Enumeration of Planar Maps

These notes are taken from


and from notes I took during courses taught by Professor Tutte.

**Rooted 2-connected planar inner triangulations.** We consider graphs which are embedded in the plane, rooted on a dart (edge+direction) on the outside face. We generally consider the rooting to be done in a clockwise fashion (although on the outer face, one has to look at the edge as if at some point in that outside face). Such a planar map is 2-connected if the graph is connected and the removal any vertex does not disconnect the graph. Under this definition, we allow $K_2$ as a 2-connected graph. A map is an inner triangulation if every face is a triangle, with the possible exception of the outer face.

Let $M$ denote a rooted 2-connected planar inner triangulation. Let $m(M)$ denote the degree of the outer face and $t(M)$ denote the number of inner triangles of $M$. Let $\mathcal{M}$ denote the class of all rooted 2-connected planar inner triangulations.

In general, Tutte would draw several initial diagrams before proceeding to the enumeration. He would then develop the enumeration via generating functions, first finding a functional equation for the generating function and from this obtain a prediction for the number of maps of slightly larger size. Then, a few cases can be checked, to make sure the functional equation does what is needed. Now, some cases can be examined to see if the numbers obtained have nice factorizations. For some problems, one can guess the final form of the coefficients if there is enough data.

Now that the world knows that Tutte was an essential part of the Bletchley part team, and that he needed to do extensive hand calculations (and, later, computer calculations on Colossus), I’m more convinced that it really was the way he approached problems. Do lots of examples, and make conjectures from there.

The reader should pause here and draw examples of rooted maps for small values of $m(M)$ and $t(M)$. Once $t(M)$ is five or six, it becomes more difficult to make sure that one has listed all possible maps for the pair $(m(M), t(M))$. Hence, we need a more organized way to count these maps.

Let

$$q(x, z) = \sum_{M \in \mathcal{M}} x^{m(M)} z^{t(M)}.$$

Our first task is to obtain a recurrence formula for $q(x, z)$. It will also be convenient to consider

$$h(z) = [x^2]q(x, z).$$

For a power series $f(x)$, we use $[x^k]f(x)$ to denote the coefficient of $x^k$ in $f(x)$.

Again, we use a decomposition technique. The map $K_2$ will not be decomposed and contributes $x^2z^0$ to $q(x, z)$. Suppose that $M \in \mathcal{M}$ and that $M \neq K_2$. Then $M$ is rooted on an edge $\overrightarrow{ac}$ such that $a$ and $c$ lie in a triangle with some other vertex $b$. 
If $M - ac$ is not 2-connected, then we can decompose $M - ac$ into two 2-connected subgraphs $M_1$ and $M_2$, rooted at $\vec{ab}$ and $\vec{bc}$, respectively. Simple calculations show

$$m(M) = m(M_1) + m(M_2) - 1 \quad \text{and} \quad t(M) = t(M_1) + t(M_2) + 1.$$  

Thus maps of this type contribute $x^{-1}zq^2(x, z)$ to $q(x, z)$.

If $M - ac$ is 2-connected, then $M' = M - ac \in \mathcal{M}$, with root $\vec{ab}$. We note that since $a \neq c$, we cannot have $m(M') = 2$. Also,

$$m(M) = m(M') - 1 \quad \text{and} \quad t(M) = t(M') + 1.$$  

Maps of this type contribute $x^{-1}z(q(x, z) - x^2h(z))$ to $q(x, z)$.

We are therefore led to the recurrence equation:

$$q(x, z) = x^2 + x^{-1}zq^2(x, z) + x^{-1}z(q(x, z) - x^2h(z)), \quad (\star)$$

a little algebra yields:

$$zq^2(x, z) + (z - x)q(x, z) + (x^3 - x^2zh(z)) = 0. \quad (*)$$

The next task is to solve $(*)$. We begin by considering

$$q(x, z) = \sum_{i=0}^{\infty} q_i(x)z^i \quad \text{and} \quad h(z) = \sum_{i=0}^{\infty} h_iz^i.$$  

Considering $[z^0](*)$ we find that $-xq_0 + x^3 = 0$, and thus that $q_0 = x^2$ and $h_0 = 1$. The term $x^2$ corresponds to the map $K_2$.

Considering $[z^1](*)$ leads to $q_0^2 + (q_0 - q_1x) - x^2h_0 = 0$, hence $q_1 = x^3$ and $h_1 = 0$. The term $x^3z$ corresponds to the map $K_3$.

Examining $[z^2](*)$ yields $2q_0q_1 + (q_1 - xq_2) - x^2h_1 = 0$, and therefore $q_2 = 2x^4 + x^2$ and $h_2 = 1$. There are two ways to root $K_4 - e$ and one way to root the other map.

In a similar vein we can compute

$$q_3 = 4x^3 + 5x^5, \quad h_3 = 0,$$

$$q_4 = 4x^2 + 15x^4 + 14x^6, \quad h_4 = 4,$$

$$q_5 = 24x^3 + 56x^5 + 42x^7, \quad h_5 = 0.$$  

The reader may check that these correctly account for the appropriate maps. In general, once we know $q_0, q_1, \ldots, q_k$, and therefore, $h_0, h_1, \ldots, h_k$, we look at $[z^{k+1}](\star)$ and we find that $q_{k+1}$ is now determined. In fact, for $k > 0$, 

\[ q_{k+1} = [z^{k+1}]q(x, z) \]
\[ = [z^{k+1}](x^2 + x^{-1}zq^2(x, z) + x^{-1}z(q(x, z) - x^2h(z))) \]
\[ = [z^{k+1}]x^{-1}zq^2(x, z) + [z^{k+1}]x^{-1}z(q(x, z) - x^2h(z)) \]
\[ = [z^k]x^{-1}q^2(x, z) + [z^k]x^{-1}(q(x, z) - x^2h(z)). \]

As another approach, we could suppose we have a partial solution \( \hat{q} \) such that
\[ q - \hat{q} = \sum_{i=k}^\infty (q_i(x) - \hat{q}_i)z^i \in \langle z^{k+1} \rangle, \] the ideal in \( R[x, z] \) generated by \( z^{k+1} \). Then, with \( \hat{h} = [x^2]\hat{q} \), we have \( \hat{q}(x, z) = \left( x^2 + x^{-1}z\hat{q}^2(x, z) + x^{-1}z(\hat{q}(x, z) - x^2\hat{q}(z)) \right) \), where \( q - \hat{q} \in \langle z^{k+2} \rangle \). Thus, (\( \bigstar \)) allows us to calculate repeated iterates of an initial guess for \( q \), say \( q = h = 0 \), with the result of each iteration being in a smaller ball of the topology generated by the ideals \( \langle z^i \rangle \). Hence, the approximations converge in the algebra of formal power series.

The effect of the first few iterates are:
\[
\begin{align*}
q &= 0 \\
q &= x^2 \\
q &= x^2 + 2x^3 \\
q &= x^2 + 2x^3 + z^2(x^2 + 2x^4) + z^3x^5 \\
q &= x^2 + 2x^3 + z^2(x^2 + 2x^4) + z^3(4x^3 + 5x^5) + z^4(3x^4 + 6x^6) + z^5(x^3 + 4x^5 + 6x^7) + z^6(2x^6 + 4x^8) + z^7x^9
\end{align*}
\]

Of course, in the last iterate, we only expect this a correct approximation up to the term in \( z^3 \). We could do similar calculations, truncating after each step.
\[
\begin{align*}
q &= 0 \\
q &= x^2 \\
q &= x^2 + 2x^3 \\
q &= x^2 + 2x^3 + z^2(x^2 + 2x^4) \\
q &= x^2 + 2x^3 + z^2(x^2 + 2x^4) + z^3(4x^3 + 5x^5) \\
q &= x^2 + 2x^3 + z^2(x^2 + 2x^4) + z^3(4x^3 + 5x^5) + z^4(4x^2 + 15x^4 + 14x^6)
\end{align*}
\]

Another technique of Tutte’s was to factor the coefficients. Here, we can see the term in \( z^{10} \):
\[
\begin{align*}
&z^{10}(1456x^2 + 9600x^4 + 28560x^6 + 48048x^8 + 43758x^{10} + 16796x^{12}) \\
= &z^{10}(2^43^47^213^21x^2 + 2^23^215^2x^4 + 2^43^215^717^1x^6 \\
&+ 2^43^217^111^113^21x^8 + 2^32^211^113^117^1x^{10} + 2^213^117^119^1x^{12})
\end{align*}
\]

The prime factors are all relatively small. This may indicate that the coefficients are products of power and factorials. One could guess that \( [x^{12}z^{10}]q \) is a fraction whose numerator has a factor of 19!, for example. (Or 20! or 21! or 22!, but probably not 23!). Analyze enough of the coefficients
and one may be able to determine where each prime first appears.

We would now like to solve (\*) more generally. Now (\*) is equivalent to

\[(2zq(x, z) + z - x)^2 = (z - x)^2 - 4z(x^3 - x^2zh(z))\]. (**) 

We will begin by determining \(h(z)\). Since \(h(z)\) is independent of \(x\), we can look for a nice choice of \(x\) to simplify (\).

Let \(L(x, z) = 2zq(x, z) + z - x\) and \(D(x, z) = (z - x)^2 - 4z(x^3 - x^2zh(z)) = z^2 - 2xz + x^2 - 4x^3z + 4x^2z^2h(z)\).

If we choose \(x = \xi(z)\) so that \(L(\xi, z) = 0\), we simplify the calculations.

We do not explicitly find \(\xi\), but we should note that there is such a power series. In fact, \(L(\xi, z) = 2zq(\xi, z) + z - \xi = 0\) implies that \(\xi = z + 2zq(\xi, z)\) and we may solve for \(\xi\) by successive approximations.

What I had in my own notes is the following, but I’m a bit skeptical of these statements. “If \(L(\xi, z) = 0\), then \(\frac{\partial}{\partial \xi}L(\xi, z) = 0\) and \(\frac{\partial^2}{\partial \xi^2}L(\xi, z) = 0\). Thus \(\frac{\partial}{\partial \xi}D(\xi, z) = 0\) and \(\frac{\partial^2}{\partial \xi^2}D(\xi, z) = 0\)”

However, we could argue as follows: \(L(\xi, z) = 0\), then \(D(\xi, z) = L^2(\xi, z) = 0\), and \(\frac{\partial}{\partial \xi}D(\xi, z) = \frac{\partial}{\partial \xi}L^2(\xi, z) = 2L(\xi, z)\frac{\partial}{\partial \xi}L(\xi, z) = 0\). This is all we actually use.

We have therefore derived that

\[D(\xi, z) = z^2 - 2z\xi + \xi^2 - 4z\xi^3 + 4z^2\xi^2h = 0,\]

and

\[\frac{1}{2} \frac{\partial}{\partial \xi}D(\xi, z) = -z + \xi - 6z\xi^2 + 4z^2\xi h = 0.\]

We now eliminate \(h\) from these equations to find that \(z^2 + 2z\xi^3 - z\xi = 0\), and assuming \(z \neq 0\), we have \(z = \xi(1 - 2\xi^2)\) and then that \(z^2h = \xi^2(1 - 3\xi^2)\).

A change of parameters \(\theta = \xi^2\) and \(y = z^2\) brings this to

\[y = \theta(1 - 2\theta)^2\]

\[y\theta = \theta(1 - 3\theta).\]

We write this in the form we need for the more general form of Lagrange’s theorem (see Goulden and Jackson, page 17):

\[\theta = a + yf(\theta) = y(1 - 2\theta)^{-2}\]

\[y\theta = F(\theta) = \theta - 3\theta^2.\]

We calculate:
\[ F'(t) = 1 - 6t, \quad f^n(t) = \left( \frac{1}{1 - 2t} \right)^{2n}, \]

\[
\left( \frac{d}{dt} \right)^{n-1} (f^n(t)F'(t)) \bigg|_{t=0} \\
= \left( \frac{d}{dt} \right)^{n-1} \left( (1 - 6t) \left( \frac{1}{1 - 2t} \right)^{2n} \right) \bigg|_{t=0} \\
= \left( \frac{d}{dt} \right)^{n-1} \left( 3 \left( \frac{1}{1 - 2t} \right)^{2n-1} - 2 \left( \frac{1}{1 - 2t} \right)^{2n} \right) \bigg|_{t=0} \\
= 2^{n-1} \left( 3(2n - 1)(2n)(2n + 1) \cdots (3n - 3) \left( \frac{1}{1 - 2t} \right)^{3n-2} \right) \bigg|_{t=0} \\
- 2^{n-1} \left( 2(2n)(2n + 1) \cdots (3n - 2) \left( \frac{1}{1 - 2t} \right)^{3n-1} \right) \bigg|_{t=0} \\
= 2^{n-1} (3(2n - 1) - 2(3n - 2)) \left( \frac{3n - 3)!}{(2n - 1)!} \right) \\
= 2^{n-1} \frac{3n - 3)!}{(2n - 1)!} \\
\]

Hence,

\[ z^2 h = \sum_{n=1}^{\infty} \frac{2^n}{n!} 2^{n-1} \frac{(3n - 3)!}{(2n - 1)!}, \]

\[ h(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(n+1)!(2n+1)!}. \]

We display the first few terms of this power series for the sake of comparison with the earlier computational results we obtained:

\[ h(z) = 1 + z^2 + 4z^4 + 24z^6 + 176z^8 + 1456z^{10} + \ldots \]

The next stage would be to solve for \( q(x, z) \) from the information we have so far.

\[
(2xz)(x, z) + z - x)^2 \\
= z^2 - 2xz + x^2 - 4x^3z + 4x^2z^2h \\
= \xi^2(1 - 2\xi^2)^2 - 2x\xi(1 - 2\xi^2) + x^2 - 4x^3\xi(1 - 2\xi^2) + 4x^2\xi^2(1 - 3\xi^2) \\
= \xi^2(1 - 2\xi^2)^2 - 2x\xi(1 - 2\xi^2) + x^2(1 + 4\xi^2 - 12\xi^4) - 4x^3\xi(1 - 2\xi^2) \\
= (1 - 2\xi^2)(\xi^2(1 - 2\xi^2) - 2x\xi + x^2(1 + 6\xi^2) - 4x^3\xi) \\
= (1 - 2\xi^2)(x - \xi^2)^2(1 - 2\xi^2 - 4x\xi) \\
= (1 - 2\xi^2)^2(x - \xi^2)^2 \left( 1 - \frac{4x\xi}{1 - 2\xi^2} \right) \\
\]

Hence,
\[ 2zq(x, z) + z - x = \pm(x - \xi)(1 - 2\xi^2) \sqrt{1 - \frac{4x\xi}{1 - 2\xi^2}}. \]

Adjusting the sign, say by looking at the coefficient of \( x \):

\[ 2zq(x, z) + z - x = -(x - \xi)(1 - 2\xi^2) \sqrt{1 - \frac{4x\xi}{1 - 2\xi^2}}. \]

We now use a familiar function, \( \gamma(t) \), where \( \gamma(t) = 1 + t\gamma^2(t) \). Hence,

\[ \gamma(t) = \frac{1 - \sqrt{1 - 4t}}{2t} = \frac{t^{-1}}{2} (1 - \sqrt{1 - 4t}). \]

(Minus sign chosen so \([t^{-1}]\gamma(t) = 0\).)

Thus,

\[ \gamma\left(\frac{x\xi}{1 - 2\xi^2}\right) = \frac{1 - 2\xi^2}{2x\xi} \left(1 - \sqrt{1 - \frac{4x\xi}{1 - 2\xi^2}}\right), \]

and

\[ 2x\xi\gamma\left(\frac{x\xi}{1 - 2\xi^2}\right) = (1 - 2\xi^2) - (1 - 2\xi^2) \sqrt{1 - \frac{4x\xi}{1 - 2\xi^2}}. \]

Rearranging we obtain:

\[ (1 - 2\xi^2) \sqrt{1 - \frac{4x\xi}{1 - 2\xi^2}} = (1 - 2\xi^2) - 2x\xi\gamma\left(\frac{x\xi}{1 - 2\xi^2}\right). \]

But then

\[ 2zq(x, z) = x - z - (x - \xi)(1 - 2\xi^2) \sqrt{1 - \frac{4x\xi}{1 - 2\xi^3}} \]

\[ = x - \xi(1 - 2\xi^2) - (x - \xi)(1 - 2\xi^2) \sqrt{1 - \frac{4x\xi}{1 - 2\xi^3}} \]

\[ = x - \xi(1 - 2\xi^2) - (x - \xi)\left(1 - 2\xi^2 - 2x\xi\gamma\left(\frac{x\xi}{1 - 2\xi^2}\right)\right). \]

Hence,

\[ zq(x, z) = x\xi^2 + x\xi(x - \xi)\gamma\left(\frac{x\xi}{1 - 2\xi^2}\right). \]

Now,

\[ q(x, z) = \frac{x\xi}{1 - 2\xi^2} + \left(\frac{x^2}{1 - 2\xi^2} - \frac{x\xi}{1 - 2\xi^2}\right)\gamma\left(\frac{x\xi}{1 - 2\xi^2}\right) \]

\[ = \frac{x\xi}{1 - 2\xi^2} + \left(\frac{x^2}{1 - 2\xi^2} - \frac{x\xi}{1 - 2\xi^2}\right) \sum_{n=0}^{\infty} \frac{(2n)!}{(n + 1)!n!} \left(\frac{x\xi}{1 - 2\xi^2}\right)^n. \]
Some checks: \[ [x]q(x, z) = \frac{\xi}{1 - 2\xi^2} - \frac{\xi}{1 - 2\xi^2} = 0, \text{ as expected.} \]
\[ [x^2]q(x, z) = \frac{1}{1 - 2\xi^2} - \frac{\xi^2}{(1 - 2\xi^2)^2} = \frac{1 - 3\xi^2}{1 - 2\xi^2} = h. \]

For \( n > 2, \)
\[ [x^n]q(x, z) = \frac{1}{1 - 2\xi^2} \cdot \frac{(2n - 4)!}{(n-2)!(n-1)!} \cdot \frac{\xi^{n-2}}{(1 - 2\xi^2)^{n-2}} - \frac{\xi}{1 - 2\xi^2} \cdot \frac{(2n - 2)!}{(n-1)!n!} \cdot \frac{\xi^{n-1}}{(1 - 2\xi^2)^{n-1}}. \]

But we have \( \xi = \frac{z}{1 - 2\theta}, \) and therefore, for \( n > 2, \)
\[ [x^n]q(x, z) = \frac{(2n - 4)!z^{n-2}}{(n-2)!(n-1)!(1 - 2\theta)^2n-3} - \frac{(2n - 2)!z^n}{(n-1)!n!(1 - 2\theta)^2n}. \]

Here’s our chance to use Lagrange’s theorem again. \( \theta = z^2(1 - 2\theta)^2 \) and \( F(\theta) = (1 - 2\theta)^{-k}. \) So \( F'(t) = 2k(1 - 2t)^{-k+1}, \) and
\[
F(\theta) = 1 + \sum_{m=1}^{\infty} \frac{z^{2m}(2m + k - 1)!2^m}{(2n+k)!m!}. \]

Substituting this into the above yields
\[
[x^n]q(x, z) = \frac{(2n - 4)!z^{n-2}}{(n-2)!(n-1)!} \cdot \frac{(2n - 3)}{(2n - 3)} \sum_{m=0}^{\infty} \frac{z^{2m}(3m + 2n - 4)!2^m}{m!(2m + 2n - 3)!} - \frac{(2n - 2)!z^n}{(n-1)!n!} \sum_{m=0}^{\infty} \frac{z^{2m-2}(3m + 2n - 4)!2^{m-1}}{m!(2m + 2n - 2)!} \frac{(2 - (2n - 2) \left( \frac{1}{n - 1} \right) (2n) \left( \frac{m}{2m + 2n - 2} \right))}{(2n - 3)!z^{n-2}} \sum_{m=0}^{\infty} \frac{z^{2m}(3m + 2n - 4)!2^{m-1}}{m!(2m + 2n - 3)!} \left(2 - \left( \frac{2m}{m + n - 1} \right) \right) \]
\[
= \frac{(2n - 3)!z^{n-2}}{(n-2)!(n-1)!} \sum_{m=0}^{\infty} \frac{z^{2m}(3m + 2n - 4)!2^{m-1}}{m!(2m + 2n - 3)!} \cdot \frac{z^{2m}(3m + 2n - 4)!2^{m-1}}{m!(2m + 2n - 3)!}. \]

Thus,
\[
[x^n z^{n+2m-2}]q(x, z) = \frac{(2n - 3)!z^{n+2m-2}}{(n-2)!(n-2)!m!(2n + 2m - 2)!}. \]

For example, \( [x^4 z^4]q(x, z) = \frac{5!7!2^2}{2!2!1!8!} = 15. \) The reader may check that there are four maps with one, two, four and eight rooting, respectively.
Earlier, we calculated \([x^{12}z^{10}] q(x, z) = 16796\). Here it is from the above formula:

\[
[x^{12}z^{10}] q(x, z) = [x^{12}z^{12+2-2}] q(x, z)
\]

\[
= \frac{(2 \cdot 12 - 3)!(2 \cdot 12 + 3 \cdot 0 - 4)!2^{0+1}}{(12 - 2)!(12 - 2)!0!(2 \cdot 12 + 2 \cdot 0 - 2)!}
\]

\[
= \frac{21!20!2!}{10!10!0!22!}
\]

\[= 16796.\]