} = [2, [1, 1, 4]] \). We note that each of the convergents \( z_0, z_1, z_2, z_3, \ldots \) is an approximation to \( \sqrt{7} \). From the values we have calculated, we can see
\[
\begin{align*}
z_0^2 - 7 & = 2^2 - 7 = -3, \\
z_1^2 - 7 & = 3^2 - 7 = 2, \\
z_2^2 - 7 & = \left(\frac{5}{2}\right)^2 - 7 = -\frac{3}{4}, \\
z_3^2 - 7 & = \left(\frac{8}{3}\right)^2 - 7 = \frac{1}{9}, \\
z_4^2 - 7 & = \left(\frac{37}{14}\right)^2 - 7 = -\frac{3}{196}, \\
z_5^2 - 7 & = \left(\frac{45}{17}\right)^2 - 7 = \frac{2}{289}, \\
z_6^2 - 7 & = \left(\frac{82}{31}\right)^2 - 7 = -\frac{3}{961}, \\
z_7^2 - 7 & = \left(\frac{127}{48}\right)^2 - 7 = \frac{1}{2304}, \\
z_8^2 - 7 & = \left(\frac{590}{223}\right)^2 - 7 = -\frac{3}{49729}.
\end{align*}
\]

The period of \([2, [1, 1, 4]]\) is four. The third convergent is \( z_3 = \frac{8}{3} \), and we obtained the equation
\[ \left(\frac{8}{3}\right)^2 - 7 = \frac{1}{9}. \]

Let’s multiply this by \( 3^2 \):
\[ 8^2 - 7 \cdot 3^2 = 1. \]

We have a solution to the Pell equation
\[ x^2 - 7 \cdot y^2 = 1. \]

In general, if \( n \) is a positive integer but not a perfect square, there is a continued fraction representation for \( \sqrt{n} \). If the period of this representation is \( p \), then the convergent \( z_{p-1} \) provides a solution to the Pell equation.
Furthermore, this solution is the smallest positive solution.

So, for our example, \( x = 8, \ y = 3 \) is the smallest positive solution to \( x^2 - 7 \cdot y^2 = 1 \). How do we find the other solutions?

Consider the factorization
\[
(8 + 3\sqrt{7})(8 - 3\sqrt{7}) = 8^2 - 7 \cdot 3^2 = 1.
\]
From this,
\[
(8 + 3\sqrt{7})^k (8 - 3\sqrt{7})^k = 1,
\]
and each \((8 + 3\sqrt{7})^k\) provides a solution. For example,
\[
(8 + 3\sqrt{7})^2 = 48\sqrt{7} + 127,
\]
and
\[
127^2 - 7 \cdot 48^2 = 1.
\]
One more example:
\[
(8 + 3\sqrt{7})^3 = 2024 + 765\sqrt{7},
\]
and
\[
2024^2 - 7 \cdot 765^2 = 1.
\]

The interested student is urged to explore the interconnections between continued fractions and solutions to Pell equations. The text by Hardy and Wright has a chapter devoted to continued fractions.

We’ve left several possible questions unanswered. For example, how does one find the continued fraction for a number like \( \sqrt{7} \)?

We could start with
\[
\sqrt{7} = 2 + \left( \sqrt{7} - 2 \right).
\]
Here, \( 0 \leq \sqrt{7} - 2 < 1 \). We then want to write \( \sqrt{7} - 2 \) as \( \frac{1}{t} \) and find the continued fraction for \( t \).
\[
\sqrt{7} - 2 = \frac{1}{\left( \frac{\sqrt{7} + 2}{3} \right)}.
\]
Now, \( 1 < \frac{\sqrt{7} + 2}{3} < 2 \), so we write
\[
\frac{\sqrt{7} + 2}{3} = 1 + \left( \frac{\sqrt{7} + 2}{3} - 1 \right) = 1 + \frac{\sqrt{7} - 1}{3}.
\]
Then,
\[
\frac{\sqrt{7} - 1}{3} = \frac{1}{\left( \frac{\sqrt{7} + 1}{2} \right)}.
\]
and \(1 < \frac{\sqrt{7} + 1}{2} < 2\). Then,
\[
\frac{\sqrt{7} + 1}{2} = 1 + \left(\frac{\sqrt{7} + 1}{2} - 1\right) = 1 + \frac{\sqrt{7} - 1}{2} = 1 + \frac{1}{\left(\frac{\sqrt{7} + 1}{3}\right)}
\]
and \(1 < \frac{\sqrt{7} + 1}{3} < 2\). Now,
\[
\frac{\sqrt{7} + 1}{3} = 1 + \left(\frac{\sqrt{7} + 1}{3} - 1\right) = 1 + \frac{\sqrt{7} - 2}{3} = 1 + \frac{1}{\sqrt{7} + 2},
\]
and \(4 < \sqrt{7} + 2 < 5\). The next step yields:
\[
\sqrt{7} + 2 = 4 + \left(\sqrt{7} - 2\right) = 4 + \frac{1}{\left(\frac{\sqrt{7} + 2}{3}\right)}
\]
but, we’ve been here before and can now see that the sequence will repeat from this point. So, 
\(\sqrt{7} = [2, [1, 1, 1, 4]]\).

You should try to find the continued fractions representation for \(\sqrt{n}\) for some other non-square positive integral \(n\).

That is, you should try to do a few before you go looking on the web to find that there are tables of these for very many values of \(n\).

There are also continued fraction expressions for numbers such a \(\pi\) and \(e\), but these are non-periodic patterns.

\(\pi = [3, [7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, 99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 1, 2, 3, 7, 1, 2, 1, 12, 1, 1, 3, 1, 1, 8, 1, 1, 2, 1, 6, 1, 1, 5, 2, 2, 3, 1, 2, 4, 4, 16, 1, 161, 45, 1, 22, 1, 2, 2, 1, 4, 1, 2, 24, 1, 2, 1, 3, 1, 2, 1, ...]]\). See: http://oeis.org/A001203

\(e = [2, [1, 2, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, 1, 1, 16, 1, 1, 18, 1, 1, 20, 1, 1, 22, 1, 1, 24, 1, 1, 26, 1, 1, 28, 1, 1, 30, 1, 1, 32, 1, 1, 34, 1, 1, 36, 1, 1, 38, 1, 1, 40, 1, 1, 42, 1, 1, 44, 1, 1, 46, 1, 1, 48, 1, 1, 50, 1, 1, 52, 1, 1, 54, 1, 1, 56, 1, 1, 58, 1, 1, 60, 1, 1, 62, 1, 1, 64, 1, 1, 66,...]]\) http://oeis.org/A003417

The continued fraction for \(e\), while not periodic, certainly seems to have a pattern. You might look at http://oeis.org/A267318, http://oeis.org/A078688. Nice expansions are known for \(\exp\left(\frac{1}{k}\right)\) and \(\exp\left(\frac{2}{k}\right)\), where \(k\) is a positive integer.