§20. CONTRADICTION AND CONTRAPOSITIVE

We address, once again, the question of how to prove a mathematical statement of the form “if A, then B”.

We have used so far the template for a direct proof: unravel A, unravel B, produce the link from A to B. So, in a direct proof, no negation is involved.

We will introduce now two more ways to prove “if A, then B”, called Contrapositive and Contradiction. They are actually two versions of a similar approach, and we will underscore their differences at the end of this discussion.

ADVICE. Since we will need to negate sentences, it is important to review sections 7 and 11 before proceeding.
CONTRAPOSITIVE. It is a simple verification (and a good way to review Truth Tables) that $x \rightarrow y$ and $\neg y \rightarrow \neg x$ have the same truth tables, and so they are equivalent Boolean expressions.

Therefore, to prove “if $A$, then $B$” we can equivalently prove “if NOT $B$, then NOT $A$”. Thus, the scheme (or “template”) to prove a statement “if $A$, then $B$” by contrapositive boils down to:

Unravel NOT $B$

Unravel NOT $A$

Produce the link from NOT $B$ to NOT $A$.

REMARK. Proofs by contrapositive and by contradiction are particular useful when trying to show that some set is empty. As a simple illustration we have.

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Prove that if \( A \cap C \subseteq B \), then \( (C - B) \cap A = \emptyset \).

**Proof.** Let us state (and prove) the contrapositive, that is, if \( (C - B) \cap A \neq \emptyset \), then \( A \cap C \) is not contained in \( B \).

Unravel the hypothesis: \( (C - B) \cap A \neq \emptyset \) means there is an element \( x \) in \( (C - B) \cap A \). Therefore \( x \in C \), \( x \notin B \) and \( x \in A \). (#)

Unravel the conclusion: \( A \cap C \) is not contained in \( B \) means we must exhibit an element of \( A \cap C \) that is not an element of \( B \). (##)

Link: From (#) we have that \( x \in C \) and \( x \in A \), so \( x \in A \cap C \). At the same time, by (#) \( x \notin B \). Hence this element \( x \) satisfies our requirement in (##).

The proof is complete.

A proof by Contradiction is the other indirect technique to be considered.

Recall that in trying to prove “if \( A \), then \( B \)” we can never assume the conclusion \( B \) to be true. However, the truth table of \( A \rightarrow B \) shows that the only possibility for the end result to be false, is that \( A \) be true and \( B \) be false.

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CONTRADICTION. The template to prove by contradiction “if A, then B” is as follows:
Assume A to be true, B to be false and try to arrive to something impossible, that is, to a contradiction.

NOTE.

a) A contradiction might be a violation of a principle, a violation of A (which was assumed to be true), a violation of a result previously shown, etc.
b) Once we arrive to a contradiction, we have shown that the negation of “if A, then B” is false, and so the statement “if A, then B” is true.

Usually, a proof by contradiction begins with the sentence “Assume, for sake of a contradiction, that....”

ILLUSTRATIONS.

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1) Prove that no integer is simultaneously even and odd.

Proof. This is rephrased as “if \( x \in \mathbb{Z} \), then \( x \) can not be both even and odd”.

Notice that this is \( A \Rightarrow B \), where
\[
A: x \in \mathbb{Z} \\
B: x \text{ can not be both even and odd.}
\]

Assume for sake of a contradiction that \( A \) is true and \( B \) is false.
Hence \( x \) is an integer that can be written simultaneously as \( x = 2k \) (some integer \( k \)) and \( x = 2w + 1 \) (some integer \( w \)).

Therefore, \( 2k = 2w + 1 \), and so \( 2(k-w) = 1 \), which gives that \( k-w \) is \( \frac{1}{2} \). This is a contradiction, since \( k-w \) is an integer, and \( \frac{1}{2} \) is not.

Having arrived to a contradiction, we have proved that the original statement is true.
2) Prove by contradiction that if 4 divides $a^2 + b^2$, then $a$ and $b$ are even.

Proof. This is $A \implies B$ where

$A$: 4 divides $a^2 + b^2$

$B$: $a$ is even and $b$ is even

Assume for sake of a contradiction that $A$ is true and $B$ is false.

A true means that $a^2 + b^2 = 4k$, for some integer $k$.

We negate $B$, so a is odd or b is odd. Let us consider, for instance, that $a$ is odd. Then $a^2$ is odd.

Since $a^2 + b^2 = 4k$, we have $b^2 = a^2 + b^2 - a^2 = 4k - a^2$, so $b$ must be odd also (because “even – odd is odd”). Thus, $b$ must be odd.

Therefore we have $a = 2t+1$, and $b = 2w+1$, with $t$ and $w$ integers. So,

$4k = a^2 + b^2 = (2t+1)^2 + (2w+1)^2 = 4t^2 + 4t + 1 + 4w^2 + 4w + 1 = 4(t^2 + t + w^2 + w) + 2 = 4y + 2$, if we set $y = t^2 + t + w^2 + w$.
Therefore, $4k = 4y + 2$, with $y$ an integer, and so $2k = 2y+1$. Solving for 1, we arrive to $1 = 2t$, so $\frac{1}{2}$ is an integer. This is a contradiction.

3) **Prove that if** $a$, $b$ and $a + b$ are prime, then $a = 2$ **or** $b = 2$.

Proof. Assume for sake of a contradiction that the hypothesis holds but the conclusion is false. Thus neither $a$ nor $b$ equals 2.

Being prime numbers (according to the hypothesis) and different from 2, they must be odd and greater than or equal to 3. Thus, $a + b \geq 6$.

Since $a$ and $b$ are odd, $a + b$ is even. But by hypothesis $a + b$ is prime, so we arrive to the following: $a + b$ is prime, $a + b \geq 6$, and $a + b$ is even.

This is a contradiction because the only even prime is 2.

**How do these two methods compare?** Observe that in a contrapositive we have only one starting point (NOT B) and must arrive to NOT A. That is done through a link. *Hence, not much information, but a clear goal.*
On the other hand, in a proof by contradiction we have more information: A and NOT B, which means an edge over the contrapositive. However, we must arrive to a contradiction, but a priori we ignore which contradiction we will end up with. *Hence, more information, but the goal is somehow undefined.*

**CAUTION.** Do not confuse “prove by contradiction” with “find a counterexample”.

**DISCUSSION PROBLEM.** Suppose we paint each point of the xy-plane by using one of either three given colors, Blue, Red and Yellow. Prove that given any positive real number d there exist two points of the same color exactly d units apart. **Note:** we have already treated the case of only two colors.