Tutte Polynomials

Let \( G \) be a connected graph with vertex set \( V(G) \) and edge set \( E(G) \). Let \( n = |V(G)| \) and \( m = |E(G)| \). Let \( F = \{e_1, e_2, \ldots, e_m\} \) be a linear ordering of \( E(G) \). There are \( m! \) choices for \( F \).

Now, let \( T \) be a spanning tree of \( G \). For each edge \( e_j \in F \), there are two possibilities: either \( e_j \in E(T) \) or \( e_j \not\in E(T) \).

If \( e_j \in E(T) \), then \( T - e_j \) has exactly two components, \( A \) and \( B \), each of which contains one endvertex of \( e_j \). Let \([A,B]\) denote the set of edges with one end in \( A \) and the other in \( B \). If \([A,B] \cap \{e_1, e_2, \ldots, e_{j-1}\} = \emptyset \), we say that \( e_j \) is internally-active with respect to \( T \).

If \( e_j \not\in E(T) \), then \( T \cup \{e_j\} \) has a unique fundamental cycle \( C \). If \( E(C) \cap \{e_1, e_2, \ldots, e_{j-1}\} = \emptyset \), we say that \( e_j \) is externally-active with respect to \( T \).

Let \( \text{in}(T,G) \) denote the number of internally-active edges of \( G \) with respect to \( T \), and let \( \text{ex}(T,G) \) denote the number of externally-active edges of \( G \) with respect to \( T \).

We now define the polynomial
\[
D_F(G;x,y) = \sum_T x^{\text{in}(T,G)}y^{\text{ex}(T,G)},
\]
where the sum is over all spanning trees \( T \) of \( G \). Obviously, \( D_F(G;1,1) = \tau(G) \), the number of spanning trees of \( G \). What might not be so obvious is that \( D_F(G;x,y) \) does not depend on the ordering \( F \) that was chosen.

We need to prove this, of course.

As with the lecture on Chromatic Polynomials, \( G - e \) is the graph obtained from \( G \) by deleting edge \( e \) deleted and \( G_e \) is the graph obtained from \( G \) by contracting the edge \( e \). For our use here, we will keep multiple copies of edges if they appear. For example, if \( G = K_3 \) and \( e \) is any edge of \( G \), then \( G_e \) will be a digon.

An edge \( e \) is a loop if \( e \) is the only edge in a cycle of length one. An edge \( e \) is a coloop if \( G - e \) has a larger number of connected components than \( G \) has. That is, \( e \) is in every spanning forest of \( G \). The interested reader might note that the term “coloop” is frequently a sign that one could rewrite things in the terminology of matroids. Spanning forests are bases, cycles correspond to circuits, etc.

The ordering \( F \) induces an ordering \( F' \) on the edge set of \( G - e \) or \( G_e \) in the obvious way. For ease in our argument, we will choose \( e = e_m \), in the following three cases.

**Special case 1.** Suppose that edge \( e \) is a coloop of \( G \). Then, \( e \) is internally active in any spanning tree \( T \) of \( G \). Also, \( T_e \) is a spanning tree of \( G_e \), and every spanning tree of \( G_e \) is \( T_e \) for some spanning tree \( T \) of \( G \). Note that \( \text{in}(T,G) = 1 + \text{in}(T_e,G_e) \) and, since \( e \) is in no fundamental cycle of \( G_e \), \( \text{ex}(T,G) = \text{ex}(T_e,G_e) \).

Hence,
\[
D_F(G;x,y) = xD_{F'}(G_e;x,y).
\]
Special case 2. Suppose that edge $e$ is a loop of $G$. Then, $e$ is externally-active in any spanning tree $T$ of $G$. Any spanning tree of $G$ is a spanning tree of $G - e$. Since $e$ is in exactly one fundamental cycle of $T$, and this cycle is not in $G - e$, $ex(T, G) = 1 + ex(T, G - e)$. Also, the edge $e \not\in [A, B]$ for any partition of the vertex set of $G$, and hence $in(T, G) = in(T, G - e)$. Thus,

$$DF(G; x, y) = yDF(G - e; x, y).$$

Case 3. Suppose that edge $e$ is neither a loop nor a coloop. Then, $e$ is in some of the spanning trees of $G$ and $e$ is not in some of the spanning trees of $G$. Let $T$ be a spanning tree of $G$.

If $e \in E(T)$, then $e$ is not internally active with respect to $T$. Here, $T_e$ is a spanning tree of $G_e$,

$$in(T, G) = in(T_e, G_e) \text{ and } ex(T, G) = ex(T_e, G_e).$$

If $e \not\in E(T)$, then $e$ is not externally-active with respect to $T$. Now, $T$ is a spanning tree of $G - e$,

$$in(T, G) = in(T, G - e) \text{ and } ex(T, G) = ex(T, G - e).$$

Therefore,

$$DF(G; x, y) = DF(G_e; x, y) + DF(G - e; x, y).$$

Combining these three cases, we have, for $e = e_m$,

$$DF(G; x, y) = \begin{cases} 
xD_F(G_e; x, y) & \text{if } e \text{ is a coloop}, \\
yDF(G - e; x, y) & \text{if } e \text{ is a loop}, \\
DF(G_e; x, y) + DF(G - e; x, y) & \text{if } e \text{ is neither a loop nor a coloop}. 
\end{cases}$$

The reader should note that this recurrence appears to be sensitive to the ordering $F$. Our goal will be to show that it is not.

Perhaps the simplest way to see that the order $F$ is not important is to compare the above recursion with the recursion for Whitney’s rank polynomial,

$$R(G; x, y) = \sum_{S \subseteq E(G)} x^{\omega(G; S) - \omega(G)} y^{v(G; S) - \omega(G; S) + \omega(G; S)} = \sum_{S \subseteq E(G)} x^{\rho(G) - \rho(G; S)} y^{v(G; S) - \omega(G; S) + \omega(G; S)},$$

where $G : S$ is the graph with vertex set $V(G)$ and edges set $S$, $\omega(H)$ is the number of connected components of $H$, $v(H)$ is the number of vertices of $H$, and $\epsilon(H)$ is the number of edges of $H$. This polynomial is actually a slight variant of the original rank polynomial, which would be $x^{\rho(G)} R(G; x^{-1}, y)$.

The quantity $\rho(H) = v(H) - \omega(H)$ is the rank of $H$, and is the rank of the $v(H) \times \varepsilon(H)$ incidence matrix $M$ of $H$ (using a column of zeroes for any loop). The cycle space of $H$ is the space $Z(H) = \{ \mathbf{x} : h^T \mathbf{x} = 0 \}$, and we let $\gamma(H) = \dim Z(H) = \varepsilon(H) - v(H) + \omega(H)$. Normally, we use the field of two elements, but in general, $Z_K(H)$ might be used to designate the cycle space of $H$ over the field (or ring) $K$. (Some don’t like to use the word “space” with the word “ring”, but I don’t mind misusing the terminology slightly.)
Let us examine the three cases we used in the Tutte recursion. Let $\rho(H) = $

If $e$ is a coloop, and $e \in S \subseteq E(G)$, then $v(G : S - e) = v(G : S)$, $\omega(G : S - e) = \omega(G : S) + 1$, $\varepsilon(G : S - e) = \varepsilon(G : S) - 1$, $\rho(G : S - e) = \rho(G : S) - 1$, and $\gamma(G : S - e) = \gamma(G : S)$. Hence,

$$ R(G; x, y) = \sum_{S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = \sum_{S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} + \sum_{e \in S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = R(G - e; x, y) + x \sum_{e \in S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = R(G - e; x, y) + x \sum_{e \in S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = (1 + x)R(G - e; x, y). $$

If $e$ is a loop, and $e \in S \subseteq E(G)$, then $v((G : S)_e) = v(G : S)$, $\omega((G : S)_e) = \omega(G : S)$, $\varepsilon((G : S)_e) = \varepsilon(G : S) - 1$, $\rho((G : S)_e) = \rho(G : S) - 1$, $\rho(G_e) = \rho(G)$, and $\gamma((G : S)_e) = \gamma(G : S) - 1$. Hence,

$$ R(G; x, y) = \sum_{S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = \sum_{S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} + \sum_{e \in S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = R(G - e; x, y) + y \sum_{e \in S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = R(G - e; x, y) + y \sum_{e \in S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = (1 + y)R(G_e; x, y). $$

Finally, suppose that $e$ is neither a loop nor a coloop. If $e \in S \subseteq E(G)$, then $v((G : S)_e) = v(G : S) - 1$, $\omega((G : S)_e) = \omega(G : S)$, $\varepsilon((G : S)_e) = \varepsilon(G : S) - 1$, $\rho((G : S)_e) = \rho(G : S) - 1$, $\rho(G_e) = \rho(G) - 1$, and $\gamma((G : S)_e) = \gamma(G : S)$. Thus,

$$ R(G; x, y) = \sum_{S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = \sum_{S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} + \sum_{e \in S \subseteq E(G)} \chi^{\rho(G) - \rho(G:S)} y^{|G:S|} $$

$$ = R(G - e; x, y) + \sum_{e \in S \subseteq E(G)} \chi^{\rho(G_e) - \rho((G:S)_e)} y^{|G:S|} $$

$$ = R(G - e; x, y) + R(G_e; x, y). $$
Combining these three results, we have

\[
R(G; x, y) = \begin{cases} 
(1 + x)R(G_e; x, y) & \text{if } e \text{ is a coloop}, \\
(1 + y)R(G - e; x, y) & \text{if } e \text{ is a loop}, \\
R(G_e; x, y) + R(G - e; x, y) & \text{if } e \text{ is neither a loop nor a coloop}.
\end{cases}
\]

We calculate

\[
R(K_1; x, y) = \sum_{S \subseteq E(K_1)} x^{\rho(K_1) - \rho(K_1; S)} y^{\gamma(K_1; S)}
\]

\[
= \sum_{S \subseteq \emptyset} x^{\rho(K_1)} y^{\gamma(K_1)}
\]

\[
= x^{\rho(K_1)} y^{\gamma(K_1)}
\]

\[= 1,
\]

and

\[
D_F(K_1; x, y) = \sum_T x^{\text{in}(T,G)} y^{\text{ex}(T,G)}
\]

\[= x^{\text{in}(K_1,K_1)} y^{\text{ex}(K_1,K_1)}
\]

\[= 1.
\]

Hence, \(D_F(K_1; x, y) = R(K_1; x - 1, y - 1)\). A simple induction now establishes \(D_F(G; x, y) = R(G; x - 1, y - 1)\).

Note that \(R(G; x - 1, y - 1)\) does not depend on the order \(F\). Hence, \(D_F(G; x, y)\) does not depend on the order \(F\), and we can now write \(D(G; x, y) = D_F(G; x, y)\). The polynomial \(D(G; x, y)\) is usually called the Tutte polynomial. (My professor [WTT] called it the dichromate.)

Since, \(D(G; x, y) = R(G; x - 1, y - 1)\), for a connected graph \(G\), we could extend our definition of \(D(G; x, y)\) to disconnected graphs in the obvious fashion.

**Evaluations of the Tutte polynomial**

Obviously, if \(G\) is a connected graph, then \(\tau(G) = D(G; 1, 1)\) is the number of spanning trees of \(G\). If \(G\) is not connected, then \(D(G; 1, 1)\) is the number of spanning forests of \(G\).

The chromatic polynomial \(P(G, \lambda)\) satisfies the recurrence \(P(G, \lambda) = P(G - e, \lambda) - P(G_e, \lambda)\). Hence, \(P(G, \lambda) = (-1)^{v(G)} \lambda D(G; 1 - \lambda, 0)\), for a connected graph \(G\). To check, assume that \(e \in E(G)\) and \(e\) is neither a loop nor a coloop. If we proceed by induction, then

\[
P(G, \lambda) = P(G - e, \lambda) - P(G_e, \lambda)
\]

\[= (-1)^{v(G-e)} \lambda D(G - e; 1 - \lambda, 0) - (-1)^{v(G_e)} \lambda D(G_e; 1 - \lambda, 0)
\]

\[= (-1)^{v(G)} \lambda D(G - e; 1 - \lambda, 0) + \lambda D(G_e; 1 - \lambda, 0)
\]

\[= (-1)^{v(G)} \lambda D(G; 1 - \lambda, 0),
\]
as needed.

If $e$ is a loop, then $G - e = G_e$, and $D(G - e; 1 - \lambda, 0) = D(G_e; 1 - \lambda, 0)$, which yields the expected $P(G, \lambda) = 0$.

Finally, if every edge of $G$ is a coloop and $G$ has no loops, then $P(G, \lambda) = \lambda(\lambda - 1)^{v(G) - 1}$. Also, every $S \subseteq E(G)$ has $\rho(G : S) = |S|$, $\rho(G) - \rho(G : S) = (v(G) - 1) - |S|$, and $\gamma(G : S) = 0$. Thus,

$$R(G, x, y) = \sum_{S \subseteq E(G)} x^{\rho(G) - \rho(G : S)} y^{\gamma(G : S)}$$

$$= \sum_{S \subseteq E(G)} x^{v(G) - 1 - |S|}$$

$$= \sum_{k=0}^{v(G) - 1} \sum_{S \subseteq E(G)} x^{v(G) - 1 - |S|}$$

$$= \sum_{k=0}^{v(G) - 1} \binom{\rho(G)}{k} x^{v(G) - 1 - k}$$

$$= (1 + x)^{v(G) - 1}.$$ 

Then, $D(G; x, y) = x^{v(G) - 1}$ and 

$$(-1)^{v(G) - 1} \lambda D(G; 1 - \lambda, 0) = (-1)^{v(G) - 1} \lambda (1 - \lambda)^{v(G) - 1} = \lambda (1 - \lambda)^{v(G) - 1},$$

as needed.

Another special case: $D(G; 2, 2) = R(G; 1, 1) = 2^{|E(G)|}$.

The flow polynomial is $(-1)^{e(G) - v(G) + 1} D(G, 0, 1 - u)$. 
