Partitions, Bell Numbers, Stirling Numbers

On Tuesday, we looked at the rook polynomial \( R_n(x) = \sum_{k=0}^{n} \rho(n,k) x^k \) for the triangle \( T_n \), the chessboard corresponding to the \( n \times n \) matrix whose \((i,j)\) entry is 1 if \( i \geq j \) and 0 if \( i < j \).

Suppose that we place a rook, marked by * in the next diagram, on one of the diagonal elements. Then, the remaining elements of \( T_n \) after deleting the row and column occupied by *, comprise a copy of \( T_{n-1} \).

Now, consider the case in which we have \( m \) rooks and we place exactly \( j \) of these rooks \( X \) on the diagonal. Any remaining rooks must appear in the copy of \( T_{n-j} \) that results from deleting the rows and columns which contained a member of \( X \). There are \( \binom{n}{j} \) ways to choose the positions for the elements of \( X \), and \( \rho(n-j,m-j) \) ways to place the remaining rooks. Thus,

\[
\rho(n,m) = \sum_{j=0}^{m} \binom{n}{j} \rho(n-j,m-j).
\]

In class, it was mentioned that \( S(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} S(j,k-1) \), where \( S(n,k) \) is a Stirling number of the second kind. More on these later. We leave the reader to show that

\[
R_n(x) = \sum_{k=0}^{n} S(n+1,n+1-k) x^k.
\]

It was also mentioned that Stirling numbers, in addition to being entries in the change of basis matrix between \( (x^n)_{n=0}^{\infty} \) and \( (x^{(n)})_{n=0}^{\infty} \), count partitions. Some sources define the Stirling number of the second kind, \( S(n,k) \) as the number of partitions of a set of \( n \) objects into exactly \( k \) parts.

Let us begin with simply counting how many partitions, \( B_n \), there are for the set \( I_n = \{1,2,\ldots,n\} \). Consider the partitions of \( I_{n+1} \): One of the parts contains the element 1 and the other parts do not.
The part $X$ containing has cardinality $k + 1$, where $0 \leq k \leq n$. There are $\binom{n}{k}$ ways to choose the elements of $X - \{1\}$, and $B_{n-k}$ ways to partition $I_{n+1} - X$. (We take $B_0 = 1$.) Therefore,

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_{n-k} = \sum_{k=0}^{n} \binom{n}{n-k} B_k = \sum_{k=0}^{n} \binom{n}{k} B_k.$$ 

We now have a recurrence equation for the series $(B_n)_{n=0}^\infty$. The next step is to embed this in a functional equation by means of a generating function. First, let’s make a small change in the presentation of the recursion:

$$\frac{B_{n+1}}{n!} = \sum_{k=0}^{n} \frac{1}{k!} \frac{B_{n-k}}{(n-k)!},$$

which then becomes

$$\frac{B_{n+1}}{n!} x^n = \sum_{k=0}^{n} \frac{x^k}{k!} \frac{B_{n-k}}{(n-k)!} x^{n-k},$$

$$\sum_{n=0}^{\infty} \frac{B_{n+1}}{n!} x^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{B_{n-k}}{(n-k)!} x^{n-k}.$$ 

Writing

$$\beta(x) = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n,$$

we have

$$\frac{d}{dx} \beta(x) = e^x \beta(x),$$

$$\frac{d \beta}{\beta} = e^x dx,$$

$$\ln \beta = e^x + C, \quad \text{for some constant } C.$$ 

Since $\beta(0) = B_0 = 1$, we find that $C = -1$. Now,

$$\ln \beta = e^x - 1,$$

$$\beta(x) = e^{e^x-1}.$$ 

We can partially check our results: $B_n$ should be $n! [x^n] \beta(x)$, where

$$\beta(x) = e^{e^x-1} = 1 + x + x^2 + \frac{5}{6} x^3 + \frac{5}{8} x^4 + \frac{13}{30} x^5 + \frac{203}{720} x^6 + \cdots.$$ 

Reading the coefficients, we find $B_0 = 1, B_1 = 1, B_2 = 2, B_3 = 5, B_4 = 15, B_5 = 52, B_6 = 203$. The
numbers $B_n$ are called the Bell numbers. (See also: http://oeis.org/A000110.)

Now, suppose that we instead wish to count how many partitions of $I_n$ have exactly $p$ parts. Call this number $n_{p}^{\, n}$.

We attempt to follow the argument given above. To calculate $n_{p}^{\, 1}$, we consider a partition of $I_{n+1}$ with exactly $p$ parts, such that the element 1 is in part $X$, and suppose that $|X| = k + 1$. There are $\binom{n}{k}$ ways to choose the other elements of $X$, and there are $n_{k}^{\, p}$ possible partitions of $I_{n+1} - X$ using exactly $p - 1$ parts. Hence,

$$
\sigma(n + 1, p) = \sum_{k=0}^{n} \binom{n}{k} \sigma(n - k, p - 1).
$$

This is the recurrence relation mentioned above for Stirling numbers of the second kind. We’ll pause to consider the change of basis property of the numbers $n_{p}^{\, n}$. First, a few values. By definition, $n_{p}^{\, 0} = 0$ if $n > 0$, $n_{n}^{\, 1} = 1$, $n_{1}^{\, 1} = 1$, and $n_{p}^{\, n} = 0$ if $p > n$. A few other values are displayed in the table below.

<table>
<thead>
<tr>
<th></th>
<th>$p = 1$</th>
<th>$p = 2$</th>
<th>$p = 3$</th>
<th>$p = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 2$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n = 3$</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$n = 4$</td>
<td>1</td>
<td>7</td>
<td>6</td>
<td>1</td>
</tr>
</tbody>
</table>

Note that, for example,

$$
x(4) + 6x(3) + 7x(2) + x(1)
= x(x - 1)(x - 2)(x - 3) + 6x(x - 1)(x - 2) + 7x(x - 1) + x
= x^4.
$$

Modifying the steps for $B_n$ again: Let

$$
\theta(x, y) = \sum_{n=0}^{\infty} \sum_{p=0}^{n} \frac{\sigma(n, p)}{n!} x^n y^p.
$$

Then,

$$
\frac{n_{p}^{\, 1}}{n!} x^n y^p = y \sum_{k=0}^{n} \frac{x^k}{k!} \frac{\sigma(n - k, p - 1)}{(n - k)!} x^{n-k} y^{p-1},
$$

$$
\sum_{n=0}^{\infty} \sum_{p=0}^{n+1} \frac{n_{p}^{\, 1}}{n!} x^n y^p = y \sum_{n=0}^{\infty} \sum_{p=1}^{n+1} \sum_{k=0}^{n} \frac{x^k}{k!} \frac{\sigma(n - k, p - 1)}{(n - k)!} x^{n-k} y^{p-1},
$$
\[
\frac{\partial}{\partial x} \theta(x, y) = ye^x \theta(x, y),
\]

\[
\ln \theta = ye^x - y,
\]

\[
\theta(x, y) = e^{(e-1)xy}
\]

\[
= 1 + xy + \frac{x^2}{2!} (y + y^2) + \frac{x^3}{3!} (y + 3y^2 + y^3)
\]

\[
+ \frac{x^4}{4!} (y + 7y^2 + 6y^3 + y^4) + \frac{x^5}{5!} (y + 15y^2 + 25y^3 + 10y^4 + y^5)
\]

\[
+ \frac{x^6}{6!} (y + 31y^2 + 90y^3 + 65y^4 + 15y^5 + y^6) + \cdots
\]

Suppose we want to distribute \( n \) distinct objects into \( r \) distinct cells, with no cell being empty. One way to do this would be to partition the set of \( n \) objects into \( r \) non-empty parts in \( \sigma(n, r) \) ways, and then assign each of these parts to a cell in \( r! \) ways.

But, we’ve also studied inclusion exclusion. The number of ways to leave a given set of \( k \) cells empty (and possibly other cells empty) is \( (r - k)^n \). There are \( \binom{r}{k} \) ways to choose the \( k \) cells. Thus, the number of ways to leave exactly zero cells empty is

\[
\sum_{k=0}^{r} (-1)^k \binom{r}{k} (r - k)^n,
\]

and therefore,

\[
r! \sigma(n, r) = \sum_{k=0}^{r} (-1)^k \binom{r}{k} (r - k)^n, \quad \text{or}
\]

\[
\sigma(n, r) = \sum_{k=0}^{r} (-1)^k \frac{1}{k!} \frac{(r - k)^n}{(r - k)!}.
\]

A simpler recurrence for these Stirling numbers can be derived. Consider a partition of \( I_{n+1} \) into \( p + 1 \) parts. The number \( n + 1 \) can either be by itself, \( \sigma(n, p - 1) \) ways, or in any one of the \( p \) parts used in a \( p \)-part partition of \( I_n \). There are \( \sigma(n, p) \) such \( p \)-part partitions of \( I_n \) and for each there is a choice of parts to which \( n + 1 \) can be assigned. Thus,

\[
\sigma(n + 1, p) = \sigma(n, p - 1) + p \sigma(n, p).
\]

We claim that

\[
x^n = \sum_{p=1}^{n} \sigma(n, p) x_{(p)}.
\]

Note that \( x^1 = x_{(1)} = \sigma(1, 1) x_{(1)} = \sum_{p=1}^{1} \sigma(1, p) x_{(p)} \). Let’s suppose that for some \( n > 0 \),
\[ x^n = \sum_{p=1}^{n} \sigma(n,p) x(p). \] Then,

\[
x^{n+1} = \sum_{p=1}^{n} \sigma(n,p) (x) x(p)
\]

\[
= \sum_{p=1}^{n} \sigma(n,p) (x - p) x(p) + \sum_{p=1}^{n} p \sigma(n,p) x(p)
\]

\[
= \sum_{p=1}^{n} \sigma(n,p) x(p+1) + \sum_{p=1}^{n} p \sigma(n,p) x(p)
\]

\[
= \sum_{p=2}^{n+1} \sigma(n,p - 1) x(p) + \sum_{p=1}^{n} p \sigma(n,p) x(p)
\]

\[
= \sum_{p=1}^{n+1} \sigma(n,p - 1) x(p) + \sum_{p=1}^{n+1} p \sigma(n,p) x(p)
\]

\[
= \sum_{p=1}^{n+1} \left( \sigma(n,p - 1) + p \sigma(n,p) \right) x(p)
\]

\[
= \sum_{p=1}^{n+1} \sigma(n+1,p) x(p).
\]

This establishes the Stirling numbers of the second kind as components of the change of basis matrix previously mentioned.

In class, we also mentioned Stirling numbers of the first kind, and that one definition of these is given by

\[ x(n) = x(x - 1)(x - 2) \cdots (x - n + 1) = \sum_{k=1}^{n} s(n,k) x^k. \]

Since \( x(n+1) = (x - n) x(n) \), equating coefficients yields

\[ s(n+1,k) = s(n,k - 1) - n s(n,k). \]

Note that

\[ x^{(n)} = x(x+1)(x+2) \cdots (x+n-1) = \sum_{k=1}^{n} \{s(n,k)\} x^k, \]

and \( \{s(n,k)\} = (-1)^{n-k} s(n,k) \) is called an unsigned Stirling number of the first kind. Note that

\[ \{s(n+1,k)\} = \{s(n,k - 1)\} + n \{s(n,k)\}. \]

It is immediately obvious that substituting \( x = 1 \) into
\[
x(x + 1)(x + 2)\cdots(x + n - 1) = \sum_{k=1}^{n} s(n, k) \cdot x^k
\]
yields
\[
n! = \sum_{k=1}^{n} s(n, k)!
\]
This hints at a link between the unsigned Stirling numbers of the first kind and elements of the permutation group.

For example, \(x(x + 1)(x + 2)(x + 3) = x^4 + 6x^3 + 11x^2 + 6x\), and the elements of \(S_4\) could be written as

\[
\begin{align*}
(1)(2)(3)(4) & \quad (1)(2)(3 4) & \quad (1 2)(3 4) & \quad (1 2 3 4) \\
(1)(3)(2 4) & \quad (1 3)(2 4) & \quad (1 2 4 3) \\
(1)(4)(2 3) & \quad (1 4)(2 3) & \quad (1 3 2 4) \\
(2)(3)(1 4) & \quad (1)(2 3 4) & \quad (1 3 4 2) \\
(2)(4)(1 3) & \quad (1)(2 4 3) & \quad (1 4 2 3) \\
(3)(4)(1 2) & \quad (2)(1 3 4) & \quad (1 4 3 2) \\
(2)(1 4 3) & \quad (3)(1 2 4) & \quad (3)(1 4 2) \\
(3)(1 2 4) & \quad (3)(1 4 2) \\
(4)(1 2 3) & \quad (4)(1 3 2)
\end{align*}
\]

Let \(q(n, k)\) denote the number of permutations of \(S_n\) which have exactly \(k\) cycles (orbits).

Suppose that we know \(q(n, k)\) for all \(k\) for a fixed value of \(n\). Then think about a permutation \(\pi \in S_{n+1}\), suppose that \(\pi(y) = n + 1\) and \(\pi(n + 1) = z\). Construct \(\pi' \in S_n\) by \(\pi'(j) = \pi(j)\) if \(j \neq y\) and, when \(y \neq n + 1\), set \(\pi'(y) = z\). Note that \(\pi'\) has the same number of cycles as \(\pi\), except when \(\pi(n + 1) = n + 1\), and in this case \(\pi'\) has one fewer cycles.

The reverse procedure takes a permutation \(\pi'\) and constructs a permutation \(\pi \in S_{n+1}\). There will be two possibilities.

(i) \(\pi'\) has \(k\) cycles. Then \(\pi = \pi' \cdot (n + 1)\).

(ii) \(\pi'\) has \(k\) cycles. Chose any \(y \in I_n\). Let \(\pi(y) = n + 1\), \(\pi(n + 1) = \pi'(y)\), and for any \(j \in I_n - \{y\}\), let \(\pi(j) = \pi'(j)\).

Type (i) gives us \(q(n, k - 1)\) permutations, and type (ii) gives us \(n \cdot q(n, k)\) permutations. Thus, \(q(n + 1, k) = q(n, k - 1) + n \cdot q(n, k)\), which is exactly the same recursion as \(s(n, k)\). Establish the proper basis, and we see that \(q(n, k) = s(n, k)\).
Recall $\delta_{n,k} = 1$ if $n = k$ and $\delta_{n,k} = 0$ if $n \neq k$. (Kronecker delta.)

Two obvious results:

$$\sum_{j=0}^{n} s(n,j) \sigma(j,k) = \delta_{n,k},$$

$$\sum_{j=0}^{n} \sigma(n,j) s(j,k) = \delta_{n,k}.$$  

Since these are the convolutions that provide the $(n,k)$ elements of the products of the change of basis matrices, the entry is zero if off the diagonal and one if on the diagonal.