Example 2.2.3, strings in $\{0,1\}^*$ avoiding substring 11.

On Thursday we discussed the composition given in Example 2.2.23, but I made a substitution a bit early.

We were counting the strings in $T = \{0,1\}^*$ with no consecutive pair of ones. So, let $S$ denote this set, and for a string $\sigma \in T$, let $w(\sigma) = (i(\sigma), j(\sigma))$, where $i(\sigma)$ is the number of ones in $\sigma$ and $j(\sigma)$ is the number of zeroes in $\sigma$. Note that $w : T \to \{0,1,\ldots\}^2$ rather than simply giving a scalar weight.

We also used the set $W = 1(1)^* = \{1,11,111,\ldots\}$.

Now, $g(x,y) = \sum_{\sigma \in T} x^{i(\sigma)} y^{j(\sigma)} = \frac{1}{1-x-y}$ is rather immediate.

Also, $h(x) = \sum_{\sigma \in W} x^{i(\sigma)} = \frac{x}{1-x}$ is obvious.

Then, it is noted that $T$ is obtained from $S$ by replacing each one in any string $\sigma$ by an element of $W$, and thus, $T = S \circ W$.

Thus, $g(u,y) = f(h(u),y)$. At this point in class, I substituted $y = u$, but that is too early.

From $g(u,y) = f(h(u),y)$, we have $\frac{1}{1-u-y} = f(u, \frac{u}{1-u}, y)$.

Now, we can set $t = \frac{u}{1-u}$, and solve for $u$, to find $u = \frac{t + 1}{1 + t}$.

Hence, $f(t,y) = g\left( \frac{t}{1+t}, y \right) = \frac{1}{1 - \frac{t}{1+t} - y} = \frac{1 + t}{1 - y(1+t)}$ and $f(t,t) = \frac{1 + t}{1 - t - t^2}$.

One possibility at this point is to use partial fractions: If $\phi = \frac{1 + \sqrt{5}}{2}$ and $\tau = \frac{1 - \sqrt{5}}{2}$, then $\phi + \tau = \frac{1 + \sqrt{5}}{2} + \frac{1 - \sqrt{5}}{2} = 1$, $\phi - \tau = \frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2} = \sqrt{5}$, and $\phi \tau = \left( \frac{1 + \sqrt{5}}{2} \right) \left( \frac{1 - \sqrt{5}}{2} \right) = -1$. Thus, $(1 - \phi t)(1 - \tau t) = 1 - t - t^2$. From this, or directly,

$1 + \phi - \phi^2 = 0 = 1 + \tau - \tau^2$.

Now,
Another approach is to let \( f(t, t) = \frac{1 + t}{1 - t - t^2} = \sum_{n=0}^{\infty} c_n t^n \), and to note that

\[
[t^{n+2}](1 - t - t^2) \sum_{n=0}^{\infty} c_n t^n = [t^{n+2}](1 + t) \text{ yields } c_{n+2} = c_{n+1} + c_n, \text{ for } n \geq 0. \text{ Together with } c_0 = 1, c_1 = 2, \text{ we recognize the Fibonacci numbers (shifted).}
\]

Let’s also take a step back to

\[
f(t, y) = \frac{1 + t}{1 - y(1 + t)} = \sum_{j=0}^{\infty} y^j(1 + t)^{j+1} = \sum_{j=0}^{\infty} \sum_{i=0}^{j+1} \binom{j + 1}{i} t^i y^j,
\]

which tells us the coefficients of \( f(t, y) \) explicitly. That is, there are \( \binom{j + 1}{i} \) strings in \( \{0, 1\}^* \) with no consecutive ones, and with exactly \( i \) ones and \( j \) zeros.

As an added bonus. setting \( y = t \) yields \( \sum_{i=0}^{\infty} \binom{n + 1}{i} - i = c_n \), proving an identity for Fibonacci numbers.